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OPERUM NEWTONI

TOMUS PRIMUS.

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ISAACI NEWTONI

OPERA

QUÆ EXSTANT OMNIA.

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COMMENTARIIS ILLUSTRABAT

SAMUEL HORSLEY, LL. D. R. S. S.

REVERENDO ADMODUM IN CHRISTO PATRI

ROBERTO EPISCOPO LONDINENSI A SACRIS.

LONDINI:

EXCUDEBAT JOANNES NICHOLS.

M DCC LXXIX.

A D

R E G E M.

QUOD eorum, Rex Augustissime, perpaucis contigit, [redacted]gno aliquo opere, Te [redacted]uum cum muneribus [redacted]veniſſe, imperare mihi [redacted]r ut ea ſit hominis le[redacted]io. Haud enim noſtr[redacted]o, ſed Newtoni. Ejus [redacted]ædam excellentia inge[redacted] exteros etiam eſt conſ[redacted]ſt natos homines, “ qu[redacted] ætatis quâ vixit, “ quàm generis humani ornamento natus eſſe videretur.” Qui primùm reconditam illam Mathematicorum ſcientiam ad illud perduxit faſtigium, quod vel principes illi viri Conon et Archimedes, ſi reviviſcere poſſent, mirarentur. Ita quæ illis difficillima erant, hic facillima reddidit; quæ verò illi nullâ arte ſuperari poſſe ſperaverant, hic, quantum vel uſus Mechanicês poſtulet, vel ad rerum naturalium inquiſitionem ſatis ſit,

ſuperavit. Tum verò Phyſicam ingreſſus, Geometriâ illâ ſuâ facem præferente, quantas ibi tenebras diſcuſſit, quæ opinionum commenta delevit! Soli ſuam in centro Mundi ſedem vindicavit. Non ut ante eum Copernicus, et Veterum nonnulli à Philolao profecti, conjecturâ tantùm et probabili ratione fretus, ſed certiſſimis rationum concluſionibus, quas ex iis ipſis deduxit, quæ Aſtronomi in cælis fieri quotidie cernunt. Importuno Vorticum onere cælos liberavit; quippe Orbes Solidos jam olim diſturbaverant machinæ Tychonis. Fibras etiam ſuſtulit, quas, magneticâ virtute præditas, in corpore ſolari aliiſque cæleſtibus Keplerus poſuerat. Planetas vi proprii ponderis in orbe quemque ſuo retineri oſtendit. Eâdem vi Cometas circa Solem in orbem agi. Vagum et multiplicem Lunæ curſum calculis etiam ſubjicere docuit; ponderis ejus tam in Solem quàm in Terram ratione habitâ. Feliciſſimam Kepleri conjecturam de cauſâ Æſtus Marini, invictiſſimis rationibus, calculis etiam ſuffragantibus, firm[illegible]it. Figuræ corporis Terræ quænam ratio ſit, oſtendit; ut, [illegible] propriam ejus circa axem ſuum vertiginem, ſub polis planior fiat, mediis partibus turgeſcat. Ut inde oriri neceſſe ſit, quod in Horologiis oſcillatoriis uſu venit, inclinato magis axe lentiùs, erecto incitatiùs incedentibus. Ut etiam à figurâ Terræ inæqualiter globoſâ cauſa arceſſenda ſit, quæ puncta æquinoctialia per ſæcula retrorſum impulit. Spatium vacuum quod Scholæ tantopere reformidârunt, illud quidem rerum Naturam non modò pati, verùm etiam poſtulare oſtendit. Nec ratione tantùm id perſuaſit, ſed rem à ſenſu remotiſſimam ſenſibus quodammodo hominum et oculis ſubjecit; ipſâ quidem experientiâ in corporibus, tum pendulis, tum cadentibus, doctrinam ejus mirè comprobante. Luminis Naturam, quæ et ipſa olim abditiſſima fuerat, in clarâ luce poſuit. Ita ut occultas omnes,

quæ vocabantur, qualitates corporibus abjudicârit, & à rerum Naturâ prorſus exulare juſſerit. Nullam enim virtutem, nullam vim, ullius corporis propriam et innatam agnovit, præter unam illam quæ ſenſibus noſtris maximè patet, quam Inertiæ nuncupavit, quamque corporibus omnibus, pro materiæ cujuſque modo, communem ſtatuit. Hâc vi, accedente mutuâ quâdam corpuſculorum omnium nunc appetentiâ nunc fugâ, omnia voluit pro notiſſimis legibus mechanicis *magnum per Inane geri*. Appetentiam illam et fugam non corporibus ipſis naturalem ſtatuit; non cum Epicuro æternam, eandem tamen fortuitam; non à Veterum quorundam Amore et Odio, Juniorum Sympathiâ et Antipathiâ profectam; ſed à cauſâ quâdam incorporeâ, quæ, ut ſenſus noſtros maximè lateat, intellectum tamen et mentis aciem haud fugit. A DEO utique hæc omnia proficiſci palam vociferatus eſt. Hujus arbitrio omnia fuiſſe procreata, hujus Imperio ſubeſſe, hujus conſilio et ſapientiâ regi. Hunc rerum omnium Opificem, Conditorem, Dominum; Unum, Potentiſſimum, Æternum, Immenſum; ſine quo non pulcherrimus hic rerum ordo, non Res ulla, non Locus rerum ullus, non Tempora et Ætates, non Spatium ipſum vacuum, non Tempus individuum Æternitatis eſſet. Hunc ex aſſiduâ Naturæ contemplatione nôſſe plurimarum ſuarum vigiliarum maximum jucundiſſimumque fructum habuit. Nec ſatis tamen habuit ex Naturâ eum nôſſe, niſi uberior ejus ſolaque ſalutaris cognitio accederet, ex Filii ejus Evangelio unicè haurienda. Igitur in libris Sacri Codicis, non legendis modò, verùm etiam illuſtrandis, plurimum temporis et ſtudii poſuit. Neque inclinatâ ſolum id ætate, quod nonnulli perſuadere volunt, et ſenio jam hebetatâ mentis acie, ſed integris adhuc et cum maximè vigentibus corporis animique viribus. Neque alio conſilio Temporum emendationem ſuſcepiſſe videtur, niſi ut

antiquiſſimis Hiſtoricis Sacris ſua conſtaret fides; tum hominum monumentis, tum cæli converſionibus comprobata. Quippe cùm indiciis rerum vetuſtiſſimarum undique conquiſitis, temporum diſtributionibus ad calculos revocatis, motuum etiam cæleſtium ratione exquiſitè habitâ, eam omninò ſe obtinuiſſe conſideret humani generis novitatem, quæ cum Telluris noſtræ Archæologiâ Sacrâ optimè conſiſteret. Sed in excutiendis tam perantiquis illis Danielis quàm recentioribus Divi Joannis vaticiniis diligentiâ haud minore uſus eſt. Haud eâ mente, quòd temporis futuri exitum, à ſapientiſſimo rerum Conditore altâ nocte merſum, improbo conatu in lucem poſſe protrahi ſperaret: quippe inſipientis id fuiſſet, ultra hominem voluiſſe ſapere. Sed ut vaticinia multis retro ſæculis edita cum hominum memoriâ conferendo, oſtenderet Naturæ de Deo judicium (quod et ipſum quidem certiſſimum et reverà divinum eſt) alio et apertiore longè Dei ipſius de ſe teſtimonio confirmari; unde docti æquè indoctique intelligerent Dei nutu et providentiâ omnia geri. Simul ut hominibus perſuaderet, ſanctos illos viros, qui ea prædixerunt futura, quæ ſæcula poſt nata fieri viderunt, Dei numine afflatos: nec vana idem eos cecinisse, quæcunque de interitu hujus Mundi cecinere, de Improborum apud Inferos ſupplicio, de ævo beato, quod Boni et Fideles, Chriſti ſanguine redempti, cum Deo ſempiternum agent. Hanc tantam tamque variam ſummi viri doctrinam, eximias hujus animi præcellentis dotes commendabat ſingularis innocentia vitæ, raraque vel priſcis etiam temporibus morum ſanctimonia. Hujus Viri ſcripta (quæ eſt hominum incuria) ſparſa hunc uſque diem et disjecta, quanquam tertius à quinquageſimo nunc vertatur annus à quo è vivis exceſſit, nunc primùm in unum corpus collecta, Tuis, Rex Optume, auſpiciis in lucem prodire incipiunt. Metuendum eſſet, ne iniquis admodum

temporibus,

temporibus, niſi Te Patronum haberent. Quis enim inter ſtrepitus armorum civilium Philoſophiæ locus? Fuiſſe tamen, in hâc atrocitate temporum, qui Te Fautore et Auctore optimis Scientiis haud ſegniter operam navârint, ut ipſis non inhoneſtum, Tibi certè apud poſteros etiam glorioſum erit. Nec vana auguror, O Rex. Etenim hanc Tui veriſſimam laudem non Malorum obtrectatio, non ævi longinquitas abolere poterit: Te regnante, Scientiam tum naturalem tum divinam tanta incrementa cepiſſe, quanta ſub Uno eorum nemine, qui Reges ante Te fuerunt, rebus etiam pacatiſſimis. Namque Newtonus longiſſimum ſuæ vitæ gloriæque curriculum quinque Principum temporibus æquavit. Nondum bis decem Anni ſunt cùm Tu regnare orſus es, jamque vidimus remotiſſimos marium tractus Navibus Tuis plus ſemel exploratos; Vidimus conſtitutam, certiſſimis obſervationibus Aſtronomicis, Solis à Tellure noſtrâ diſtantiam; definitam ideo ſpatii omnis cis Saturnum ſtellam amplitudinem. Quod repertum etſi ipſa rerum Natura noſtris temporibus deſtinaverit, quippe quæ ſiderum motus eis legibus aſtrinxerat, ut expectatiſſimi illi ſtellæ Veneris per diſcum ſolarem tranſcurſus in aliam hominum ætatem haud inciderent; quòd pleniſſimus tamen ejus ſpectaculi fructus in Philoſophiam redundàrit, Tuæ, Rex Inclyte, Munificentiæ nemo non acceptum refert. Gravitatis Doctrinam novis obſervationibus, Te annuente, in ultimâ Caledoniâ inſtitutis confirmatam vidimus. Perſcrutatam ſubtiliſſimis experimentis Aëris naturam. Vidimus, à Te ſuſtentatas, in Britanniâ Tuâ adoleſcere inſolitas hoc cœlo artes: Urbem novâ extructionum magnificentiâ ſuperbire: Celſa ſurgere Procerum palatia: Intùs etiam ea ſpectandis, non ut olim exterorum, ſed civium operibus condecorari. Quod verò hiſce omnibus majus eſt et divinius, præcipua religionis noſtræ dogmata, praviſſimis ſtudiis, proh dolor! impugnata, animoſè tamen ac naviter de-

fenſa

fenſa vidimus. Vidimus ſcriptos codices ſacri textûs Hebraici undique conquiſitos, penitùs excuſſos; magno certè perennique literarum ſacrarum emolumento. Quod Opus ut labore conſtitit! ut ſumptibus! quibus populorum omnium Reges Proceresſque ut impensè ſe præbuerunt! Ita Tu ſcilicet propriæ largitatis exemplo omnium ſtudia accenderas. Videmus Prophetam diſertiſſimum Iſaiam, ejus Viri curis quem Tu Eccleſiæ Tuæ Londinenſi præfeciſti, modò non reviviſcere, & ſermone populari luculentè converſum myſteria Dei clarè et apertè loqui. Hæc Te regnante Ætas Noſtra vidit. Plura, credo, majoraque videbit. Ita omnes bonis literis, quarum Te nôrunt amantiſſimum, impigri et alacres incumbunt. Faxit Deus O. M. creſcat indies in hoc populo ſcientiæ amor. Te, Pater, tantum Doctrinarum Artiumque omnium Patronum, diu nobis ſoſpitet, tueatur, cuſtodiat. Tibi verò gratum precor ſit Munus, quod reverentiæ et officii cauſſâ Tibi dicat, qui in ſubditorum Tuorum fideliſſimorum numero nomen ſuum profiteri geſtit, unus idem ex humillimis

SAMUEL HORSLEY.

LECTORIS.

SOLENT ii, qui in aliquâ re magnam ſui expectationem concitaverint, primùm omnibus modis oſtendere, officio ſe non defuiſſe; deinde apud æquos judices ſuffragia eorum bonâ cum ſpe opperiri. Quod etiamſi non artis eſſet, et conſuetudinis, tamen à nobis commodè faciendum eſt, qui Newtoni Opera cum inſigniore quodam apparatu in lucem edere decrevimus. Etenim, ſi illud verum ſit, quo gravior doctrinæ locus aliquis et ſplendidior, tanto magis cognitione atque induſtriâ egere, qui eum tradituri ſint; omnino nos neque exile munus, neque otioſum ſumus adepti.—Tum verò Auctorem ſequimur, ſuper graviſſimis rebus preſſè ſæpiùs et anguſtè locutum, præ ingenii amplitudine aſperum aliquando et perdifficilem; ita ut magnum Interpreti negotium faceſſerit, qui non tam omnia demonſtraviſſe, quàm velut ex oraculo edidiſſe, quæ mente animoque rata ſibi penitus habuerat, arbitrandus eſt. Igitur omnibus conſiliis, et quâcunque demum ratione enitendum eſt, quo plenior fiat et illuſtrior tanti hujus Viri explicatio. Ita cùm in decurſu operis ubique ea attulimus, quæ diſcentium induſtriæ ſuppetias eant, ſparſa tamen atque uſu et genere diverſa, anteire debet totius rei, et ordinis quem ſecuti ſumus tractatio. Quod et Tironibus magno certè adjumento futurum eſt, ſi quis manum porrigat, et iter ingreſſuris etiam viæ ſubſidia palam oſtendat.

Ex quo itaque publici juris fuit hujuſce Operis proſpectus, ita rem univerſam partiri ſtatuimus, ut primum è quinque Tomis abſolverent Newtoni tractatus omnes ad puram, quæ dicitur, Matheſin quodammodo pertinentes. Quorum etiam initio poſuimus Arithmeticam Univerſalem; poſt eam, cætera quæ ad Series et Fluxionum Doctrinam ſpectant Opuſcula. De ſingulis ſingulatim dicendum.

ARITHMETICAM UNIVERSALEM *perfecti Operis formâ caruiſſe nemo omnium eſt, qui non intelligat; cùm vix aliud præ ſe ferat, quàm*

Præ-

Prælectionum Capita, quas Newtonus Cantabrigiæ jam olim è Cathedrá tradebat. Ita monita et præcepta, quæ ibi deprehendas, Aphoriſmis ferè omnino concluſa ſunt, neque ratiociniis firmata, neque demonſtrationibus legitimis abſoluta—Quin et de Algebræ inventoribus, aliiſve qui, ante Newtonum, rem quadantenus promoverant, altum ubique ſilentium. Quæ omnia et in ſecundá iſtius libri editione pariter deſideramus; magno poſteris infortunio, quòd, annis jam ingraveſcentibus, præterita repetere, et ultimam illis manum imponere Newtonus ipſe non tulerit.

Igitur nuſquam aliás ab Interprete magis elaborandum erat; ubi indaganda omnia: quæ nimis anguſtè tradita ſunt, fuſiùs efferenda: et ſubtilis artificii Regulæ ferè ab integro ad examen revocandæ. Neque ſimplicem profectò ſuper his impendimus laborem—Quæ levioris erant operæ, ſubjectis ubique annotationibus pro re natá effecimus, explendo ea quæ breviùs apud Newtonum ſcripta ſunt, et inventa, quæ aliunde originem traxerant, ſuis Auctoribus adjudicando. Graviora alia, et majoris momenti in Appendicem contulimus undeviginti capitibus expoſitam, quam in animo erat ad calcem ſtatuiſſe, nì mole ſuá ultra juſtam hujuſce Tomi magnitudinem accreverat. Olim fortaſſe et ſeorſim ab hoc opere proditura eſt. Tantùm hîc loci ſubjecto Indice innuimus, quænam ibi præcipuè tractare viſum eſt.

ELENCHUS CAPITUM

Appendicis in Arithmeticam Univerſalem.

I.

Traditur Methodus inveniendi numeros primos per cribrum Eratoſthenis.

II.

Inventio diviſorum ſimplicium ex mente Newtoni, demonſtratione munitur.

III.

Agitur de inventione diviſorum compoſitorum. Recenſentur Cl. Virorum Carteſii, De Beaume, Bartholini, Van Hudde, et Waſſanaeri

ſanaeri in hâc re progreſſus. Oſtenditur quantùm cæteros omnes Newtonus antecelluit.

IV, V, VI, VII.

In ſe habent demonſtrationes Newtoni regularum, quæ diviſoribus compoſitis inveniendis inſerviant. Quas et fatemur è Graveſandi Specimine Commentarii in Arithmeticam Univerſalem *petitas eſſe; exili libello, cui tamen omne illud retuliſſe debuerant, quicquid in hoc opere Maclaurinus, Bernouillius et Clairaultius uſpiam profecerint.*

VIII.

Inſtituitur demonſtratio Newtoni Regulæ, quam ex Bombellio et aliis expreſſerat, de extrahendâ Radice quadraticâ Multinomii è quantitatibus integris et radicalibus compoſiti—Subjicitur et alterius regulæ, quam in eandem rem ex Stevino attulerat, explicatio—Harum ultimam oſtenditur in irritum plerumquè cadere, niſi priùs adbibitâ quâdam multinomii tractatione, quæ et ſæpiùs operoſior, quàm ut quis libenter ad eam confugiat.

IX.

Regula Newtoniana extrahendi altiores Radices ex quantitatibus numeralibus demonſtratione confirmatur, quam etiam Graveſandi libellus ſufficere potuiſſet. Subjicitur hujuſce artificii origo; quantùm in illo, præ aliis Waſſanaerus profecerat; et quàm longè in eodem opere cæteros omnes poſt ſe reliquit Newtonus.

X.

Traditur methodus generalis quantitatem incognitam, ubi binæ pluresve ſint, exterminandi; quæ nuſquam non ſufficiet, ubi ad alias omnes fruſtra confugeris: modò ſatis artificioſè ſit adhibita.

XI.

Exponitur methodus Fermatiana quantitates quotcunque ſurdas ex æquationibus tolli, ubi Climactiſmi ope Aſymmetriâ eas liberare non datur.

XII.

Demonſtratur Carteſii Regula, cujus ope cognoſcitur quot ſint Radices negativæ, ubi æquationis radices nullæ ſint impoſſibiles. Cujus rei cùm Principia poſuerat Kæſtnerus è Gottingiæ Profeſſoribus, tamen Marte proprio eam proveximus, et omnino Geometricè abſolvimus.

XIII.

Ad mentem Cl. Viri Campbelli, apud Acta Regiæ Societatis, demonſtratione confirmatur Newtoni Regula, per quam ferè cognoſci poteſt, quot ſunt Radices impoſſibiles: ubi aliunde deprompta plenius tamen et perfectiùs ipſi abſolvimus.

XIV.

Newtoni Regulà de colligendo ſummam poteſtatum Radicis incognitæ luculentè exponitur. Oſtenditur illam ipſam Albertum Girardum jam antea tradidiſſe, magni nominis virum, et in Analyſeos ſpecioſæ inventis modò non Vietæ æqualem, utcunque incuriâ hominum ferè in deſuetudinem abierit.

XV.

Adducitur Limitum demonſtratio, intra quos radices æquationum cenſendæ ſunt conſiſtere.

XVI.

Additamentum D^{ni} D^{ris} B. Taylori de Æquationum proprietatibus, Commentarii loco, ſubjicitur iis, quæ Newtonus in hâc re ſtatuerat.

XVII.

A principiis ſuis excutitur Regula de Æquationum Reductione per diviſores ſurdos. Quem locum ubi Caſtillioneus attigerat, nuſquam tamen expoſuit, quod apud Newtonum tam ſubtile atque ele-

 gans,

gans, quomodo inveniantur quantitates Græcis Litteris deſignatæ. Idem neque aliùs quiſquam, ut opinor, ante nos perfecit.

XVIII.

Breviter abſolvuntur ea, quæ à Vietâ et Alberto Girardo tradita ſunt, de proferendis Radicibus Æquationis Cubicæ $x^3-qx\mp r=0$ *per anguli triſectionem, ubi Cardani Regulæ defecerint; ita ut impeditam rem Geometriæ ope pulcherrimè aliquis promoveat.*

XIX.

Quæritur cauſa, unde Cardani Regulæ unquam deficiant, vel, ut aliter dicam, unde quantitates quæ reverâ et actu ſunt, unicè deſignari poſſint Algebraico ſermone, ut quæ exiſtentiam non habeant.

Et hæc quidem de Appendice noſtrâ dicta ſunto—Quam ſi flagitent homines Mathematici, abſoluto hoc Opere, et negotiis vacuus, ut et alia fortaſſe ad Newtonum ſpectantia, in lucem edere non gravabor.

Transeo *ad alteram hujuſce Tomi partem; quæ ut ſuprà dictum eſt, congeſta habet* Newtoni *de Seriebus et Fluxionum Doctrinâ Opuſcula. Quarum rerum, ſiqua apud nos ſit intelligentia, illam in univerſum omnem ex ipſius Newtoni ſcriptis conquiſitam eſſe edicimus; aliis relicturi—*erroremque hoſtibus illum—*velle minimo ſudore rem agere, Interpretum genus omne aſſectari, et aliunde potiùs quàm ab ipſo Auctore veram illius doctrinam exquirere. Quod ſi in aliis rebus inſcitiæ eſt, in iſtiuſmodi diſciplinâ graviùs vituperandum eſt, ubi ſubtilia omnia, et in acie novaculæ, quod aiunt, vera terminorum poſita ſit æſtimatio.*

Nihil itaque defuturum arbitramur, modò penitus perſpecta et plenè cognita ſint ea, quæ Newtonus ipſe in arte ſuâ tradiderit. Neque aliò ſpectant, quæ ipſi adjicimus, quàm ut hæc ſtudia difficultate rei non tantopere homines abſterreant. Nimirum tam gravis argumenti tractationem univerſam in ordinem redegimus; etiam diſcendi ſubſidia ubique comparavimus; prætermiſſa exprimendo; explicando ea quæ anguſtè ponebantur, et ubique commonendo lecto-

rem, quò intuens, a verâ Newtoni ſententiâ minùs aberret—*Eo denique conſilio omnia concinnavimus, ut in poſterum nuſquam aliàs confugiant, qui ſublimioris Geometriæ Elementa penitus complecti ſecum ſtatuerint.*

Jam verò brevi Monito *primùm oſtendimus, quali ordine Tirones ad legendum accedant. Quod ut ritè fiat, initio poſuimus Divini illius* Principiorum *Operis Sectionem primam, quaſi anteſignanum aliquod, et ipſum cardinem in quo univerſa Doctrinæ Fluxionalis demonſtratio verſatur. Porrò illi Annotationes ſubjecimus, quæ probandi media ubique ſuppleant; Propoſitiones etiam Corollariis aliquot adauximus. Quæ et denuò apud ipſum in Principiis locum deprehenderis*—*Annotationes ſemel protuliſſe ſatis eſto.*

Hæc excipit Analyſis per Æquationes numero terminorum infinitas; *ad quam, inter alia, prima illius Regula demonſtratione Geometricâ è Fermatio munita eſt, et oſtenſum inſuper, quomodo Quadratura ad pag. 7. Edit. Joneſii det longitudinem Arcûs Ellipſeos.*

Tertium habent Locum Excerpta ex Epiſtolis Newtoni, *eadem ſanè, quæ ad calcem hujuſce tractatûs protulit Joneſius, tantùm ordine paululum immutato, quo Tironibus fiat accommodatior. Quod ſi minùs legitimè factum eſſe quis exiſtimet, Epiſtolas etiamnum integras ſuo loco invenerit. Apud nos Excerpta ita ordinamus, ut primum ſit*

Ex Epiſtolâ Newtoni ad Oldenburgum primâ, *de formulis Algebraicis in Series infinitas reſolvendis, ab initio ſcilicet ad verba* "aliam excogitare adactus ſum" *(p. 25, Edit. Joneſ.)*

Secundum et tertium quæ et Joneſio ibidem poſita, alterum ſcilicet ex Epiſtolâ Newtoni ad Oldenburgum poſteriore, *de ſeriebus condendis et invertendis*—*Alterum ex* Epiſtolâ Newtoni ad Walliſium *de radicibus æquationum fluxionalium extrahendis.*

Quartum denique ex parte aliquâ primi apud Joneſium conſtare volumus, ex Epiſtolâ Newtoni ad Oldenburgum primâ, *de Problematis per ſeries infinitas reſolvendis.*

Ita, ni fallor, accuratiùs et magis dilucidè ſe exceptura ſunt hæc fragmenta. Quænam opis noſtræ ſint, ad ſingula proferendum eſt.

Ad

Ad primum igitur, Theorematis Binomii demonſtrationem adhibuimus—Ad ſecundum, Newtoniana duo Theoremata tertio, neque minùs utili, Moivræano adduximus, de Infinitæ Æquationis radice extrahendâ, ſive, ut aliter dicam, de duabus ſeriebus ad unam redigendis. Quorum omnium demonſtrationem qui elicere velit, ad locum adeat Maclaurinum de Seriebus convergentibus, ubi et perſpicuè traditur, quidnam Parallelogrammum Newtonianum ad Series inveniendas conferat, et Regularum etiam, in Excerpto tertio, ſatis luculenta ad manus ſit explicatio.

Excerptorum ultimum plura comitantur, quæ noſtræ ſunt indaginis. Inter alia, recondita Jacobi Gregorii inventa Geometricè demonſtravimus; qui Tangentem et Secantem Logarithmicam definire novit ex Arcu, et Arcum viciſſim ex illis, nullâ Naturalium numerorum ratione habitâ. Omnino enim viſum eſt tam ſubtilis artificii Series, in hujus argumenti tractatione, non prætermittendas eſſe, præſertim etiam quòd in Commercio Epiſtolico inveniendæ ſint earum formulæ. Seriem Newtonianam pro arcu circulari in datâ ratione multiplicando ſive dividendo, mutatis tantùm ſignis, idem poſſe apud hyperbolicos ſectores oſtendimus; et in Circuli cum Hyperbolâ ſimilitudine, per hanc Seriem patefactâ, poni ea quæ Coleſius in Harmoniâ Menſurarum tam ſcienter inſtituit.

Ad locum expoſita eſt methodus arcum Ellipſeos *æſtimandi, in hoc Excerpto tradita, quam* ad Parabolam etiam et Hyperbolam *accommodavimus—ubi et ad manum fuit illam Ellipſeos et Hyperbolæ cognationem explicare, quam Algebraici innuunt, Hyperbolam alio nomine Ellipſin appellantes, cui axis ſit negativus; unde factum eſt, ut formulæ generales ad quamlibet è Sectionibus Conicis, nullo fere negotio ad alias transferantur. Adhæc Arcûs Parabolici æſtimationi Coteſianæ in tantum immorati ſumus, ut Geometrica illius ratiocinia dilucidè ponerentur.*

Itaque omnia ſingulatim eruendo, ad Seriem Newtoni, quâ Sphæroidis ſegmenta ſecunda exponi poſſint, jam deventum eſt. Cujus profectò rationem univerſam, quos ortus habuit, et quænam artificia, parvo molimine indaganda eſſe nemo æquior ſtatuerit. Utcunque hoc ſit, per omnes hujus Excerpti ſeries, ſedulò fecimus, ne lateret uſpiam earum conformatio, ex ipſius Curvæ, quæ æſtimanda erat, affectubus aſſequenda.

Finem

Finem ex hâc parte fecimus, explicatis apertiſſimè Problematum Conſtructionibus, quæ veris ſaltem proximæ ex Infinitis æquationibus derivari poſſint: cujus rei nuda tantummodò exempla in Epiſtolis ſuprà memoratis extiterint.

Proximè collocatus eſt de Quadraturâ Curvarum *Libellus, præ cæteris quidem ſcientiæ ubertate inſignis, idem tamen rerum compreſſione et amplitudine præ cæteris obſcurus et gravis. Ita contentionem animi, et ſtudium poſcere acerrimum, et ingenii quaſi formâ quâdam lucere, quæ prenſantes tamen miſerè effugeret. Quæ cùm ita ſint, pluſquam mediocri ſtudio locum tam difficilem exercere viſum eſt. Primò itaque Problema illud* Datâ æquatione quotcunque fluentes quantitates involvente, invenire fluxiones, *prorſus aliâ loquendi ratione explicuimus; eo ſermone uſi, qui in Methodi Indiviſibilium ſuſpicionem omnino cadere non poterit, quam et è Geometriâ exulare penitus, et ſub umbras ire jam pridem oportebat.*

Demonſtrationem Propoſitionis Quartæ, ut et omnium quæ in Quintæ præceptis paulo magis ſpinoſa, et ſextæ quoque luculentè abſolvimus. Quarum ultimam jam ante nos Stewartus concinnaverat.

Explicuimus ea alioqui parùm intellecta, quæ monuit Newtonus in Propoſitionis ſeptimæ Caſu primo de coefficientibus ϑ, $\theta+\lambda\eta$, &c. *ubi aliquis eorum defecerit. E Maclaurino Caſum ſecundum aliter demonſtravimus, et univerſam hujus loci doctrinam Æquationibus Canonicis expoſuimus ad uſum Computandi, ni fallor, quàm maximè accommodatis.*

Propoſitionem octavam duplici probatione confirmavimus; et Nonæ Corollaria demonſtrationibus partim è Stewarto partim noſtris ſingulatim adauximus.

Undecimæ, coronidis loco, ſubjecimus Robineſii demonſtrationem Latinè redditam.

Librum de Quadraturis excipiet Opus Poſthumum de Fluxionibus; *modico tamen, ſi ad cætera ſpectes, Annotationum apparatu, ubi neque neceſſe erat fuſiùs rem agere.*

Denique Tomum hunc claudent Methodus Differentialis *et* Enumeratio Linearum Tertii Ordinis. *Neque dubitavimus binos de proprio ſubjicere Libellos, alterum* de Geometriâ Fluxionum, *ſive*

ſive Additamentum tractatûs Newtoniani de rationibus primis et ultimis; *alterum, quem* Logiſticam Infinitorum *appellare libeat. Primo utique Elementa Doctrinæ Fluxionum ita abſolvimus, ut integris et intaminatis Geometriæ ratiociniis omnia eluceant, eâtenus re provectâ, ut exponatur fluxio quantitatis* $\frac{x^{\frac{m}{n}}}{y^{\frac{p}{q}}}$.

Logiſtica Infinitorum *formulas habet, quas in computandi ſubſidium excogitavimus, ubi in ſeries numero terminorum infinitas operationes Arithmeticæ inſtituendæ ſint. Inter alia, Theorema Infinitomium Cl. Viri Moivræi aliter à nobis declaratum eſt, omnino etiam in majorem computandi utilitatem, quàm ex formâ auctoris, vel immutatione ipſius Cheynii.*

Quod reliquum eſt, omnia expoſui, quæ ad inſequentem Tomum, qualiacunque demum Matheſeos incrementa, protulero. Quæ ſi graviter ſatis, et ſubtiliter à nobis diſputata ſunt, mitto quàm ornatè, et ad vulgi aucupium. Etenim is ego ſum, qui in Philoſophiæ tractatione, ſuccum et ſanguinem, non coloratam quandam ſpeciem requiro. Omnino neque eos ſalutatum prodeo, qui abſque labore ullo et induſtriæ periculo, in Epicuri hortulis, quod aiunt, omne ſcientiæ genus, et edita ſapientum templa *ſomniant. Alacres, et exercitatos, et hiſce ſtudiis ardentes volo, qui cum Newtono philoſophari cupiunt. Iidem etiam, ut æquum de me ſtatuant judicium, nullus vereor; quanquam quæ hoc Libro concluſa ſunt, neque numero neque pondere æquiparanda ſunt iis, quæ poſthac edituri ſumus. Quod fauſtum felixque ſit, abſolutis hiſce, Divinum* PRINCIPIORUM *Opus jampridem ingreſſi ſumus, brevi ſub eâdem formâ in lucem proditurum, modò Deus O. M. vitam nobis et vires integras in tantum indulſerit.*

IN HOC TOMO CONTINENTUR

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ARITHMETICA UNIVERSALIS

SIVE

DE COMPOSITIONE

ET

RESOLUTIONE ARITHMETICA

LIBER.

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ARITH-

ARITHMETICA UNIVERSALIS,

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ARITHMETICA LIBER.

PROŒMIUM.

COMPUTATIO vel fit per *numeros*, ut in vulgari Arithmeticâ, vel per *ſpecies*, ut Analyſtis mos eſt. Utraque iiſdem innititur fundamentis, & ad eandem metam collimat: *Arithmetica* quidem definite & particulariter, *Algebraica* autem indefinitè & univerſaliter: ita ut enuntiata fere omnia quæ in hâc computatione habentur, & præſertim concluſiones, *Theoremata* dici poſſint. Verùm Algebra maximè præcellit, quòd cum in Arithmeticâ Quæſtiones tantùm reſolvantur progrediendo a datis ad quæſitas quantitates, hæc a quæſitis tanquam datis ad datas tanquam quæſitas quantitates plerumque regreditur; ut ad concluſionem aliquam, ſeu Æquationem, quocunque demum modo perveniatur, ex quâ quantitatem quæſitam elicere liceat. Eoque pacto conficiuntur difficillima Problemata, quorum reſolutiones ex Arithmeticâ ſolâ fruſtra peterentur. Arithmetica tamen Algebræ in omnibus ejus operationibus ita ſubſervit, ut non niſi unicam perfectam *computandi Scientiam* conſtituere videantur; & utramque propterea conjunctim explicabo.

CAPUT PRIMUM.

Quiſquis hanc Scientiam aggreditur, imprimis vocum & notarum ſignificationes intelligat, & fundamentales addiſcat operationes, Additionem nempe, Subductionem, Multiplicationem, Diviſionem, Extractionem Radicum, Reductiones fractionum & radicalium quantitatum, & modos ordinandi terminos Æquationum, ac incognitas quantitates (ubi plures ſunt) exterminandi. Deinde has operationes, reducendo Problemata ad æquationes, exerceat; & ultimò naturam & reſolutionem æquationum contempletur.

C A P. I.

DE VOCUM QUARUNDAM ET NOTARUM SIGNIFICATIONE.

PER *Numerum* non tam multitudinem unitatum, quàm abſtractam quantitatis cujuſvis ad aliam ejuſdem generis quantitatem, quæ pro unitate habetur, rationem intelligimus. Eſtque triplex; integer, fractus, & ſurdus: *Integer* quem unitas metitur, *Fractus* quem unitatis pars ſubmultiplex metitur, & *Surdus* cui unitas eſt incommenſurabilis [a].

Integrorum numerorum notas (0, 1, 2, 3, 4, 5, 6, 7, 8, 9,) & notarum, ubi plures inter ſe nectuntur, valores nemo non intelligit. Quemadmodum verò numeri, in primo loco ante unitatem, ſive ad ſiniſtram ſcripti, denotant denas unitates, in ſecundo

[a] Hominibus ſcilicet, qui fictas ſuas rerum imagines ac notas magis quam res ipſas contemplari conſueverunt, paulatim mos obrepſit, quantitates eas omnes numeros dicendi, quæ per notas numerales ſignificari ſolent. Hinc locutiones abſonæ, doctiſſimorum etiam ſcriptorum uſu frequentatæ, *numerus unitate minor*, et *numerus ſurdus*: cum ea quidem *numeri* ſit natura ut unitate major ſit, utpote ex unitatum multitudine conflatus; cum *ſurdi* natura in eo ipſo poſita ſit, ut numerus non ſit, vel tanquam numerus intelligi nequeat. Melius eſſet dicere *quantitates fractas ſurdaſve*; vel ſi per neutra loquentes, *ſurda*, *fracta*, diceremus. Quorſum enim ſermo rerum naturæ repugnans?

[b] Apud veteres Arithmeticos frequens eſt hujuſmodi ſcriptura 064 ③, 732569 ③, 572083 ④. Interpretatio ſatis obvia. Numeri cujuſlibet propoſiti tot poſtremæ, dextram verſus, notæ pro decimarum notis ſunt habendæ, quot nota circulo incluſa indicat. Hæc autem decimarum notatio propter ambiguitatem merito rejecta eſt: nimirum cum ante Vietam quantitatis incognitæ et poteſtatum ejus, in operationibus algebraicis, uſitatiſſima eſſet deſignatio per notas numerales circulis incluſas; nempe hoc modo, ⓪, ①, ②, ③, &c. pro 1, x, x^2, x^3, &c. Junioribus notæ decimarum ab integrorum notis, commate vel puncto interpoſito, diſcriminari ſolent: 732,569 vel 732.569. Comma autem ſuperne poſitum notarum decimalium uſque iterationem indicat. 732,569' vel 732,5694'94'. Mihi ſolemne eſt, in numeris uti loquuntur naturalibus comma adhibere, puncto ad

ſecundo centenas, in tertio millenas, &c. ſic numeri in primo loco poſt unitatem ſcripti denotant decimas partes unitatis, in ſecundo centeſimas, in tertio milleſimas, &c. Hos autem dicimus Fractos *Decimales*, quòd in ratione decimali perpetuò decreſcant. Et ad diſtinguendum integros a decimalibus interjici ſolet comma, vel punctum, vel etiam lineola. Sic numerus 732'|569, denotat ſeptingentas triginta duas unitates, unà cum quinque decimis, ſex centeſimis, & novem milleſimis partibus unitatis. Qui & ſic 732,|569, vel ſic 732·569, vel etiam ſic 732⌞569, nonnunquam ſcribitur. Atque ita numerus 57104'2083, denotat quinquaginta ſeptem mille, centum & quatuor unitates; unà cum duabus decimis, octo milleſimis, & tribus decimis milleſimis partibus unitatis. Et numerus 0'064 denotat ſex centeſimas & quatuor milleſimas partes [b]. Surdorum & aliorum fractorum notæ in ſequentibus habentur. Notatio.

Cum rei alicujus quantitas ignota eſt vel indeterminatè ſpectatur, ita ut per numeros non liceat exprimere, ſolemus per ſpeciem *aliquam ſeu literam deſignare.* Et ſi quando cognitas quantitates tanquam indeterminatas ſpectemus, diſcriminis cauſâ, deſignamus initialibus Alphabetæ literis *a*, *b*, *c*, *d*, & incognitas finalibus *z*, *y*, *x*, &c. Aliqui pro cognitis ſubſtituunt conſonantes, vel majuſculas literas, & vocales, vel minuſculas, pro incognitis.

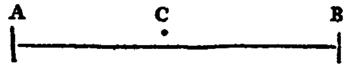
Quantitates vel Affirmativæ *ſunt ſeu majores nihilo*; *vel* Negativæ *ſeu nihilo minores* [c]. Sic in rebus humanis poſſeſſiones dici poſſunt

ad ſeparationem indicis Logarithmici a notis reliquis reſervato: eum ſcilicet in finem, ut in operationibus arithmeticis numeri naturales et logarithmi certâ aliquâ diſtinctione, oculis manifeſtâ, ſemper internoſcantur.

[c] Omnem nimirum quantitatem Arithmeticus tanquam ex aliis compoſitam conſiderare ſolet. Duplex autem compoſitionis modus, additionis alter, alter ſubductionis, hic illi contrarius. Auget enim additio, ſubductio minuit. Quantitates igitur, quæ novæ alicui procreandæ per additionem inſerviunt, affirmativæ dicuntur; negativæ quæ ſubductæ acervum minuunt: quas ob vim illam minuendi in compoſitione arithmeticâ, quaſi naturam veræ quantitatis contrariam inducerent, Albertus Girardus, ni fallor, omnium primus (quem ſummum interea mathematicum agnoſco) durâ quâdam verborum figurâ, Diophanto et Vietæ prorſus ignotâ, quam vellem Carteſius et noſtrates minus avide arripuiſſent, *nihilo minores* dixit. Hinc novam, neſcio quam, nonnulli ſomniârunt quantitatis ſpeciem, cujus naturam in eo ponunt quod quantitatis notioni quam maxime repugnat. Hiſce Newtonus ſuffragari nolens, de quantitatibus numero multitudinis locutus eſt. Neque enim dicit *quantitatem* vel affirmativam eſſe, ſeu majorem nihilo, vel negativam ſeu nihilo minorem; uti omnino ei loquendum fuiſſet, ſi de partitione aliquâ univerſi generis τοῦ ποσοῦ cogitâſſet, qualem iſti cogitare exiſtimandi ſunt; ſed cum *quantitates* pluraliter

dicat,

poſſunt bona affirmativa, debita verò bona negativa. Inque motu locali progreſſus dici poteſt motus affirmativus, & regreſſus motus negativus, quia prior auget & poſterior diminuit iter confectum. Et ad eundem modum in Geometriâ, ſi linea verſus plagam quamvis ducta pro affirmativâ habeatur, negativa erit quæ verſus plagam oppoſitam ducitur. Veluti ſi AB dextrorſum ducatur, & BC ſiniſtrorſum; ac AB ſtatuatur affirmativa, tunc BC pro negativâ habebitur, eò quòd inter ducendum diminuit AB; redigitque vel ad breviorem AC; vel ad nullam, ſi forte C inciderit in ipſum A; vel ad minorem nullâ,[d] ſi BC longior fuerit quam AB de quâ aufertur. *Negativæ* quantitati deſignandæ ſignum −, *Affirmativæ* ſignum + præfigi ſolet. Signum ± incertum eſt, & ſignum ∓ etiam incertum, ſed priori contrarium.



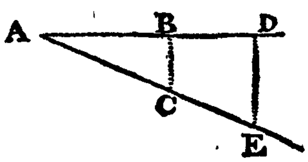
In aggregato quantitatum nota + ſignificat quantitatem ſuffixam eſſe cæteris addendam, *& nota − eſſe* ſubducendam. Et has notas vocabulis *plus* & *minus* exprimere ſolemus. Sic $2+3$, ſive 2 plus 3, valet ſummam numerorum 2 & 3, hoc eſt 5. Et $5-3$, ſive 5 minus 3, valet differentiam quæ oritur ſubducendo 3 à 5, hoc eſt 2. Et $-5+3$ valet differentiam quæ oritur ſubducendo 5 à 3, hoc eſt -2. Et $6-1+3$ valet 8. Item $a+b$ valet ſummam quantitatum a & b: Et $a-b$ valet differentiam, quæ oritur ſubducendo b ab a. Et $a-b+c$ valet ſummam iſtius differentiæ & quantitatis c. Puta ſi a ſit 5; b, 2; & c, 8; tum $a+b$ valebit 7; & $a-b$, 3; & $a-b+c$, 11. Item $2a+3a$ valet $5a$. Et $3b-2a-b+3a$ valet $2b+a$; nam $3b-b$ valet $2b$ & $-2a+3a$ valet a, quorum aggregatum eſt $2b+a$. Et ſic in aliis. Hæ autem notæ + & − dicuntur *Signa*. Et ubi neutrum initiali quantitati præfigitur ſignum + ſubintelligi debet.

dicat, ſatis utique ſignificat, quas concretas dialectici vocarent, quales Arithmetici tractare ſolent, eas animo ſibi obverſari, et technicas earum Arithmeticorum compoſitiones. In Arithmeticâ igitur contrariis compoſitionum modis voces contrariæ, affirmatio et negatio, reſpondent. In aliis autem omnibus quibus phraſis algebraica accommodari ſolet, affirmatio et negatio, neutiquam diverſa rerum genera indicant, ſed relationes rerum in eodem genere contrarias.

[d] Hoc eſt, vel magis quam ad nullam redigit: novam ſcilicet apponit, a puncto A ſiniſtrorſum exporrectam, ſi BC longior fuerit quam AB.

MULTI-

Multiplicatio propriè dicitur, quæ fit per numeros integros; Notatio.
utpote quærendo novam quantitatem toties majorem quantitate multiplicandâ, quoties numerus multiplicans ſit major unitate. Sed, aptioris vocabuli defectu, Multiplicatio etiam dici ſolet, quæ fit per fractos aut ſurdos numeros; quærendo novam quantitatem in eâ quâcunque ratione ad quantitatem multiplicandam, quam habet multiplicator ad unitatem. Neque tantùm fit per abſtractos numeros ſed etiam per concretas quantitates, ut per lineas, ſuperficies, motum localem, pondera, &c. quatenus hæ ad aliquam ſui generis notam quantitatem tanquam unitatem relatæ, rationes numerorum exprimere poſſunt, & vices ſupplere. Quemadmodum ſi quantitas A multiplicanda ſit per lineam duodecim pedum, poſito quod linea bipedalis ſit unitas, producentur per iſtam multiplicationem 6 A, ſive ſexies A, perinde ac ſi A multiplicaretur per abſtractum numerum 6; ſiquidem 6 A ſit in eâ ratione ad A, quam habet linea duodecim pedum ad unitatem bipedalem [e]. Atque ita ſi duas quaſvis lineas AC & AD per ſe multiplicare oportet, capiatur AB unitas, & agatur BC eique parallela DE, & AE productum erit hujus multiplicationis, eo quod ſit ad AD ut AC ad unitatem AB. Quinetiam mos obtinuit, ut geneſis ſeu deſcriptio ſuperficiei, per lineam ſuper aliâ lineâ ad rectos angulos moventem, dicatur multiplicatio iſtarum linearum.

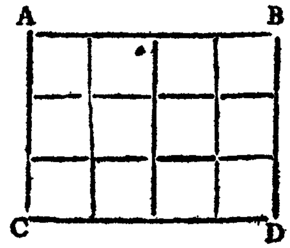


Nam quàmvis linea utcunque multiplicata non poſſit evadere ſuperficies, adeoque hæc ſuperficiei e lineis generatio longè alia ſit a multiplicatione, in hoc tamen conveniunt, quòd numerus unitatum in alterutrâ lineâ multiplicatus per numerum unitatum in alterâ producat abſtractum numerum unitatum in ſuperficie lineis iſtis comprehenſâ, ſi modò Unitas ſuperficialis definiatur, ut ſolet, Quadratum cujus latera ſunt

[e] Sic pecuniæ pecunias nonnunquam multiplicare dicuntur, ut in quæſtione illâ quæ tironibus proponi ſolet, quanta ſit pecuniæ ſumma quam ſumma quælibet data, puta 4*l.* 19*s.* 11 $\frac{3}{4}$*d.* ſemet ipſa multiplicans effecerit? Hujuſmodi quæſtioni generaliter reſpondere licet, eam fore ſummam multiplicatione effectam, quæ ad ſummam propoſitam rationem habeat quam propoſita ad unitatem nummariam. Unitatem autem nummariam in quânam nummorum ſpecie conſtitutam velit, dicendum eſt ei qui quæſtionem ponit. Poteſt enim ea in alio atque alio genere ad arbitrium ſumi, & prout varie ſumpta fuerit, aliæ atque aliæ pecuniarum ſummæ e multiplicatione provenient.

CAPUT PRIMUM.

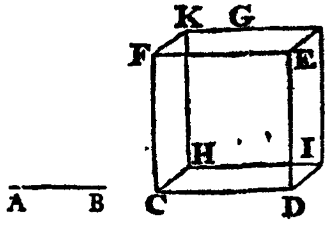
ſunt unitates lineares [f]. Quemadmodum ſi recta AB conſtet quatuor unitatibus & AC tribus, tum rectangulum AD conſtabit quater tribus ſeu duodecim unitatibus quadratis, ut inſpicienti Schema patebit. Eſtque ſimilis analogia ſolidi & ejus quod continuâ trium quantitatum multiplicatione producitur. Et hinc viciſſim evenit quod vocabula *ducere*, *contentum*, *rectangulum*, *quadratum*, *cubus*, *dimenſio*, *latus*, & ſimilia quæ ad Geometriam ſpectant, Arithmeticis tribuantur operationibus. Nam per *quadratum*, vel *rectangulum*, vel *quantitatem duarum dimenſionum* non ſemper intelligimus ſuperficiem, ſed ut plurimum quantitatem alterius cujuſcunque generis quæ multiplicatione aliarum duarum quantitatum producitur, & ſæpiſſime lineam quæ producitur multiplicatione aliarum duarum linearum.



Atque

venient. Etenim ſi unam libram Anglicanam (*one pound ſterling*) unitatem ſtatueris, ſumma propoſita, 4*l.* 19*s.* 11 $\frac{3}{4}$*d.* ſemet ipſa multiplicans faciet 2[illegible] 19*s.* 9 $\frac{1}{2}$*d.* cum parte $\frac{1}{960}$ unius ſemioboli. Atqui ſi nummum argenteum (Anglice 1 *ſhilling*) pro unitate ſumas, conſimilis pecuniæ propoſitæ multiplicatio ſummam longè majorem procreabit; nempe 499*l.* 15*s.* 10*d.* cum parte $\frac{1}{48}$ unius ſemioboli: majorem adhuc, ſi denarium (Anglice *a penny*) pro unitate haberi velis; provenient enim 5997*l.* 10*s.* cum parte $\frac{1}{4}$ unius ſemioboli: tum vero maximam, ſi unitatem nummariam, in infimâ nummorum ſpecie conſtitutam, uni tantum ſemiobolo (*a farthing*) æquaveris. Ita enim ſummæ propoſitæ in ſeipſam multiplicatio libras Anglicanas 23990 cum uno ſemiobolo fecerit: 23990 *l.* 0 *s.* 0 $\frac{1}{4}$*d.*

[f] Parallelogramma ſcilicet, vel parallelepipeda æquiangularia inter ſe rationem habent e laterum rationibus compoſitam. Numeri quoque, e duobus tribuſve numeris mutuâ multiplicatione facti, inter ſe rationem habent e rationibus numerorum, quorum multiplicationibus facti ſunt, compoſitam. Sit igitur recta AB ad rectam CD ut unitas ad numerum quemlibet *m*. Eadem vero recta AB ad aliam rectam DE proportionem habeat quam unitas ad alium aliquem numerum *n*. Numeri autem *n*, *m*, inter ſe multiplicati numerum *p* faciant. Rectis autem, CD, DE, ad perpendiculum compoſitis rectangulum DF compleatur. Quadratum ex AB ad rectangulum DF proportionem habet ex proportionibus rectæ AB ad CD, ejuſdemque AB ad DE compoſitam: compoſitam igitur e rationibus unitatis ad numerum *m*, et unitatis ad numerum *n*. Habet autem unitas ad numerum *p* rationem e rationibus unitatis ad numeros *m* et *n* compoſitam. Quadratum igitur ex AB ad rectangulum DF proportionem habet quam unitas ad numerum *p*. Unitas autem metitur numerum *p*. Quadratum igitur ex AB rectangulum DF toties metietur. Atque eadem erit argumentatio, quæcunque ſint longitudines rectarum CD, DE, modò ſingulæ ad rectam AB proportionem habeant numeri alicujus ad unitatem. Quadratum utique ex AB rectangula omnia, ejuſmodi rectis comprehenſa, metietur, toties unumquodque, quoties unitas numerum *p* metiatur, e numeris illis, *m*, *n* factum, qui ad unitatem rationes habeant quam rectæ CD, DE, ad rectam AB. Planum rectanguli DF, in



1 *m* *n*
p
q
r

Atque ita dicimus *Cubum* vel *Parallelepipedum*, vel *quantitatem trium dimenſionum* pro eo quod binis multiplicationibus producitur, *latus* pro radice, *ducere* pro multiplicare ; & ſic in aliis. NOTATIO.

Numerus ſpeciei alicui immediatè præfixus denotat ſpeciem illam toties ſumendam eſſe. Sic 2 *a* denotat duo *a*, 3 *b* tria *b*, 15 *x* quindecim *x*. [g]

Duæ vel plures ſpecies immediatè connexæ deſignant factum, *ſeu quantitatem quæ ſit per multiplicationem omnium in ſe invicem.* Sic *a b* denotat quantitatem quæ ſit multiplicando *a* per *b*. Et *a b x*, denotat quantitatem quæ ſit multiplicando *a* per *b*, & factum illud per *x*. Puta ſi *a* ſit 2, & *b* ſit 3, & *x* ſit 5, tum *a b* erit 6, & *a b x*, 30.

Inter quantitates ſeſe multiplicantes, nota ×, vel vocabulum *in*, ad factum deſignandum nonnunquam interſcribitur. Sic 3 × 5, vel 3 in 5, denotat 15. Sed uſus harum notarum præ-

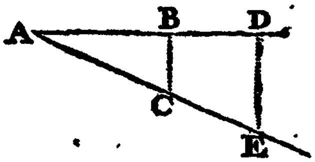
in puncto D ad perpendiculum inſiſtat recta D I, quæ ſit ad A B ut numerus quidam *q* ad unitatem. Numeri autem *p*, *q* inter ſe multiplicati numerum *r* faciant, et parallelepipedon D K compleatur. Cubus ex A B ad parallelepipedon D K proportionem habet ex proportionibus quadrati ex A B ad rectangulum D F, et rectæ A B ad rectam D I compoſitam : Compoſitam igitur e rationibus unitatis ad numerum *p*, et unitatis ad numerum *q*. Habet autem unitas ad numerum *r* rationem e rationibus unitatis ad numeros *p* et *q* compoſitam. Cubus igitur ex A B ad parallelepipedon D K proportionem habet quam unitas ad numerum *r*. Unitas autem metitur numerum *r*. Cubus igitur ex A B parallelepipedon D K toties metietur. Atque ſimilis erit argumentatio, quæcunque demum rectæ D I longitudo ſit, modo talis ea ſit quæ ad rectam A B proportionem habeat numeri alicujus ad unitatem. Cubus ſcilicet ex A B parallelepipeda omnia rectangulo D F et rectâ D I comprehenſa metietur, toties unumquodque, quoties unitas numerum illum *r* metiatur, e numeris *m*, *n*, *q* factum, qui ad unitatem rationes habeant rectarum C D, D E, D I ad rectam A B.

Hinc ſi ſcala quædam longitudinum a rectâ A B originem et initium trahat, quæ proinde illius ſcalæ unitas linearis dicenda eſt, e longitudinibus hujus ſcalæ ad perpendiculum compoſitis aliam quandam *ſuperficierum* ſcalam formare licebit, quæ a quadrato rectæ A B initium ſumet. Item e rectis ſcalæ primæ, cum rectangulis ſcalæ ſuperficiariæ ad perpendiculum compoſitis, formari poteſt *ſolidorum* ſcala, quæ ab ejuſdem A B cubo originem trahet. Hanc ſcilicet ob cauſam quadratum ex A B unitatis ſuperficiariæ nomen ſortietur, et cubus ex A B unitas ſolida quodam modo dici merebitur. Atque hâc analogiâ moti Veteres voces Geometriæ proprias *planum*, *ſolidum*, *quadratum*, *cubum*, malo exemplo in Arithmeticam tranſtulerunt, et Juniores viciſſim verba Arithmeticês, *multiplicare*, *dividere*, in Geometriam. Quæ tamen non ita ſiunt accipienda, ac ſi figuræ planæ et ſolidæ multiplicatione aliquâ, quæ ne intelligi quidem poteſt, e lineis procreatæ eſſent ; ſed quòd ſpatii ſuperficiarii ſolidive ex datâ menſurâ aliquâ lineari æſtimatio numeralis homini Arithmetico per multiplicationem ineunda ſit. E contrario linearum longitudines vel ſuperficierum planitiæ, ex datis menſuris ſuperficiariis ſolidiſve, per diviſionem numeris definiendæ ſunt.

[g] Antiqui moris erat coefficientem numeralem literæ ſuffigere ($x\,3$ pro $3\,x$). Præpoſitus eſt, confuſionis evitandæ gratiâ, ex quo Carteſius inſtituit notis numeralibus literæ radicali ſuperne ſuffixis, (ad hunc modum x^2. x^3. x^4.) quantitatum poteſtates deſignare, quas ante vel notâ numerali circulo incluſâ deſignabant. (Vide not. *b*) vel vocibus curtatis, vel iteratâ quoties opus eſſet literâ radicali. A quad. A. cub. A. biquad. vel *aa*, *aaa*, *aaaa*, &c. pro a^2, a^3, a^4, &c.

CAPUT PRIMUM. cipuus eſt, ubi compoſitæ quantitates ſeſe multiplicant. Veluti ſi $y-2b$ multiplicet $y+b$, terminos utriúſque multiplicatoris lineolâ ſuperimpoſitâ connectimus, & ſcribimus $\overline{y-2b}$ in $\overline{y+b}$, vel $\overline{y-2b} \times \overline{y+b}$.

DIVISIO propriè eſt quæ fit per numeros integros, quærendo novam quantitatem toties minorem quantitate dividendâ quoties unitas ſit minor Diviſore. Sed ob analogiam vox etiam uſurpari ſolet, cum nova quantitas in ratione quâcunque ad quantitatem dividendam quæritur, quam habet unitas ad diviſorem; ſive diviſor ille ſit fractus, aut ſurdus numerus, aut alia cujuſvis generis Quantitatis. Sic ad dividendum lineam AE per lineam AC, exiſtente AB unitate; agenda eſt ED parallela CB, & erit AD Quotiens. Imo & Diviſio, propter ſimilitudinem quandam, dicitur, cum rectangulum ad datam lineam tanquam Baſem applicatur, ut inde noſcatur altitudo.



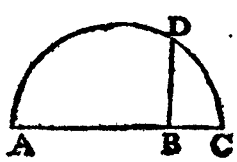
Quantitas infra quantitatem cum lineolâ interjectâ denotat quotum, *ſeu quantitatem quæ oritur ex diviſione ſuperioris quantitatis per inferiorem.* Sic $\frac{6}{2}$ denotat quantitatem quæ oritur dividendo 6 per 2, hoc eſt 3: & $\frac{5}{8}$ quantitatem quæ oritur dividendo 5 per 8, hoc eſt octavam partem numeri 5: & $\frac{a}{b}$ denotat quantitatem quæ oritur dividendo a per b; puta ſi a ſit 15, & b, 3, tum $\frac{a}{b}$ denotat 5. Et ſic $\frac{ab-bb}{a+x}$ denotat quantitatem quæ oritur dividendo $ab-bb$ per $a+x$. Atque ita in aliis. Hujuſmodi autem quantitates *fractiones* dicuntur, parſque ſuperior *Numerator*, ac inferior *Denominator*.

Aliquando Diviſor quantitati diviſæ, interjecto arcu, præfigitur. Sic ad deſignandum quantitatem quæ oritur ex diviſione $\frac{axx}{a+b}$ per $a-b$, ſcribi poteſt $\overline{a-b)}\frac{axx}{a+b}$.

Etſi multiplicatio per immediatam quantitatum conjunctionem denotari ſolet, tamen numerus integer ante numerum fractum denotat ſummam utriuſque. Sic $3\frac{1}{2}$ denotat tria cum ſemiſſe.

Si quantitas ſeipſam multiplicet, numerus factorum, compendii gratiâ, ſuffigi ſolet. Sic pro *aaa* ſcribimus a^3, pro *aaaa* ſcribimus a^4, pro *aaaaa* ſcribimus a^5, & pro *aaabb* ſcribimus a^3bb vel a^3b^2. Puta ſi *a* ſit 5, & *b* ſit 2, tum a^3 erit $5 \times 5 \times 5$, ſive 125; & a^4 erit $5 \times 5 \times 5 \times 5$, ſive 625; atque a^3b^2 erit $5 \times 5 \times 5 \times 2 \times 2$, ſive 500. Ubi nota, quòd numerus inter duas ſpecies immediatè ſcriptus, ad priorem ſemper pertinet. Sic 3 in quantitate a^3bb non denotat *bb* ter capiendum eſſe, ſed *a* in ſe bis ducendum. Nota etiam, quòd hæ quantitates tot *dimenſionum* vel *poteſtatum* vel *dignitatum* eſſe dicuntur, quot factoribus ſeu quantitatibus ſe multiplicantibus conſtant; & numerus ſuffixus vocatur *Index* poteſtatum vel dimenſionum. Sic *aa* eſt duarum dimenſionum vel poteſtatum, & a^3 trium, ut indicat ſuffixus numerus 3. Dicitur etiam *aa* quadratum, a^3 cubus, a^4 quadrato-quadratum, a^5 quadrato-cubus, a^6 cubo-cubus, a^7 quadrato-quadrato-cubus, & ſic porro. Et quantitas *a*, ex cujus in ſe multiplicatione hæ poteſtates generantur, dicitur earum *Radix*, nempe radix quadratica quadrati *aa*, cubica cubi a^3, &c. NOTATIO.

Cùm autem radix per ſeipſam multiplicata producat quadratum, & quadratum illud iterum per radicem multiplicatum producat cubum, *&c.* erit (ex definitione Multiplicationis) ut unitas ad radicem, ita radix ad quadratum, & quadratum ad cubum, *&c.* Adeoque quantitatis cujuſcunque *radix quadratica* erit medium proportionale inter unitatem & quantitatem illam, & *radix cubica* primum è duobus mediè proportionalibus, & *radix quadrato-quadratica* primum è tribus, & ſic præterea. Duplici igitur affectione radices innoteſcunt, tum quòd ſeipſas multiplicando producant ſuperiores poteſtates, tum quòd ſint è mediis proportionalibus inter iſtas poteſtates & unitatem. Sic numeri 64 radicem quadraticam eſſe 8, & cubicam 4, vel ex eo patet, quòd 8×8, & $4 \times 4 \times 4$ valeant 64; vel quòd ſit 1 ad 8 ut 8 ad 64, & 1 ad 4 ut 4 ad 16, & 16 ad 64. Et hinc ſi lineæ alicujus AB radix quadratica extrahenda eſt, produc eam ad C, ut ſit BC unitas; dein ſuper AC deſcribe ſemicirculum, & ad B erige perpendiculum huic circulo occurrens in D, eritque BD radix, quia media proportionalis eſt inter AB & unitatem BC.



Ad deſignandam radicem *alicujus quantitatis præfigi ſolet nota,* $\surd$, *ſi radix ſit quadratica, &* $\surd 3:$ *ſi ſit cubica, &* $\surd 4:$ *ſi quadrato-quadratica,* &c. Sic $\surd 64$ denotat 8; & $\surd 3: 64$ denotat 4; & $\surd aa$ denotat a; & $\surd ax$ denotat radicem quadraticam ex ax; & $\surd 3: 4axx$ radicem cubicam ex $4axx$. Ut ſi a ſit 3, & x, 12; tum $\surd ax$ erit $\surd 36$, ſeu 6; & $\surd 3: 4axx$ erit $\surd 3: 1728$, ſeu 12. Et hæ radices, ubi non licet extrahere, dicuntur *ſurdæ quantitates,* ut $\surd ax$; vel *ſurdi numeri,* ut $\surd 12$.

Nonnulli pro deſignandâ quadraticâ poteſtate uſurpant q, pro cubicâ c, pro quadrato-quadraticâ qq, pro quadrato-cubicâ, qc, &c. Et ad hunc modum pro quadrato, cubo, & quadrato-quadrato ipſius A, ſcribitur Aq, Ac, Aqq, &c. Et pro radice cubicâ ex $abb - x^3$ ſcribitur $\surd c: abb - x^3$. Alii alias notas adhibent, ſed quæ jam ferè exoleverunt[b].

Nota

[b] Commodiſſimè autem radices deſignantur adhibitis indicibus fractis, Alberti Gerardi more, pleriſque etiam hodie uſitato. Nempe pro radice quadraticâ quantitatis ab ſcribere ſolemus $\overline{ab}|^{\frac{1}{2}}$, pro radice cubicâ quantitatis a^2b, $\overline{a^2b}|^{\frac{1}{3}}$: pro radice quadraticâ cubi quantitatis a, $a^{\frac{3}{2}}$: pro radice cubicâ quadrati quantitatis a, $a^{\frac{2}{3}}$: et ſimiliter pro ſuperiorum poteſtatum radicibus. Cujus quidem notationis ea eſt vis, ut cujuſque ſive poteſtatis ſive radicis index locum ejus ſignificet in perpetuâ quâdam, quæ ab unitate initium ſumat, proportione convenientium ſerie. Sic quantitatis a^3 index ternarius ſignificat quantitatem illam proportione convenientium tertiam poſt unitatem eſſe, quarum quantitas ſimplex a poſt unitatem prima eſt. Quantitatis a^5 index quinarius ſignificat quantitatem illam quintam poſt unitatem eſſe, in eâdem proportione convenientium ulterius exporrectâ ſerie. Quantitatis autem $a^{\frac{1}{3}}$ index $\frac{1}{3}$ ſignificat quantitatem illam unitati proximam, in perpetuâ quâdam proportione convenientium ſerie, cujus quantitas ſimplex a poſt unitatem tertia eſt. Quantitatis $a^{\frac{2}{3}}$ index $\frac{2}{3}$ ſignificat quantitatem illam unitati proximam eſſe, in perpetuâ proportione convenientium ſerie, cujus tertium poſt unitatem locum quantitatis a poteſtas quadratica occupat. Denique quatitatis $a^{\frac{3}{2}}$ index $\frac{3}{2}$ ſignificat quantitatem illam unitati proximam eſſe, in continuâ proportione convenientium ſerie, cujus ſecundum poſt unitatem locum quantitatis a poteſtas cubica occupat. In ſummâ, index radicalis $\frac{m}{n}$ ſignificat radicem à numero n denominatam, ex poteſtate à numero m denominatâ, extrahendam. — Indices autem illi, ſive poteſtatum ſive radicum, Logarithmorum quodammodo ſortiti ſunt naturam. Etenim ſi pro Logarithmo rationis unitatis ad quantitatem aliquam ſumatur 1, Logarithmus rationis unitatis ad poteſtatem illius quantitatis quadraticam binarius erit; ad cubicam ternarius; ad biquadraticam quaternarius; ad ulteriorem denique omnem poteſtatem numerus poteſtatis illius indici æqualis. Logarithmus autem rationis unitatis ad radicem ejuſdem quantitatis quadraticam erit, $\frac{1}{2}$; ad radicem cubicam $\frac{1}{3}$; ad radicem biquadraticam; $\frac{1}{4}$ ad radicem $\frac{1}{n}$, erit $\frac{1}{n}$; ad radicem $\frac{2}{3}$, erit $\frac{2}{3}$; ad radicem $\frac{3}{4}$, erit; $\frac{3}{4}$ ad radicem $\frac{m}{n}$, erit $\frac{m}{n}$. Hinc haud inepta indicis definitio erit, eum eſſe quantitatem illam, quæ eam ad unitatem proportionem habeat, quam Logarithmus rationis inter unitatem & quantitatem poteſtati vel radici indicatæ æqualem ad Logarithmum rationis inter unitatem & quantitatem ſimplicem, unde poteſtas vel radix indicata ortum duxit. Hinc indicum eadem quæ Logarithmorum erit Logiſtica. Hinc etiam quantitates

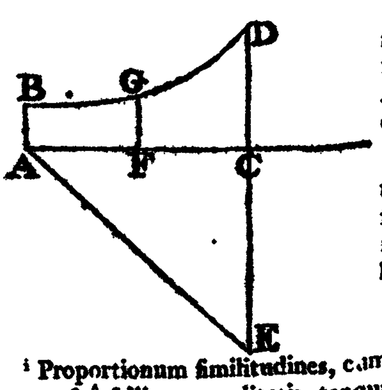
Nota = *deſignat quantitates hinc inde* æquales *eſſe*. Sic $x = b$ deſignat x æqualem eſſe b. NOTATIO.

Nota :: *ſignificat quantitates hinc inde* proportionales *eſſe*. Sic $a.\ b :: c.\ d$, ſignificat eſſe a ad b ut c ad d. Et $a.\ b.\ e :: c.\ d.\ f$ eſſe a, b & e inter ſe, ut ſunt c, d & f inter ſe reſpectivè; vel eſſe a ad c, b ad d & e ad f in eâdem ratione [i].

Denique notarum, quæ ex his componuntur, interpretatio per Analogiam facilè innoteſcit. Sic enim $\frac{3}{4}a^3bb$ denotat tres quartas partes ipſius a^3bb; & $3\frac{a}{c}$, ter $\frac{a}{c}$; & $7\sqrt{ax}$, ſepties $\sqrt{ax}$. Item $\frac{a}{b}x$ denotat id quod fit multiplicando x per $\frac{a}{b}$; & $\frac{5ee}{4a+9e}Z^3$ id quod fit multiplicando Z^3 per $\frac{5ee}{4a+9e}$; hoc

tates $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^{\frac{1}{2}}}$, $\frac{1}{a^{\frac{1}{3}}}$, $\frac{1}{a^{\frac{m}{n}}}$ quæ, ipſâ quidem unitate ad initium ſcalæ Logarithmicæ conſtitutâ, Logarithmis negativis gaudent, indicibus etiam negativis haud incommodè exprimantur; hoc modo: $a^{-1}, a^{-2}, a^{-3}, a^{-\frac{1}{2}}, a^{-\frac{1}{3}}$. Indices autem illi negativi poſitivis ita ſcilicet reſpondent, ut in perpetuâ quantitatum analogiâ, æqualia quaſi ab unitate locorum intervalla indices æquales, poſitivi negativique, ſignificent, ſed a partibus contrariis. Ut ſi quantitas $a^{\frac{m}{n}}$ unitate major ſit, quantitas $a^{-\frac{m}{n}}$ unitate ſecundam eandem rationem erit minor. E diverſo ſi quantitas $a^{\frac{m}{n}}$ unitate minor ſit, erit $a^{-\frac{m}{n}}$ unitate ſecundum eandem rationem major. Duarum vero $a^{\frac{m}{n}}$, $a^{-\frac{m}{n}}$, in utroque caſu, unitas ipſa proportione media erit. Generaliter igitur, indicum æqualium affirmatio et negatio analogiæ continuationem ſignificat, gradibus iiſdem, in partes contrarias.

Ex hâc denique notatione novum quoddam irrationalium genus in notitiam venit, quæ irrationaliter irrationalia haud immeritò dicantur, indicibus nimirum ſurdis deſignata. Hujuſmodi ſunt $\overline{ab}|^{\sqrt{2}}$ $\overline{a^3b+9a^2xx}|^{\sqrt{3}}$, modo quantitates ſimplices a, b, x, rationales ſint.



Exponatur curva Logarithmica B G D, cujus ordinata A B ſit unitas, ordinata autem C D quantitati ab æquetur. In D C productâ capiatur C E = 2 A C, junctâque A E, ſecetur A C in puncto F, ut ſit A F ad A C ut C E ad A E. Per F ducatur ordinata; quæ curvæ in G occurrat. Recta F G quantitas erit notis $\overline{ab}|^{\sqrt{2}}$ ſignificata, quæ haud aliter, ut opinor, ſub phantaſiam cadit, niſi tanquam ordinata curvæ logarithmicæ, vel tanquam area hyperbolica, vel per aliquam ſaltem logarithmorum lineationem.

[i] Proportionum ſimilitudines, cum exteris, ſic deſignare ſoleo $a : b = c : d$ rationibus conferendis interpoſitâ ſcilicet æqualitatis, tanquam [illegible] notâ.

 eſt

eſt per Quotum exortum diviſione $5ee$ per $4a+9e$; & $\frac{2a^3}{9c}\sqrt{ax}$ id quod fit multiplicando $\sqrt{ax}$ per $\frac{2a^3}{9c}$; & $\frac{7\sqrt{ax}}{c}$ quotum exortum diviſione $7ax$ per c; & $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ quotum exortum diviſione $8a\sqrt{cx}$ per ſummam quantitatum $2a+\sqrt{cx}$. Et ſic $\frac{3axx-x^3}{a+x}$ denotat quotum exortum diviſione differentiæ $3axx-x^3$ per ſummam $a+x$; & $\sqrt{\frac{3axx-x^3}{a+x}}$ radicem ejus Quoti; & $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ id quod fit multiplicando radicem illam per ſummam $2a+3c$. Sic etiam $\sqrt{\frac{1}{4}aa+bb}$ denotat radicem ſummæ quantitatum $\frac{1}{4}aa$ & bb; & $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ radicem ſummæ quantitatum $\frac{1}{2}a$ & $\sqrt{\frac{1}{4}aa+bb}$; & $\frac{2a^3}{aa-zz}\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ radicem illam multiplicatam per $\frac{2a^3}{aa-zz}$. Et ſic in aliis.

Cæterùm nota, quòd in hujuſmodi complexis quantitatibus, non opus eſt ad ſignificationem ſingularum literarum ſemper attendere; ſed ſufficit in genere tantum intelligere, *e. g.* quòd $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ ſignificat radicem aggregati $\frac{1}{2}a+\sqrt{\frac{1}{4}aa\times bb}$; quodcunq; tandem prodeat illud aggregatum, cùm numeri vel lineæ pro literis ſubſtituuntur. Atque ita quòd $\frac{\sqrt{\frac{1}{2}a\times\sqrt{\frac{1}{4}aa\times bb}}}{a-\sqrt{ab}}$ ſignificat quotum exortum diviſione quantitatis $\sqrt{\frac{1}{2}a\times\sqrt{\frac{1}{4}aa\times bb}}$ per quantitatem $a-\sqrt{ab}$, perinde ac ſi quantitates illæ ſimplices eſſent & cognitæ, etſi quænam ſint impræſentiarum prorſùs ignoretur, & ad ſingularum partium conſtitutionem aut ſignificationem neutiquàm attendatur. Id quod monendum eſſe duxi, ne complexione terminorum Tirones quaſi conterriti in limine hæreant.

CAP.

CAPUT SECUNDUM.

Additio.

DE ADDITIONE.

NUmerorum, ubi *non ſunt admodum compoſiti*, Additio per ſe manifeſta eſt. Sic quòd 7 & 9, ſeu 7 + 9, faciunt 16, & quòd 11 + 15 faciunt 26 primâ fronte patet. At in *magis compoſitis* opus peragitur *ſcribendo numeros ſerie deſcendente, & ſummas columnarum ſigillatim colligendo.* Quemadmodum ſi numeri 1357 & 172 addendi ſunt, ſcribe alterutrum 172 infra alterum 1357, ita ut hujus unitates 2 alterius unitatibus 7 ſubjiciantur, cæterique numeri numeris correſpondentibus, nempe deni 7 denis 5, & centenus 1 centenis 3. Tum incipiendo ad dextram, dic 2 & 7 faciunt 9; quem ſcribe infrà. Item 7 & 5 faciunt 12, cujus poſteriorem numerum 2 ſcribe infrà, priorem vero 1 aſſerva proximis numeris, 1 & 3, adjiciendum. Dic itaque præterea 1 & 1 faciunt 2, cui 3 adjectus facit 5; & ſcribe 5 infrà; & manebit tantùm 1 prima figura ſuperioris numeri, quæ etiam infrà ſcribenda eſt, & ſic habebitur ſumma 1529.

1357
172
1529

Sic numeros 87899 + 13403 + 885 + 1920, quo in unam ſummam redigantur, ſcribe in ſerie deſcendente, ita ut unitates unam columnam, deni numeri aliam, centeni tertiam, milleni quartam conſtituant, & ſic præterea. Deinde dic 5 + 3 valent 8; & 8 + 9 valent 17; ſcribeque 7 infrà, & 1 adjice proximis numeris, dicendo 1 + 8 valent 9; 9 + 2 valent 11; ac 11 + 9 valent 20: Subſcriptoque 0, dic iterum ut antè, 2 + 8 valent 10; 10 + 9 valent 19; 19 + 4 valent 23; & 23 + 8 valent 31; adeoque aſſervato 3, ſubſcribe 1 ut antè; & iterum dic 3 + 1 valent 4; 4 + 3 valent 7; & 7 + 7 valent 14. Quare ſubſcribe 4, denuoque dic 1 + 1 valent 2; & 2 + 8 va-

87899
13403
1920
885
104107

lent

lent 10; quem ultimò ſubſcribe, & omnium ſummam habebis 1.04107.

Ad eundem modum numeri decimales adduntur ut in annexo paradigmate videre eſt.

630'953
51'0807
305'27
987'3037

In terminis Algebraicis Additio *fit connectendo quantitates addendas cum ſignis propriis, & inſuper uniendo quæ poſſunt uniri.* Sic a & b faciunt $a+b$; a & $-b$ faciunt $a-b$; $-a$ & $-b$ faciunt $-a-b$; $7a$ & $9a$ faciunt $7a+9a$; $-a\sqrt{ac}$ & $b\sqrt{ac}$ faciunt $-a\sqrt{ac}+b\sqrt{ac}$, vel $b\sqrt{ac}-a\sqrt{ac}$, nam perinde eſt quo ordine ſcribantur.

Quantitates affirmativæ, quæ ex parte ſpecierum conveniunt, uniuntur addendo numeros præfixos, quibus ſpecies multiplicantur. Sic $7a+9a$ faciunt $16a$. Et $11bc+15bc$ faciunt $26bc$. Item $3\frac{a}{c}+5\frac{a}{c}$ faciunt $8\frac{a}{c}$; & $2\sqrt{ac}+7\sqrt{ac}$ faciunt $9\sqrt{ac}$; & $6\sqrt{ab-xx}+7\sqrt{ab-xx}$ faciunt $13\sqrt{ab-xx}$. Et ad eundem modum $6\sqrt{3}+7\sqrt{3}$ faciunt $13\sqrt{3}$. Quinetiam $a\sqrt{ac}+b\sqrt{ac}$ faciunt $\overline{a+b}\sqrt{ac}$, additis nempe a & b tanquam ſi eſſent numeri multiplicantes $\sqrt{ac}$. Et ſic $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}+3a\sqrt{\frac{3axx-x^3}{a+x}}$ faciunt $\overline{5a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$, eo quòd $2a+3c$ & $3a$ faciant $5a+3c$.

Fractiones affirmativæ quarum idem eſt denominator, uniuntur addendo numeratores. Sic $\frac{1}{5}+\frac{2}{5}$ faciunt $\frac{3}{5}$; & $\frac{2ax}{b}+\frac{3ax}{b}$ faciunt $\frac{5ax}{b}$; & $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}+\frac{17a\sqrt{cx}}{2a+\sqrt{cx}}$ faciunt $\frac{25a\sqrt{cx}}{2a+\sqrt{cx}}$; & $\frac{aa}{c}+\frac{bx}{c}$ faciunt $\frac{aa+bx}{c}$.

Negativæ quantitates eodem modo adduntur ac affirmativæ. Sic

Sic -2 & -3 faciunt -5; $-\frac{4ax}{b}$ & $-\frac{11ax}{b}$ faciunt $-\frac{15ax}{b}$; $-a\sqrt{ax}$ & $-b\sqrt{ax}$ faciunt $-a-b\sqrt{ax}$. ADDITIO.

Ubi verò *negativa quantitas affirmativæ adjicienda eſt* oportet affirmativam negativâ diminuere. Sic 3 & -2 faciunt 1; $\frac{11ax}{b}$ & $-\frac{4ax}{b}$ faciunt $\frac{7ax}{b}$; $-a\sqrt{ac}$ & $b\sqrt{ac}$ faciunt $\overline{b-a}\sqrt{ac}$. Et nota, quòd ubi negativa quantitas excedit affirmativam, aggregatum erit negativum. Sic 2 & -3 faciunt -1; $-\frac{11ax}{b}$ & $\frac{4ax}{b}$ faciunt $-\frac{7ax}{b}$; ac $2\sqrt{ac}$ & $-7\sqrt{ac}$ faciunt $-5\sqrt{ac}$.

In additione aut plurium aut magis compoſitarum quantitatum, convenit obſervare formam operationis ſuprà in additione numerorum expoſitam. Quemadmodum ſi $17ax-14a+3$, & $4a+2-8ax$, & $7a-9ax$ addendæ ſunt, diſpono eas in ſerie deſcendente, ita ſcilicet ut termini maximè affines ſtent in iiſdem columnis. Nempe numeri 3 & 2 in unâ columnâ, ſpecies $-14a$ & $4a$ & $7a$ in aliâ columnâ, atque ſpecies $17ax$ & $-8ax$ & $-9ax$ in tertiâ. Dein terminos cujuſque columnæ ſigillatim addo; dicendo 2 & 3 faciunt 5; quod ſubſcribo: dein $7a$ & $4a$ faciunt $11a$; & inſuper $-14a$ facit $-3a$; quod iterum ſubſcribo: denique $-9ax$ & $-8ax$ faciunt $-17ax$, & inſuper $17ax$ facit 0. Adeoque prodit ſumma $-3a+5$.

$$\begin{array}{rrr} 17ax & -14a & +3 \\ -8ax & +4a & +2 \\ -9ax & +7a & \\ \hline * & -3a & +5 \end{array}$$

Eâdem methodo res in ſequentibus exemplis abſolvitur.

$$\begin{array}{lll} 12x+7a & 11bc+7\sqrt{ac} & -\frac{4ax}{b}+6\sqrt{3+\frac{1}{5}} \\ 7x+9a & 15bc+2\sqrt{ac} & +\frac{11ax}{b}-7\sqrt{3+\frac{2}{5}} \\ \hline 19x+16a & 26bc-5\sqrt{ac} & \frac{7ax}{b}-\sqrt{3+\frac{1}{5}} \end{array}$$

$$\begin{array}{r} -6xx+\frac{3}{7}x \\ 5x^3\quad+\frac{5}{7}x \\ \hline 5x^3-6xx+\frac{8}{7}x \end{array} \qquad \begin{array}{r} aay+2a^3-\frac{a^4}{2y} \\ -2ayy-4aay+\quad a^3 \\ y^3+2ayy-\frac{1}{2}aay \qquad\qquad \\ \hline y^3\quad * -3\frac{1}{2}aay+3a^3-\frac{a^4}{2y} \end{array}$$

$$\begin{array}{l} 5x^4+2ax^3 \\ -3x^4-2ax^3+\ 8\frac{1}{4}a^3\sqrt{aa+xx} \\ -2x^4+5bx^3-20\ a^3\sqrt{aa-xx} \\ \qquad\quad -4bx^3-\ 7\frac{1}{4}a^3\sqrt{aa+xx} \\ \hline \quad * \ +\ bx^3+\quad a^3\sqrt{aa+xx} \\ \qquad\qquad\ -\ 20a^3\sqrt{aa-xx}. \end{array}$$

CAPUT TERTIUM.

DE SUBDUCTIONE.

NUMERORUM *non nimis compoſitorum* inventio etiam Differentiæ per ſe patet. Quemadmodum quòd 9 de 17 relinquat 8. At in *magis compoſitis* Subductio fieri ſolet *ſubſcribendo numerum ablativum, & ſigillatim auferendo figuras inferiores de ſuperioribus*. Sic ad auferendum 63543 de 782579, ſubſcripto 63543, dic 3 de 9 relinquit 6, quod ſcribe infrà: Dein 4 de 7 relinquit 3, quod pariter ſcribe infrà: Tum 5 de 5 relinquit 0, quod itidem ſubſcribe: Poſtea 3 de 2 auferendum eſt: ſed cùm 3 ſit majus, figura 1 à proximâ figurâ 8 mutuò ſumi debet, quæ unà cum 2 faciat 12; à quo auferri poteſt 3, & reſtat 9; quod inſuper ſubſcribe: Adhæc cùm præter 6 etiam 1 de 8 auferendum ſit, adde 1 ad 6, & ſumma 7 de 8 relinquet 1; quod etiam ſubſcribe. Denique cùm in inferiori numero nihil reſtet auferendum de ſuperiori 7, ſubſcribe etiam 7, & ſic tandem habes differentiam 719036.

$$\begin{array}{r} 782579 \\ 63543 \\ \hline 719036 \end{array}$$

Cæterùm

I

Cæterùm omnino cavendum eſt, ut figuræ numeri ablativi ſubſcribantur in locis homogeneis; nempe unitates infra alterius numeri unitates, deni numeri infra denos, decimæ partes infra decimas, *&c.* Sicut in Additione dictum eſt. Sic ad auferendum decimalem o'63 ab integro 547, non diſpones numeros hoc modo $\begin{smallmatrix}5\,47\\0{,}63\end{smallmatrix}$; ſed ſic $\begin{smallmatrix}547\\0{,}63\end{smallmatrix}$; ita nempe ut circulus, qui locum unitatum in decimali occupat, ſubjiciatur unitatibus alterius numeri. Tum, circulis in locis vacuis ſuperioris numeri ſubintellectis, dic 3 de o auferendum eſſe; ſed cùm nequeat, debet 1 de loco anteriori mutuò ſumi, ut o evadat 10; à quo 3 auferri poteſt, & dabit 7, quod infrà ſcribe. Dein illud 1 quod mutuò ſumitur, adjectum 6 facit 7, & hoc de ſuperiore o auferendum eſt; ſed cùm nequeat, debet iterum 1 de loco anteriori ſumi, ut o evadat 10; & 7 de 10 relinquet 3, quod ſimiliter infrà ſcribendum eſt. Tum illud 1 adjectum o facit 1, & hoc 1 de 7 relinquit 6; quod itidem ſubſcribe. Denique figuras etiam 54, ſiquidem de illis nihil amplius auferendum reſtat, ſubſcribe, & habebis reſiduum 546'37. SUBDUCTIO.

547
o'63
546'37

Exercitationis gratiâ plura, tum in integris tum in decimalibus numeris, exempla ſubjecimus.

1673	1673	'458074	35'72	46,5003	308,7
1541	1580	9205	14'32	3,078	25,74
132	93	448869	21'4	43,4223	282,96

Siquando major numerus de minori auferendus eſt, oportet minorem de majore auferre, & reſiduo præfigere negativum ſignum. Veluti ſi auferendum ſit 1673 de 1541, è contra aufero 1541 de 1673, & reſiduo 132 præfigo ſignum $-$.

In terminis Algebraicis Subductio *fit connectendo quantitates, cum ſignis omnibus quantitatis ſubducendæ mutatis, & inſuper uniendo quæ poſſunt uniri*, perinde ut in Additione factum eſt. Sic $+7a$ de $+9a$ relinquit $+9a-7a$, ſive $2a$; $-7a$ de $+9a$ relinquit $+9a+7a$, ſive $16a$; $+7a$ de $-9a$ relinquit $-9a-7a$, ſive $-16a$; & $-7a$ de $-9a$ relinquit $-9a+7a$, ſive $-2a$. Sic $3\frac{a}{c}$ de

$5\frac{a}{c}$ relinquit $2\frac{a}{c}$; $7\sqrt{ac}$ de $2\sqrt{ac}$ relinquit $-5\sqrt{ac}$; $\frac{2}{9}$ de $\frac{5}{9}$ relinquit $\frac{3}{9}$; $-\frac{4}{7}$ de $\frac{3}{7}$ relinquit $\frac{7}{7}$; $-\frac{2ax}{b}$ de $\frac{3ax}{b}$ relinquit $\frac{5ax}{b}$; $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ de $\frac{-17a\sqrt{cx}}{2a+\sqrt{cx}}$ relinquit $\frac{-25a\sqrt{cx}}{2a+\sqrt{cx}}$; $\frac{aa}{c}$ de $\frac{bx}{c}$ relinquit $\frac{bx-aa}{c}$; $a-b$ de $2a+b$ relinquit $2a+b-a+b$ ſive $a+2b$; $3az-zz+ac$ de $3az$ relinquit $3az-3az+zz-ac$ ſive $zz-ac$; $\frac{2aa-ab}{c}$ de $\frac{aa+ab}{c}$ relinquit $\frac{aa+ab-2aa+ab}{c}$ ſive $\frac{-aa+2ab}{c}$: Et $\overline{a-x}\sqrt{ax}$ de $\overline{a+x}\sqrt{ax}$ relinquit $\overline{a+x-a+x}\sqrt{ax}$ ſive $2x\sqrt{ax}$. Et ſic in aliis.

Cæterùm ubi quantitates pluribus terminis conſtant, operatio perinde ac in numeris inſtitui poteſt. Id quod in ſequentibus exemplis videre eſt.

$$\begin{array}{r} 12x+7a \\ 7x+9a \\ \hline 5x-2a \end{array} \qquad \begin{array}{r} 15bc+2\sqrt{ac} \\ -11bc+7\sqrt{ac} \\ \hline 26bc-5\sqrt{ac} \end{array} \qquad \begin{array}{l} 5x^3 \quad +\frac{5}{7}x \\ \quad 6xx-\frac{3}{7}x \\ \hline 5x^3-6xx+\frac{8}{7}x \end{array}$$

$$\begin{array}{l} \frac{11ax}{b}-7\sqrt{3}+\frac{2}{5} \\ \frac{4ax}{b}-6\sqrt{3}-\frac{1}{5} \\ \hline \frac{7ax}{b}-\sqrt{3}+\frac{3}{5} \end{array}$$

CAPUT QUARTUM.

Multiplicatio.

DE MULTIPLICATIONE.

NUMERI qui ex Multiplicatione duorum quorumvis numerorum non majorum quàm 9 oriuntur, memoriter addiſcendi ſunt. Veluti quod 5 in 7, facit 35; quòdque 8 in 9 facit 72, &c. Deinde majorum numerorum multiplicatio ad horum exemplorum normam inſtituetur.

Si 795 per 4 multiplicare oportet, ſubſcribe 4, ut vides. Dein dic, 4 in 5 facit 20, cujus poſteriorem figuram 0, ſcribe infrà 4, priorem vero 2 reſerva in proximam operationem. Dic itaque præterea 4 in 9 facit 36, cui adde præfatum 2, & fit 38; cujus poſteriorem figuram 8 ut antè ſubſcribe, & priorem 3 reſerva. Denique dic 4 in 7 facit 28 cui adde prædictum 3 & fit 31. Eoque pariter ſubſcripto, habebitur 3180, numerus qui prodit multiplicando totum 795 per 4.

```
 795
   4
----
3180
```

Porrò ſi 9043 multiplicandus eſt per 2305, ſcribe alterutrum 2305 infra alterum 9043 ut antè, & multiplica ſuperiorem 9043, primò per 5 pro more oſtenſo; & emerget 45215: dein per 0, & emerget 0000: tertio per 3, & emerget 27129: denique per 2, & emerget 18086. Hoſque ſic emergentes numeros in ſerie deſcendente ita ſcribe, ut cujuſque inferioris ultima figura ſit uno loco proprior ſiniſtræ, quàm ultima ſuperioris. Tandem hos omnes adde & orietur 20844115, numerus qui fit multiplicando totum 9043 per 2305.

```
    9043
    2305
--------
   45215
   0000
 27129
18086
--------
20844115
```

Decimales numeri per integros vel per alios decimales perinde multiplicantur, ut vides in his exemplis.

Caput Quartum.

72,4	50,18	3,9025
29	2,75	0,0132
6516	25090	78050
1448	35126	117075
2099,6	10036	39025
	137,9950	0,05151300

Sed nota, quòd *in prodeunte numero tot ſemper figuræ ad dextram pro decimalibus abſcindi debent, quot ſunt figuræ decimales in utroque numero multiplicante.* Et ſi fortè non ſint tot figuræ in prodeunte numero, deficientes loci circulis adimplendi ſunt, ut hîc fit in exemplo tertio.

Simplices termini Algebraici multiplicantur ducendo numeros in numeros, & ſpecies in ſpecies, ac ſtatuendo factum Affirmativum, *ſi ambo factores ſint affirmativi aut ambo negativi, &* Negativum *ſi ſecus.*

Sic $2a$ in $3b$, vel $-2a$ in $-3b$, facit $6ab$; vel $6ba$: Nihil enim refert quo ordine ponantur. Sic etiam $2a$ in $-3b$, vel $-2a$ in $3b$, facit $-6ab$. Et ſic [k] $2ac$ in $8bcc$ facit $16abccc$, ſive $16abc^3$; & $7axx$ in $-12aaxx$ facit $-84a^3x^4$; & $-16cy$ in $31ay^3$ facit $-496acy^4$; & $-4z$ in $-3\sqrt{az}$ facit $12z\sqrt{az}$. Atque ita 3 in -4 facit -12, & -3 in -4 facit 12.

Fractiones multiplicantur ducendo numeratores in numeratores, ac denominatores in denominatores.

Sic $\frac{2}{5}$ in $\frac{3}{7}$ facit $\frac{6}{35}$; & $\frac{a}{b}$ in $\frac{c}{d}$ facit $\frac{ac}{bd}$; & $2\frac{a}{b}$ in $3\frac{c}{d}$ facit $6 \times \frac{a}{b} \times \frac{c}{d}$ ſeu $6\frac{ac}{bd}$; & $\frac{3acy}{2bb}$ in $\frac{-7cyy}{4b^3}$ facit $\frac{-21accy^3}{8b^5}$; & $\frac{-4z}{c}$ in $\frac{-3\sqrt{az}}{a}$ facit $\frac{12z\sqrt{az}}{cc}$; & $\frac{a}{b}x$ in $\frac{c}{d}xx$ facit

ac

[k] Quantitates potentiales et radicales diverſarum denominationum ab eâdem quantitate ſimplici, adhibitis indicibus fractis, multiplicantur indicum additione: ſiquidem logarithmi ſunt

MULTIPLICATIO.

$\frac{ac}{bd}x^3$. Item 3 in $\frac{2}{5}$ facit $\frac{6}{5}$; ut pateat, ſi 3 reducatur ad formam fractionis $\frac{3}{1}$, adhibendo unitatem pro Denominatore. Et ſic $\frac{15aaz}{cc}$ in $2a$ facit $\frac{30a^3z}{cc}$. Unde obiter nota, quòd $\frac{ab}{c}$ & $\frac{a}{c}b$ idem valent; ut & $\frac{abx}{c}$, $\frac{ab}{c}x$, & $\frac{a}{c}bx$; nec non $\frac{\overline{a+b}\sqrt{cx}}{a}$ & $\frac{a+b}{a}\sqrt{cx}$, & ſic in aliis.

Quantitates radicales ejuſdem denominationis (hoc eſt, ſi ſint ambæ radices quadraticæ, aut ambæ cubicæ, aut ambæ quadrato-quadraticæ, &c.) multiplicantur ducendo terminos in ſe invicem ſub eodem ſigno radicali. Sic $\sqrt{3}$ in $\sqrt{5}$ facit $\sqrt{15}$; & $\sqrt{ab}$ in $\sqrt{cd}$ facit $\sqrt{abcd}$. Et $\sqrt[3]{5ayy}$ in $\sqrt[3]{7ayz}$ facit $\sqrt[3]{35aay^3z}$. Et $\sqrt{\frac{a^3}{c}}$ in $\sqrt{\frac{abb}{c}}$ facit $\sqrt{\frac{a^4bb}{cc}}$, hoc eſt * $\frac{aab}{c}$. Et $2a\sqrt{az}$ in $3b\sqrt{az}$ facit $6ab\sqrt{aazz}$, hoc eſt $6aabz$. Et $\frac{3xx}{\sqrt{ac}}$ in $\frac{-2x}{\sqrt{ac}}$ facit $\frac{-6x^3}{\sqrt{aacc}}$, hoc eſt $\frac{-6x^3}{ac}$. Et $\frac{-4x\sqrt{ab}}{7a}$ in $\frac{-3dd\sqrt{5cx}}{10ee}$ facit $\frac{12ddx\sqrt{5abcx}}{70aee}$ &c.

* Vide Cap. De Notatione.

Quantitates pluribus partibus conſtantes multiplicantur ducendo ſingulas unius partes in ſingulas alterius, perinde ut in Multiplicatione numerorum oſtenſum eſt. Sic $c-x$ in a facit $ac-ax$, & $aa+2ac-bc$ in $a-b$ facit $a^3+2aac-aab-3bac+bcc$. Nam $aa+2ac-bc$ in $-b$ facit $-aab-2acb+bbc$, & in a facit $a^3+2aac-abc$, quorum ſumma eſt $a^3+2aac-aab-3abc+bbc$. Hujus multiplicationis ſpecimen, unà cum aliis conſimilibus exemplis, ſubjectum habes.

ſunt indices (vide cap. 1. not. [h]). Sic $a^2\times a^3=a^5$, et $a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{5}{6}}$, et $a^2\times a^{\frac{1}{3}}=a^{\frac{7}{3}}$, et $a^3\times a^{\frac{1}{2}}=a^{\frac{7}{2}}$, et $a^3\times a^{-1}=a^2$, et $a^3\times a^{-5}=a^{-2}$, et $a^{\frac{2}{3}}\times a^{-\frac{1}{2}}=a^{\frac{1}{6}}$, & $a^{-\frac{2}{3}}\times a^{-\frac{4}{5}}=a^{-\frac{22}{15}}$, et $a^3\times a^2\times a^{-5}=a^0=1$, et $a^{\frac{2}{5}}\times a^{-\frac{2}{5}}=a^0=1$.

Caput Quartum.

$$\begin{array}{r} aa+2ac-bc \\ a-b \\ \hline -aab-2abc+bbc \\ a^3+2aac-abc \quad\quad \\ \hline a^3+2aac-aab-3abc+bbc \end{array} \qquad \begin{array}{r} a+b \\ a+b \\ \hline ab+bb \\ aa+ab \quad\; \\ \hline aa+2ab+bb \end{array}$$

$$\begin{array}{r} a+b \\ a-b \\ \hline -ab-bb \\ aa+ab \quad\; \\ \hline aa \;\; * -bb \end{array} \qquad \begin{array}{r} yy+2ay-\frac{1}{2}aa \\ yy-2ay+aa \\ \hline aayy+2a^3y-\frac{1}{2}a^4 \\ -2ay^3-4aayy+a^3y \quad\quad\; \\ y^4+2ay^3-\frac{1}{2}aayy \quad\quad\quad\quad\quad\quad \\ \hline y^4 \;\; * -3\frac{1}{2}aayy+3a^3y-\frac{1}{2}a^4 \end{array}$$

$$\begin{array}{r} \frac{2ax}{c}-\sqrt{\frac{a^3}{c}} \\ 3a+\sqrt{\frac{abb}{c}} \\ \hline \frac{2ax}{c}\sqrt{\frac{abb}{c}}-\frac{aab}{c} \\ \frac{6aax}{c}-3a\sqrt{\frac{a^3}{c}} \quad\quad\; \\ \hline \frac{6aax}{c}-3a\sqrt{\frac{a^3}{c}}+\frac{2ax}{c}\sqrt{\frac{abb}{c}}-\frac{aab}{c}. \end{array}$$

CAPUT

CAPUT QUINTUM.

DE DIVISIONE.

Diviſio.

DIVISIO *in numeris inſtituitur, quærendo quot vicibus Diviſor in Dividendo continetur, totieſque auferendo, & ſcribendo totidem unitates in Quoto. Idque iteratò ſi opus eſt, quamdiu diviſor auferri poteſt.*

Sic ad dividendum 63 per 7, quære quoties 7 continetur in 63, & emergent 9 pro quoto præcisè; adeoque $\frac{63}{7}$ valet 9. Inſuper ad dividendum 371 per 7, præfige diviſorem 7, & imprimis opus inſtituens in initialibus figuris Dividendi proximè majoribus Diviſore, nempe in 37, dic quoties 7 continetur in 37? Reſp. 5. Tum, ſcripto 5 in Quoto, aufer 5×7, ſeu 35, de 37, & reſtabit 2; cui adnecte ultimam figuram Dividendi, nempe 1, & fit 21, reliqua pars Dividendi, in quâ proximum opus inſtituendum eſt. Dic itaque ut ante quoties 7 continetur in 21? Reſp. 3. Quare ſcripto 3 in Quoto, aufer 3×7, ſeu 21, de 21, & reſtabit 0. Unde conſtat 53 eſſe numerum præcisè, qui oritur ex diviſione 371 per 7.

```
7)371(53
  35
  ——
   21
   21
   ——
    0
```

Atque ita ad dividendum 4798 per 23, opus primò inſtituens in initialibus figuris 47, dic quoties 23 continetur in 47? Reſp. 2. Scribe ergo 2 in Quoto; & de 47 ſubduc 2×23, ſeu 46, reſtatque 1; cui ſubjunge proximum numerum Dividendi, nempe 9, & fit 19 in ſubſequens opus. Dic itaque quoties 23 continetur in 19? Reſp. 0. Quare ſcribe 0 in Quoto; & de 19, ſubduc 0×23, ſeu 0, & reſtat 19; cui ſubjunge ultimum numerum 8, & fit 198 in proximum opus. Quamobrem dic ultimò quoties 23 continetur in 198, (id quod ex initialibus numeris,

CAPUT QUINTUM.

2 & 19, conjici poteſt, animadvertendo quoties 2 continetur in 19) ? Reſp. 8. Quare ſcribe 8 in Quoto; & de 198 ſubduc 8 × 23, ſeu 184; reſtabitque 14 adhuc dividendus per 23. Adeoque Quotus erit $208\frac{14}{23}$. Quod ſi hujuſmodi fractio minùs placeat, poſſis Diviſionem in Fractionibus decimalibus ultrà ad libitum proſequi, ſemper adnectendo circulum numero reſiduo. Sic reſiduo 14 adnecte 0, fitque 140. Tum dic quoties 23 ſit in 140 ? Reſp. 6. Scribe ergo 6 in Quoto; & de 140 ſubduc 6 × 23, ſeu 138, & reſtabit 2; cui adnecte 0 ut ante. Et ſic, opere ad arbitrium continuato, emerget tandem Quotus 208,6086, &c.

```
23)4798(208,6086,&c.
   46
   --
    19
    00
    --
    198
    184
    ---
     140
     138
     ---
       20
       00
       --
       200
       184
       ---
       160
```

Ad eundum modum fractio decimalis 3,5218 per fractionem decimalem 46,1 dividitur, & prodit 0, 07639, &c. *Ubi nota, quòd in Quoto tot figuræ pro decimalibus abſcindendæ ſunt, quot ſunt in ultimo dividuo plures quàm in diviſore:* Ut in hoc exemplo quinque, quia ſex ſunt in ultimo dividuo 0,004370 & una in Diviſore 46,1.

```
46,1)3,5218(0,07639
      322,7
      -----
       2948
       2766
       ----
        1820
        1383
        ----
         4370
```

Exempla plura lucis gratiâ ſubjunximus.

```
9043)20844115(2305.          72,4)2099,6(29
     18086                        1448
     -----                        ----
      27581                       6516
      27129                       6516
      -----                       ----
        45215                        0
        45215
        -----
            0
```

Diviſio.

```
50,18)137,995(2,75.     0,0132)0,051513(3,9025
      10036                    396
      -----                    ----
      37635                    1191
      35126                    1188
      -----                    ----
       25090                    330
       25090                    264
       -----                    ---
           0                     660
                                 660
                                 ---
                                   0
```

In terminis Algebraicis Diviſio fit reſolvendo quicquid per multiplicationem conflatur. Sic ab diviſ. per a dat b pro quoto, $6ab$ div. per $2a$ dat $3b$; & div. per $-2a$ dat $-3b$. $-6ab$ div. per $2a$ dat $-3b$; & div. per $-2a$ dat $3b$. $16abc^3$ div. per $2ac$ dat $8bcc$. $-84a^3x^4$ div. per $-12aaxx$ dat $7axx$. Item $\frac{6}{35}$ div. per $\frac{2}{5}$ dat $\frac{3}{7}$. $\frac{ac}{bd}$ div. per $\frac{a}{b}$ dat $\frac{c}{d}$. $\frac{-21accy^3}{8b^5}$ div. per $\frac{3acy}{2bb}$ dat $\frac{-7cyy}{4b^3}$. $\frac{6}{5}$ div. per 3 dat $\frac{2}{5}$; & viciſſim $\frac{6}{5}$ div. per $\frac{2}{5}$ dat $\frac{3}{1}$, ſeu 3. $\frac{30a^3z}{cc}$ div. per $2a$ dat $\frac{15aaz}{cc}$; & viciſſim diviſ. per $\frac{15aaz}{cc}$ dat $2a$. Item $\sqrt{15}$ div. per $\sqrt{3}$ dat $\sqrt{5}$. $\sqrt{abcd}$ div. per $\sqrt{cd}$ dat $\sqrt{ab}$. $\sqrt{a^3c}$ per $\sqrt{ac}$ dat $\sqrt{aa}$, ſeu a. $\sqrt[3]{35aay^3z}$ div. per $\sqrt[3]{5ayy}$ dat $\sqrt[3]{7ayz}$. $\sqrt{\frac{a^4bb}{cc}}$ div. per $\sqrt{\frac{a^3}{c}}$ dat $\sqrt{\frac{abb}{c}}$. $\frac{12ddx\sqrt{5abcx}}{70aee}$ div. per $\frac{-3dd\sqrt{5cx}}{10ee}$ dat $\frac{-4x\sqrt{ab}}{7a}$. Atque ita $\overline{a+b}\sqrt{ax}$ div. per $a+b$ dat $\sqrt{ax}$, & viciſſim div. per $\sqrt{ax}$ dat $a+b$. Et $\frac{a}{a+b}\sqrt{ax}$ div. per $\frac{1}{a+b}$ dat $a\sqrt{ax}$; vel div. per a dat $\frac{1}{a+b}\sqrt{ax}$, ſive $\frac{\sqrt{ax}}{a+b}$; & viciſſim div. per $\frac{\sqrt{ax}}{a+b}$ dat a. Cæterùm in hujuſmodi reſolutionibus omninò cavendum eſt, ut

Caput Quintum. quantitates ſint ejuſdem ordinis quæ ad invicem applicantur. Nempe ut numeri applicentur ad numeros, ſpecies ad ſpecies, radicales ad radicales, numeratores Fractionum ad Numeratores, ac Denominatores ad Denominatores, nec non in Numeratoribus, Denominatoribus, & Radicalibus quantitates cujuſque generis ad quantitates homogeneas.

Quòd ſi quantitas dividenda nequeat ſic per diviſorem reſolvi, ſufficit, ubi ambæ quantitates ſunt integræ, ſubſcribere Diviſorem cum lineolâ interjectâ. Sic ad dividendum ab per c, ſcribitur $\frac{ab}{c}$; & ad dividendum $\overline{a+b}\sqrt{cx}$ per a, ſcribitur $\frac{\overline{a+b}\sqrt{cx}}{a}$, vel $\frac{a+b}{a}\sqrt{cx}$. Et ſic $\sqrt{ax-xx}$ diviſ. per $\sqrt{cx}$ dat $\frac{\sqrt{ax-xx}}{\sqrt{cx}}$, ſive $\sqrt{\frac{ax-xx}{cx}}$. Et $\overline{aa+ab}\sqrt{aa-2xx}$ diviſ. per $\overline{a-b}\sqrt{aa-xx}$ dat $\frac{aa+ab}{a-b}\sqrt{\frac{aa-2xx}{aa-xx}}$. Et $12\sqrt{5}$ div. per $4\sqrt{7}$ dat $3\sqrt{\frac{5}{7}}$.

Ubi verò fracta ſunt illæ quantitates, Duc Numeratorem Dividendæ quantitatis in Denominatorem Diviſoris, ac Denominatorem in Numeratorem; & *factus prior* erit Numerator, ac *poſterior* Denominator Quoti. Sic ad dividendum $\frac{a}{b}$ per $\frac{c}{d}$ ſcribitur $\frac{ad}{bc}$, multiplicato ſcilicet a per d, & b per c. Parique ratione $\frac{3}{7}$ diviſ. per $\frac{5}{4}$ dat $\frac{12}{35}$: & $\frac{3a}{4c}\sqrt{ax}$ diviſ. per $\frac{2c}{5a}$ dat $\frac{15aa}{8cc}\sqrt{ax}$; diviſ. autem per $\frac{2c\sqrt{aa-xx}}{5a\sqrt{ax}}$ dat $\frac{15a^2x}{8cc\sqrt{aa-xx}}$. Et ad eundem modum $\frac{ad}{b}$ diviſ. per c (ſive per $\frac{c}{1}$) dat $\frac{ad}{bc}$. Et c (ſive $\frac{c}{1}$) diviſ. per $\frac{ad}{b}$ dat $\frac{bc}{ad}$. Et $\frac{3}{7}$ div. per 5 dat $\frac{3}{35}$. Et 3 div. per $\frac{5}{4}$ dat $\frac{12}{5}$. Et $\frac{a+b}{c}\sqrt{cx}$ div. per a dat $\frac{a+b}{ac}\sqrt{cx}$. Et $\overline{a+b}\sqrt{cx}$

div.

div. per $\frac{a}{c}$ dat $\frac{ac+bc}{a}\sqrt{cx}$. Et $2\sqrt{\frac{axx}{c}}$ diviſ. per $3\sqrt{cd}$ dat $\frac{2}{3}\sqrt{\frac{axx}{ccd}}$; Div. autem per $3\sqrt{\frac{cd}{x}}$ dat $\frac{2}{3}\sqrt{\frac{ax^3}{ccd}}$. Et $\frac{1}{5}\sqrt{\frac{7}{11}}$ diviſ. per $\frac{3}{8}\sqrt{\frac{3}{7}}$ dat $\frac{8}{15}\sqrt{\frac{49}{33}}$. Et ſic in aliis. DIVISIO.

Quantitas ex pluribus terminis compoſita dividitur applicando ſingulos ejus terminos ad Diviſorem. Sic $aa+3ax-xx$ diviſum per a dat $a+3x-\frac{xx}{a}$. At ubi Diviſor etiam ex pluribus terminis conſtat, diviſio perinde ac in Numeris inſtitui debet. Sic ad dividendum $a^3+2aac-aab-3abc+bbc$ per $a-b$, Dic quoties a continetur in a^3, nempe primus terminus Diviſoris in primo Dividendi? Reſp. aa. Quare ſcribe aa in Quoto, & ablato $a-b$ in aa, ſive a^3-aab, de Dividendo, reſtabit $2aac-3abc+bbc$ adhuc dividendum. Dic itaque rurſus quoties a continetur in $2aac$? Reſp. $2ac$. Quare ſcribe etiam $2ac$ in Quoto, & ablato $a-b$ in $2ac$, ſive $2aac-2abc$, de præfato Reſiduo, reſtabit etiamnum $-abc+bbc$. Quamobrem dic iterum quoties a continetur in $-abc$? Reſp. $-bc$. Et proinde ſcribe $-bc$ in Quoto, & ablato denuo $a-b$ in $-bc$, ſive $-abc+bbc$, de noviſſimo Reſiduo, reſtabit nihil. Quod indicat Diviſionem peractam eſſe, prodeunte Quoto $aa+2ac-bc$.

Cæterùm ut hujuſmodi operationes ad formam, quâ in Diviſione numerorum uſi ſumus, debitè reducantur, *termini tum dividendæ quantitatis tum Diviſoris juxta dimenſiones literæ alicujus, quæ ad hanc rem maximè idonea judicabitur, in ordine diſponendi ſunt*, ita nempe ut illi primum locum occupent, in quibus litera iſta eſt plurimarum dimenſionum, iique ſecundum in quibus dimenſiones ejus ad maximas proximæ ſunt; Et ſic deinceps uſque ad terminos qui per literam iſtam non omnino multiplicantur, adeoque ultimum locum occupabunt. Sic in allato Exemplo ſi termini ordinentur juxta dimenſiones literæ a, formam operis exhibebit adjunctum Diagramma.

$$\begin{array}{l} a-b)\,a^3 \begin{smallmatrix}+2aac\\-aab\end{smallmatrix} -3abc+bbc\,(aa+2ac-bc \\ \underline{a^3-aab} \\ 0+2aac-3abc \\ \quad\underline{2aac-2abc} \\ \qquad 0-abc+bbc \\ \qquad\underline{\;\;-abc+bbc} \\ \qquad\quad 0 \quad 0 \end{array}$$

Ubi videre eſt quòd terminus a^3, ſive a trium dimenſionum, occupat primum locum dividendæ quantitatis, terminique $\begin{smallmatrix}2aac\\-aab\end{smallmatrix}$, in quibus a eſt duarum dimenſionum, ſecundum occupant, & ſic præterea. Potuit etiam dividenda quantitas ſic ſcribi; $a^3 \begin{smallmatrix}+2c\\-b\end{smallmatrix} aa-3bca+bbc$: ubi termini ſecundum locum occupantes uniuntur, aggregando factores literæ juxta quam fit ordinatio. Et hoc modo ſi termini juxta dimenſiones literæ b diſponerentur, opus ſicut in proximo Diagrammate inſtitui deberet: Cujus explicationem adnectere viſum eſt.

$$\begin{array}{l} -b+a)\,cbb \begin{smallmatrix}-3ac\\-aa\end{smallmatrix} b \begin{smallmatrix}+a^3\\+2aac\end{smallmatrix} \left(-cb \begin{smallmatrix}+2ac\\+aa\end{smallmatrix}\right. \\ \underline{cbb-acb} \\ 0 \begin{smallmatrix}-2ac\\-aa\end{smallmatrix} b \begin{smallmatrix}+a^3\\+2aac\end{smallmatrix} \\ \;\;\underline{\begin{smallmatrix}-2ac\\-aa\end{smallmatrix} b \begin{smallmatrix}+2aac\\+a^3\end{smallmatrix}} \\ \qquad 0 \quad 0 \end{array}$$

Dic quoties $-b$ continetur in cbb? Reſp. $-cb$. Quare ſcripto $-cb$ in Quoto, aufer $-b$[illegible]$-cb$, ſeu $bbc-acb$, & reſtabit in ſecundo loco $\begin{smallmatrix}-2ac\\-aa\end{smallmatrix} b$. Re[illegible]ic adnecte, ſi placet, quantitates

DIVISIO.

tes in ultimo loco, nempe $\begin{matrix} a^3 \\ +2aac \end{matrix}$; & dic iterum quoties $-b$ continetur in $\begin{matrix} -2ac \\ -aa \end{matrix} b$? Reſp. $\begin{matrix} +2ac \\ +aa \end{matrix}$. Quare his in Quoto ſcriptis, aufer $-b+a$ in $\begin{matrix} +2ac \\ +aa \end{matrix}$, ſeu $\begin{matrix} -2ac \\ -aa \end{matrix} b \begin{matrix} +2aac \\ +a^3 \end{matrix}$ & reſtabit nihil. Unde conſtat diviſionem peractam eſſe, prodeunte Quoto $-cb+2ac+aa$ ut ante.

Atque ita ſi dividere oportet $aay^4-aac^4+yyc^4+y^6-2y^4cc-a^6-2a^4cc-a^4yy$ per $yy-aa-cc$: Quantitates juxta literam y ad hunc modum ordino, $yy\begin{matrix}-aa\\-cc\end{matrix}\Big)\,y^6\begin{matrix}+aa\\-2cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc.\\-aac^4\end{matrix}$ Dein Diviſionem ut in ſubjecto Diagrammate inſtituo. Adjiciuntur & alia exempla, de quibus inſuper obſervandum eſt, quod ubi dimenſiones literæ, ad quam ordinatio fit, non in eâdem ubique progreſſione Arithmeticâ, ſed per ſaltum alicubi procedunt, locis vacuis ſubſtituitur nota *

$$
\begin{array}{l}
yy\begin{matrix}-aa\\-cc\end{matrix}\Big)\,y^6\begin{matrix}+aa\\-2cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc\\-aac^4\end{matrix}\\
\hline
y^6\begin{matrix}-aa\\-cc\end{matrix}y^4 \qquad \Big(\,y^4\begin{matrix}+2aa\\-cc\end{matrix}yy\begin{matrix}+a^4\\+aacc\end{matrix}\\
\hline
0\quad\begin{matrix}+2aa\\-cc\end{matrix}y^4\\
\quad\begin{matrix}+2aa\\-cc\end{matrix}y^4\begin{matrix}-2a^4\\-aacc\\+c^4\end{matrix}yy\\
\hline
\qquad 0\quad\begin{matrix}+a^4\\+aacc\end{matrix}yy\\
\qquad\quad\begin{matrix}+a^4\\+aacc\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc\\-aac^4\end{matrix}\\
\hline
\qquad\quad 0 \qquad\qquad 0
\end{array}
$$

$$
\begin{array}{l}
a+b)\,aa\,*\,-bb\,(a-b\\
\hline
aa\ +ab\\
\hline
0\ \ -ab\\
\quad -ab-bb\\
\hline
\quad\ 0\qquad 0
\end{array}
$$

CAPUT SEXTUM.

$$\begin{array}{l}
yy - 2ay + aa) \quad y^4 \quad * \quad - 3\tfrac{1}{2}aayy + 3a^3y - \tfrac{1}{2}a^4 \quad (yy + 2ay - \tfrac{1}{2}aa. \\
\quad y^4 - 2ay^3 + \quad aayy \\
\quad 0 + 2ay^3 - 4\tfrac{1}{2}aayy \\
\quad\quad + 2ay^3 - 4\ aayy + 2a^3y \\
\quad\quad\quad 0 - \tfrac{1}{2}aayy + a^3y \\
\quad\quad\quad\quad - \tfrac{1}{2}aayy + a^3y - \tfrac{1}{2}a^4 \\
\quad\quad\quad\quad\quad 0 \quad\quad 0 \quad 0
\end{array}$$

$$\begin{array}{l}
aa + ab\sqrt{2} + bb) \quad a^4 \quad * \quad * \quad * \quad + b^4 \quad (aa - ab\sqrt{2} + bb \\
\quad a^4 + a^3b\sqrt{2} + \quad aabb \\
\quad\quad - a^3b\sqrt{2} - \quad aabb \\
\quad\quad - a^3b\sqrt{2} - 2aabb - ab^3\sqrt{2} \\
\quad\quad\quad + \quad aabb + ab^3\sqrt{2} \\
\quad\quad\quad + \quad aabb + ab^3\sqrt{2} + b^4 \\
\quad\quad\quad\quad 0 \quad\quad 0 \quad\quad 0
\end{array}$$

Aliqui Diviſionem incipiunt ab ultimis terminis; ſed eodem recidit, ſi, inverſo terminorum ordine, incipiatur à prioribus. Sunt & aliæ methodi dividendi, ſed facillimam & commodiſſimam nôſſe ſufficit.

CAPUT SEXTUM.

DE EXTRACTIONE RADICUM.

CUM numeri alicujus *radix quadratica* extrahi debet, *is in locis alternis, incipiendo ab unitate, punctis notandus eſt*; *Dein figura*

[1] Neſcio an vulgò notum ſit, quod tament eſte Theone refero, radicis quadraticæ inquiſitionem eodem prorſus modo Veteres inſtituiſſe. Petenda eſt operationis ratio à Theoremate formationis quadrati à radice binominali: $\overline{A + B}|$quad. $=$ A quad. $+$ 2AB $+$ B quad. Conſimilem radicis cubicæ extractionem à Theoremate formationis cubi derivatam Arithmetici paſſim tradunt. Nam eâdem quidem viâ Nicolaus Tartaglia, et Tartagliæ diſcipulus Raphael Bombelli extractionum opus in gradibus etiam elatiſſimis abſolvebant, operationem utique pro radice quâque peculiarem

figura in Quoto, ſeu Radice, ſcribenda, cujus quadratum figuræ vel figuris, ante primum punctum, aut æquale ſit, aut proximè minus. Et ablato illo quadrato, cæteræ radicis figuræ ſigillatim invenientur, dividendo reſiduum per duplum radicis eatenus extractæ, & ſingulis vicibus auferendo è reſiduo illo factum à figurâ noviſſimè prodeunte & decuplo prædicti Diviſoris figurâ illâ aucti [l]. EXTRACTIO RADICUM.

Sic ad extrahendam radicem ex 99856, imprimis nota cum punctis ad hunc modum 9·98·56. Dein quære numerum, cujus quadratum æquatur primæ figuræ 9, nempe 3; ſcribeque in Quoto. Et de 9 ablato quadrato 3 × 3 ſeu 9, reſtabit 0; cui adnecte figuras ante proximum punctum, nempe 98, pro ſequente opere. Tum neglectâ ultimâ figurâ 8, dic quoties duplum 3, ſeu 6, continetur in priori 9? Reſp. 1. Quare ſcripto 1 in Quoto, aufer factum 1 × 61, ſeu 61, de 98: reſtabit 37; cui adnecte ultimas figuras 56, & fiet 3756 numerus in quo opus denuo inſtitui debet. Quare & hujus ultimâ figurâ, 6, neglectâ, dic quoties duplum 31, ſeu 62, continetur in 375 (id quod ex initialibus figuris 6 & 37 conjici poteſt, animadvertendo quoties 6 continetur in 37?) Reſp. 6. Et ſcripto 6 in Quoto, aufer factum 6 × 626, ſeu 3756, & reſtabit nihil. Unde conſtat opus peractum eſſe; prodeunte Radice 316.

```
9·98·56(316
9
———
098
 61
————
 3756
 3756
————
  0
```

Atque ita ſi radicem ex 22178791 extrahere oportet, imprimis factâ punctatione, quære numerum cujus quadratum (ſiquidem id nequeat æquari) ſit proximè minus figuris 22 antecedentibus primum punctum, & invenies eſſe 4. Nam 5 × 5, ſive 25, major eſt quam 22, & 4 × 4, ſive 16, minor. Quare 4 erit pri-

liarem ordinando ſecundum theorema, quod vocant, ſyntheticum poteſtatis homologæ. Atque horum veſtigiis inſiſtens Franciſcus Vieta poteſtatum utcunque affectarum reſolutiones ſimilibus operationibus expediebat. Praxin Tartaglianam miro quodam acumine inventam, ordine luculento eximiâ præceptorum perſpicuitate et elegantiâ à Vietâ traditam, Harrioto et Oughtredo ſummopere excultam, licet ob immenſum in gradibus ultra-cubicis computationum laborem, evulgatis Newtoni inventis, brevi exoleverit, nihilominus ſedulo velim ediſcat ſiquis in hiſce diſciplinis perfectus eſſe velit. Siquidem in omni ſcientiarum genere, nôſſe quibus gradatim incrementis adoleverint, per ſe ſcitu jucundiſſimum eſt, et ad rerum ipſarum intelligentiam plurimum confert.

22·17·87·91(4709,43637 &c.
16
617
609
88791
84681
4110.00
3767 36
3426400
2825649
60079100
56513196
356190400
282566169
73624231

ma figura radicis. Et hâc itaque in Quoto ſcriptâ, de 22 aufer quadratum 4 × 4, ſeu 16; reſiduoque 6 adjunge deſuper proximas figuras 17, & habebitur 617; cujus diviſione per duplum 4 elicienda eſt ſecunda figura radicis. Nempe, neglectâ ultimâ figurâ 7, dic quoties 8 continetur in 61? Reſp. 7. Quare ſcribe 7 in Quoto, & de 617 aufer factum 7 in 87, ſeu 609; & reſtabit 8, cui adjunge proximas duas figuras 87, & habebitur 887; cujus diviſione per duplum 47, ſeu 94, elicienda eſt tertia figura. Utpote dic quoties 94 continetur in 88? Reſp. 0. Quare ſcribe 0 in quoto, adjungeque ultimas duas figuras 91, & habebitur 88791, cujus diviſione per duplum 470, ſeu 940, elicienda eſt ultima figura. Nempe dic quoties 940 continetur in 8879? Reſp. 9. Quare ſcribe 9 in Quoto, & radicem habebis 4709.

Cæterùm cùm factus 9 × 9409, ſeu 84681, ablatus de 88791 relinquat 4110, id indicio eſt numerum 4709 non eſſe radicem numeri 22178791 præciſè, ſed eâ paulo minorem exiſtere. Et in hoc caſu, aliiſque ſimilibus, ſi veram radicem magis appropinquare placeat, proſequenda eſt operatio in decimalibus numeris, adnectendo ad reſiduum circulos duos in ſingulis operationibus. Sic reſiduum 4110, adnexis circulis, evadit 411000; cujus diviſione per duplum 4709, ſeu 9418, elicietur figura prima decimalis, nimirum 4. Dein ſcripto 4 in Quoto, aufer 4 × 94184, ſeu 376736, de 411000, & reſtabit 34264. Atque ita adnexis iterum duobus circulis, opus pro lubitu continuari poteſt, prodeunte tandem radice 4709,43637, *&c.*

Ubi

Ubi vero radix ad medietatem, aut ultra, extracta eſt, cæteræ figuræ per diviſionem ſolam obtineri poſſunt. Ut in hoc exemplo, ſi radicem ad uſque novem figuras extrahere animus eſſet, poſtquam quinque priores 4709,4 extractæ ſunt, quatuor poſteriores 3637 elici poſſent dividendo reſiduum 34264 per duplum 4709,4.

Extractio Radicum.

Et ad hunc modum ſi radix ex 32976 ad uſque quinque figuras extrahi debet; poſtquam figuræ punctis notantur, ſcribe 1 in Quoto, utpote cujus quadratum 1 × 1, ſeu 1, maximum eſt quod in 3, figurâ primum punctum antecedente, continetur. Ac de 3 ablato quadrato illo 1, reſtabit 2. Dein huic, 2, annexis proximis figuris, 29, quære quoties duplum 1, ſeu 2, continetur in 22, & invenies quidem pluſquam 10: ſed nunquam licet diviſorem vel decies ſumere, imo neque novies in hoc caſu, quia factus 9 × 29, ſive 261, major eſt quàm 229, unde deberet auferri. Quare pone tantùm 8. Et perinde ſcripto 8 in Quoto, & ablato 8 × 28, ſive 224, reſtabit 5. Huic inſuper annexis figuris 76, quære quoties duplum 18, ſeu 36, continetur in 57, & invenies 1; adeoque ſcribe 1 in Quoto; ac de 576 ablato 1 × 361, ſeu 361, reſtabit 215. Denique, ad cæteras figuras eliciendas, divide hunc 215 per duplum 181, ſeu 362, & exibunt figuræ 59; quibus etiam ſcriptis in Quoto, habebitur Radix 181,59.

```
3·29·76(181,59
1
———
229
224
———
   576
   361
   ———
362)215(59
```

Eâdem methodo radices etiam è decimalibus numeris extrahuntur. Sic ex 329,76 radix eſt 18,159. Et ex 3,2976 radix eſt 1,8159. Et ex 0,032976 radix eſt 0,18159. Et ſic præterea. Sed ex 3297,6 radix eſt 57,4247. Et ex 32,976 radix eſt 5,74247. Atque ita ex 9.9856 radix eſt 3,16. Sed ex 0,99856 radix eſt 0,999279, *&c.* Quemadmodum è ſubjectis Diagrammis conſtare poteſt.

CAPUT SEXTUM.

32·97;6(57,4247, &c.
25
797
749
4860
4576
1148) 284(247

0;99·85·6(0,999279, &c.
81
1885
1701
18460
17901
1998) 559 (279

Extractionem radicis cubicæ, & aliarum omnium, regulâ generali comprehendam, praxi potiùs intellectu facili quàm expeditæ conſulens, ne moram in eâ quod rarò uſu veniet, diſcentibus inferam. *Nimirum* tertia *quæque figura, incipiendo ab unitate, primò punctis notanda eſt, ſi radix ſit* cubica ; *aut unaquæque* quinta *ſi ſit* qûadrato-cubica, *&c. Dein figura in Quoto ſcribenda eſt, cujus maxima poteſtas (hoc eſt* cubica *ſi radix ſit* cubica, *aut* quadrato-cubica *ſi radix ſit* quadrato-cubica, *&c.) aut æquetur figuræ vel figuris ante primum punctum, aut proximè minor ſit. Et ablatâ illâ poteſtate, figura proxima elicietur, dividendo reſiduum, proximâ numeri reſolvendi figurâ auctum, per poteſtatem Quoti pene-maximam ductam in indicem maximæ poteſtatis, hoc eſt, per* triplum Quadratum *Quoti ſi radix ſit* cubica, *aut per* quintuplum quadrato-quadratum *ſi radix ſit* quadrato-cubica, *&c. Rurſuſque à numero reſolvendo ablatâ maximâ Quoti poteſtate, figura tertia invenietur, dividendo reſiduum illud, proximâ numeri reſolvendi figurâ auctum, per poteſtatem Quoti pene-maximam ductam in indicem maximæ poteſtatis.* Et ſic in infinitum.

Sic ad extrahendam radicem cubicam ex 13312053, numerus ille primò punctis ad hunc modum 13·312·053 notandus eſt. Deinde in Quoto ſcribenda eſt illa figura 2, cujus cubus 8, ſiqui-

Extractio Radicum.

dem æquari nequeat, proximè minor ſit figuris 13 antecedentibus primum punctum. Et ablato illo cubo reſtabit 5; quod proximâ numeri reſolvendi figurâ 3 auctum, & per triplum quadratum quoti 2 diviſum (quærendo nempe quoties 3×4, ſeu 12, continetur in 53) dat 4 pro ſecundâ figurâ Quoti.

```
            13·312·053 (237
aufer cub.   8
   12) reſtat 53 (4. aut 3.
    aufer c. 12167
 1587) reſtat 11450(7.
    aufer c. 13312053
             reſtat 0
```

Sed cùm Quoti, 24, prodiret cubus 13824 major quàm qui auferri poſſet de figuris 13312 antecedentibus ſecundum punctum, ſcribi debet tantùm 3 in Quoto. Tum Quotus 23, in chartâ aliquâ ſeorſim per 23 multiplicatus, dat quadratum 529; quod iterum per 23 multiplicatum dat cubum 12167; & hic de 13312 ablatus relinquit 1145; quod proximâ reſolvendi numeri figurâ 0 auctum, & per triplum quadratum Quoti 23 diviſum (quærendo nempe quoties 3×529, ſeu 1587, continetur in 11450) dat 7 pro tertiâ figurâ Quoti. Tum Quotus 237 per 237 multiplicatus dat quadratum 56169; quod iterum per 237 multiplicatum dat cubum 13312053; & hic de reſolvendo numero ablatus relinquit nihil. Unde patet radicem quæſitam eſſe 237.

Atque ita ad extrahendam radicem quadrato-cubicam ex 364·30820, punctum ponitur ad quintam figuram; & figura 3, cujusquadrato-cubus, 243, proximè minor eſt figuris 364 antecedentibus punctum iſtud, ſcribitur in Quoto. Dein quadrato-cubo 243 de 364 ablato, reſtat 121; quod proximâ reſolvendi numeri figurâ 3 auctum, & per quinquies quadrato-quadratum Quoti diviſum (quærendo nempe quoties 5×81, ſeu 405, continetur in 1213) dat 2 pro ſecundâ figurâ. Quotus ille, 32, in ſe ter ductus efficit quadrato-quadratum 1048576, & hoc iterum in 32 ductum efficit quadrato-cubum 33554432; qui à numero reſolvendo ablatus relinquit 2876388. Itaque 32

```
            364·30820 (32,5
            243
      405)  1213 (2
            33554432
  5242880)  2876388,0 (5
```


eſt

CAPUT SEXTUM.

eſt integra pars radicis, ſed non juſta radix; & proinde ſi opus in decimalibus numeris proſequi animus eſt, reſiduum circulo auctum dividi debet per quinquies prædictum quadrato-quadratum Quoti, quærendo quoties 5×1048576 ſeu 5242880 continetur in 2876388,0; & prodibit tertia figura, ſive prima decimalis, 5. Atque ita auferendo quadrato-cubum Quoti 32,5 de numero reſolvendo, ac dividendo reſiduum per quinquies quadrato-quadratum ejus, erui poteſt quarta figura. Et ſic in infinitum.

Cum radix quadrato-quadratica extrahenda eſt, oportet bis extrahere radicem quadraticam, eo quod $\sqrt{}^{4}$ *valeat* $\sqrt{}^{2 \times 2}$. *Et cum radix cubo-cubica extrahenda eſt, oportet extrahere radicem cubicam, & ejus radicis radicem quadraticam, eo quod* $\sqrt{}^{6}$ *valeat* $\sqrt{}^{2 \times 3}$: *unde aliqui radices haſce non cubo-cubicas ſed quadrato-cubicas dixêre.* Et idem in aliis radicibus, quarum indices non ſunt numeri primi, obſervandum eſt.

E ſimplicibus quantitatibus Algebraicis extractio radicum ex ipsâ Notatione patet. Quemadmodum quòd $\sqrt{aa}$ ſit a, & quòd $\sqrt{aacc}$ ſit ac, & quòd $\sqrt{9aacc}$ ſit $3ac$, & quòd $\sqrt{49a^4xx}$ ſit $7aax$. Atque ita quòd $\sqrt{\frac{a^4}{cc}}$, ſeu $\frac{\sqrt{a^4}}{\sqrt{cc}}$, ſit $\frac{aa}{c}$; & quòd $\sqrt{\frac{a^4bb}{cc}}$ ſit $\frac{aab}{c}$; & quòd $\sqrt{\frac{9aazz}{25bb}}$ ſit $\frac{3az}{5b}$; & quòd $\sqrt{\frac{4}{9}}$ ſit $\frac{2}{3}$. Et quòd $\sqrt{}^{3}\frac{8b^6}{27a^3}$ ſit $\frac{2bb}{3a}$. Et quòd $\sqrt{}^{4}aabb$ ſit $\sqrt{ab}$. Quinetiam quòd $b\sqrt{aacc}$, ſeu b in $\sqrt{aacc}$, valeat b in ac, ſive abc. Et quòd $3c\sqrt{\frac{9aazz}{25bb}}$ valeat $3c \times \frac{3az}{5b}$ ſive $\frac{9acz}{5b}$. Et quòd $\frac{a+3x}{c}\sqrt{\frac{4bbx^4}{81aa}}$ valeat $\frac{a+3x}{c} \times \frac{2bxx}{9a}$ ſive $\frac{2abxx+6bx^3}{9ac}$.

Hæc inquam patent, ſiquidem propoſitas quantitates è radicibus in ſe ductis produci (ut aa ex a in a, $aacc$ ex ac in ac, $9aacc$ ex $3ac$ in $3ac$, &c.) primâ fronte conſtare poteſt. Ubi verò quantitates pluribus terminis conſtant, opus perinde ac in numeris abſolvitur. Sic ad extrahendam radicem quadraticam

EXTRACTIO RADICUM.

ex $aa+2ab+bb$, imprimis radicem primi termini aa, nempe a, ſcribe in Quoto. Et ablato ejus quadrato $a \times a$, reſtabit $2ab+bb$ pro eliciendâ reliquâ parte radicis. Dic itaque quoties duplum quoti, ſeu $2a$, continetur in primo reſidui termino $2ab$? Reſp. b. Adeoque ſcribe b in Quoto, & ablato facto b in $\overline{2a+b}$, ſeu $2ab+bb$, reſtabit nihil. Quod indicat opus peractum eſſe, prodeunte radice $a+b$.

$$\begin{array}{l} aa+2ab+bb(a+b \\ \underline{aa} \\ 0 \quad \overline{2ab+bb} \\ \overline{0 \quad\; 0} \end{array}$$

Et ſic ad extrahendam radicem ex $a^4+6a^3b+5aabb-12ab^3+4b^4$, imprimis pone in Quoto radicem primi termini a^4 nempe aa, & ablato ejus quadrato $aa \times aa$, ſeu a^4, reſtabit $6a^3b+5aabb-12ab^3+4b^4$ pro reliquâ radice eliciendâ. Dic itaque quoties $2aa$ continetur in $6a^3b$? Reſp. $3ab$. Quare ſcribe $3ab$ in Quoto: & ablato facto $3ab$ in $\overline{2aa+3ab}$, ſeu $6a^3b+9aabb$, reſtabit etiamnum $-4aabb-12ab^3+4b^4$ pro opere proſequendo.

$$\begin{array}{l} a^4+6a^3b+5aabb-12ab^3+4b^4(aa+3ab-2bb \\ \underline{a^4} \\ 0 \\ \quad \underline{6a^3b+9aabb} \\ \quad\quad 0 \; -4aabb \\ \quad\quad\quad \underline{-4aabb-12ab^3+4b^4} \\ \quad\quad\quad\quad 0 \quad\quad 0 \quad\quad 0 \end{array}$$

Adeoque dic iterum quoties duplum Quoti, nempe $2aa+6ab$ continetur in $-4aabb-12ab^3$; ſive, quod perinde eſt, dic quoties duplum primi termini Quoti, ſeu $2aa$, continetur in primo reſidui termino $-4aabb$? Reſp. $-2bb$. Et proinde ſcripto $-2bb$ in Quoto, & ablato facto $-2bb$ in $2aa+6ab-2bb$, ſeu $-4aabb-12ab^3+4b^4$, reſtabit nihil. Unde conſtat radicem eſſe $aa+3ab-2bb$.

Atque ita quantitatis $xx-ax+\frac{1}{4}aa$ radix eſt $x-\frac{1}{2}a$; & quantitatis y^4+4y^3-8y+4 radix $yy+2y-2$; & quantitatis $16a^4-24aaxx+9x^4+12bbxx-16aabb+4b^4$ radix $3xx-4aa+2bb$, ut è ſubjectis diagrammis conſtare poteſt.

$xx-$

CAPUT SEXTUM.

$$\begin{array}{l} xx - ax + \frac{1}{4}aa(x - \frac{1}{2}a. \\ \underline{xx} \\ 0 \quad -ax + \frac{1}{4}aa \\ \quad \underline{\quad\quad\quad\quad} \\ \quad 0 \quad 0 \end{array}$$

$$\begin{array}{l} 9x^4 \begin{matrix} -24aa \\ +12bb \end{matrix} xx \begin{matrix} +16a^4 \\ -16aabb \\ +4b^4 \end{matrix} \left(3xx \begin{matrix} -4aa \\ +2bb \end{matrix}\right. \\ \underline{9x^4} \\ 0 \\ \quad \begin{matrix} -24aa \\ +12bb \end{matrix} xx \begin{matrix} +16a^4 \\ -16aabb \\ +4b^4 \end{matrix} \\ \quad\quad 0 \quad\quad\quad 0 \end{array}$$

$$\begin{array}{l} y^4 + 4y^3 * - 8y + 4(yy + 2y - 2 \\ \underline{y^4} \\ 0 \\ \quad \underline{4y^3 + 4yy} \\ \quad 0 - 4yy \\ \quad\quad \underline{-4yy - 8y + 4} \\ \quad\quad\quad 0 \quad 0 \quad 0 \end{array}$$

Si radicem cubicam ex $a^3 + 3aab + 3abb + b^3$ oportet extrahere, operatio eſt hujuſmodi. Extrahe radicem cubicam primi

$$\begin{array}{l} a^3 + 3aab + 3abb + b^3 (a + b. \\ \underline{a^3} \\ 3aa)0 + 3aab(b \\ \overline{a^3 + 3aab + 3abb + b^3} \\ 0 \quad 0 \quad 0 \quad 0 \end{array}$$

termini a^3 nempe a, & pone in Quoto. Tum ablato ejus cubo a^3, dic quoties triplum quadratum ejus, ſeu $3aa$, continetur in proximo reſidui termino $3aab$; & prodit b. Quare ſcribe etiam b in Quoto, & cubo Quoti $a + b$ ablato reſtabit nihil. Radix itaque eſt $a + b$.

Eodem modo radix cubica, ſi extrahatur ex $z^6 + 6z^5 - 40z^3 + 96z - 64$, prodit $zz + 2z - 4$. Atque ita in altioribus radicibus.

CAPUT SEPTIMUM.

De REDUCTIONE FRACTIONUM ET RADICALIUM.

PRæcedentibus operationibus inſervit reductio fractarum & radicalium quantitatum, idque vel ad minimos terminos vel ad eandem denominationem.

SECT. I.

De REDUCTIONE FRACTIONUM *ad minimos terminos.*

Deductio Fractionum.

FRactiones ad minimos terminos reducuntur dividendo numeratores ac denominatores per maximum communem diviſorem. Sic fractio $\frac{aac}{bc}$ reducitur ad ſimpliciorem, $\frac{aa}{b}$, dividendo utrumque aac & bc per c; & $\frac{203}{667}$ reducitur ad ſimpliciorem, $\frac{7}{23}$, dividendo utrumque 203 & 667 per 29; & $\frac{203aac}{667bc}$ reducitur ad $\frac{7aa}{23b}$ dividendo per $29c$. Atque ita $\frac{6a^3-9acc}{6aa+3aa}$ evadit $\frac{2aa-3cc}{2a+c}$ dividendo per $3a$. Et $\frac{a^3-aab+abb-b^3}{aa-ab}$ evadit $\frac{aa+bb}{a}$ dividendo per $a-b$.

Et hâc Methodo termini poſt Multiplicationem vel Diviſionem plerumque abbreviari poſſunt. Quemadmodum ſi multiplicare oportet $\frac{2ab^3}{3ccd}$ per $\frac{9acc}{bdd}$, vel id dividere per $\frac{bdd}{9acc}$, prodibit $\frac{18aab^3cc}{3bccd^3}$, & per reductionem $\frac{6aabb}{d^3}$. Sed in hujuſmodi caſibus præſtat ante operationem concinnare terminos, dividendo per maximum communem diviſorem, quos poſtea dividere oporteret. Sic in allato exemplo, ſi dividam $2ab^3$ & bdd per communem diviſorem b, & $3ccd$ ac $9acc$ per communem diviſorem $3cc$; emerget fractio $\frac{2abb}{d}$ multiplicanda per $\frac{3a}{dd}$, vel dividenda per $\frac{dd}{3a}$, prodeunte tandem $\frac{6aabb}{d^3}$ ut ſupra. Atque ita $\frac{aa}{c}$ in $\frac{c}{b}$ evadit $\frac{aa}{1}$ in $\frac{1}{b}$, ſeu $\frac{aa}{b}$. Et $\frac{aa}{c}$ diviſ. per $\frac{b}{c}$ evadit aa diviſ. per b, ſeu $\frac{aa}{b}$. Et $\frac{a^3-axx}{xx}$ in $\frac{cx}{aa+ax}$ evadit $\frac{a-x}{x}$ in $\frac{c}{1}$, ſeu $\frac{ac}{x}-c$. Et 28 diviſ. per $\frac{7}{3}$ evadit 4 diviſ. per $\frac{1}{3}$, ſeu 12.

SECT.

NEWTONI

SECT. II.

De inventione Diviſorum.

HUC ſpectat inventio diviſorum, per quos quantitas aliqua dividi poſſit. *Si quantitas ſimplex eſt, divide eam per minimum ejus diviſorem, & quotum per minimum diviſorem ejus, donec quotus reſtet indiviſibilis, & omnes quantitatis diviſores primos habebis. Dein horum diviſorum ſingulos binos, ternos, quaternos,* &c. *duc in ſe, & habebis etiam omnes diviſores compoſitos.* Ut ſi numeri 60 diviſores omnes deſiderentur, divide eum per 2, & quotum 30 per 2, & quotum 15 per 3, & reſtabit quotus indiviſibilis 5. Ergo diviſores primi ſunt 1,2,2,3,5 : ex binis compoſiti 4,6,10,15 : ex ternis 12,20,30; ex omnibus 60. Rurſus ſi quantitatis $21abb$ diviſores omnes deſiderentur, divide eam per 3, & quotum $7abb$ per 7, & quotum abb per a, & quotum bb per b, & reſtabit quotus primus b. Ergo diviſores primi ſunt $1,3,7,a,b,b$; ex binis compoſiti 21 $3a$, $3b$, $7a$, $7b$, ab, bb; ex ternis $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; ex quaternis $21ab$, $21bb$, $3abb$, $7abb$; ex quinis $21abb$. Eodem modo ipſius $2abb-6aac$ diviſores omnes ſunt $1, 2, a$, $bb-3ac$, $2a$, $2bb-6ac$, $abb-3aac$, $2abb-6aac$.

Si quantitas poſtquam diviſa eſt per omnes ſimplices diviſores manet compoſita, & ſuſpicio eſt eam compoſitum aliquem diviſorem habere, diſpone eam ſecundum dimenſiones literæ alicujus quæ in eâ eſt; & pro literâ illâ ſubſtitue ſigillatim tres vel plures terminos hujus progreſſionis Arithmeticæ, 3, 2, 1, 0, − 1, − 2, *ac terminos totidem reſultantes, unà cum omnibus eorum diviſoribus, ſtatue è regione correſpondentium terminorum progreſſionis, poſitis diviſorum ſignis tàm affirmativis quàm negativis. Dein è regione etiam ſtatue progreſſiones arithmeticas, quæ per omnium numerorum diviſores percurrunt, pergentes à majoribus terminis ad minores eodem ordine quo termini progreſſionis* 3, 2, 1, 0, − 1, − 2 *pergunt, & quarum termini differunt vel*

unitate,

unitate, vel numero aliquo, qui dividit altiſſimum terminum propo- INVENTIO DIVISORUM.
ſitæ quantitatis. Siqua occurrit ejuſmodi progreſſio, iſte terminus ejus qui ſtat è regione termini o *progreſſionis primæ, diviſus per differentiam terminorum, & cum ſigno ſuo annexus literæ præfatæ, componet quantitatem per quam diviſio tentanda eſt.*

Ut ſi quantitas ſit $x^3 - xx - 10x + 6$, pro x ſubſtituendo ſigillatim terminos progreſſionis 1, 0, − 1, orientur numeri − 4, 6, + 14 quos, cum omnibus eorum diviſoribus, colloco è regione terminorum progreſſionis 1, 0, − 1 hoc modo. Dein quoniam altiſſimus terminus x^3 per nullum numerum præter unitatem diviſibilis eſt; quæro in diviſoribus progreſſionem, cujus termini differunt unitate, &c, à ſuperioribus ad inferiora pergendo, decreſcunt, perinde ac termini progreſſionis lateralis 1, 0, − 1. Et hujuſmodi progreſſionem unicam tantùm invenio, nempe 4, 3, 2, cujus itaque terminum + 3 ſeligo, qui ſtat è regione termini o progreſſionis primæ 1, 0, − 1, tentoque diviſionem per $x + 3$: et res ſuccedit, prodeunte $xx - 4x + 2$.

1	4	1.2.4.	+ 4.
0	6	1.2.3.6	+ 3.
− 1	14	1.2.7.14	+ 2.

Rurſus ſi quantitas ſit $6y^4 - y^3 - 21yy + 3y + 20$, pro y ſubſtituo ſigillatim 2, 1, 0, − 1, − 2; & numeros reſultantes 30, 7, 20, 3, 34, cum omnibus eorum diviſoribus, è regione colloco, ut ſequitur: et in diviſoribus hanc ſolam eſſe animadverto decreſcentem progreſſionem arithmeticam, + 10, + 7, + 4, + 1, − 2. Hujus terminorum differentia, 3, dividit altiſſimum quantitatis terminum, $6y^4$. Quare terminum + 4, qui ſtat è regione termini o, diviſum per differentiam terminorum, 3, adjungo literæ y, tentoque diviſionem per $y + \frac{4}{3}$, vel, quod perinde eſt, per $3y + 4$; & res ſuccedit prodeunte $2y^3 - 3yy - 3y + 5$.

2	30	1.2.3.5.6.10.15.30	+ 10.
1	7	1.7	+ 7.
0	20	1.2.4.5.10.20	+ 4.
− 1	3	1.3	+ 1.
− 2	34	1.2.17.34	− 2.

Atque ita ſi quantitas ſit $24a^5 - 50a^4 + 49a^3 - 140aa + 64a + 30$; operatio erit ut ſequitur. Tres occurrunt hîc progreſſiones, quarum termini, − 1. − 5. − 5, diviſi per differentias terminorum 2, 4, 6, dant tres diviſores tentandos,

2	42	1.2.3.6.7.14.21.42.	+ 3. + 3. + 7.
1	23	1.23.	+ 1. − 1. + 1.
0	30	1.2.3.5.6.10.15.30.	− 1. − 5. − 5.
− 1	297	1.3.9.11.27.33.99.297.	− 3. − 9. − 11.

 $a - \frac{1}{2}$,

 $a-\frac{1}{2}$, $a-\frac{5}{4}$ & $a-\frac{5}{6}$. Et diviſio per ultimum diviſorem, $a-\frac{5}{6}$, ſeu $6a-5$, ſuccedit, prodeunte $4a^4-5a^3+4aa-20a-6$.

Si nullus occurrit hâc methodo diviſor, vel nullus qui dividit propoſitam quantitatem, concludendum erit quantitatem illam non admittere diviſorem unius dimenſionis. Poteſt tamen fortaſſe, ſi plurium [m] ſit quàm trium dimenſionum, diviſorem admittere duarum. Et ſi ita, diviſor ille inveſtigabitur hâc methodo. *In quantitate illâ pro literâ ſubſtitue, ut antè, quatuor vel plures terminos progreſſionis hujus,* 3, 2, 1, 0, − 1, − 2, − 3. *Diviſores omnes numerorum reſultantium ſigillatim adde & ſubduc quadratis correſpondentium terminorum progreſſionis illius ductis in diviſorem aliquem numeralem altiſſimi termini quantitatis propoſitæ, & ſummas differentiaſque è regione progreſſionis colloca. Dein progreſſiones omnes collaterales nota, quæ per iſtas ſummas differentiaſque percurrunt. Sit* ± C *terminus iſtiuſmodi progreſſionis, qui ſtat è regione termini* 0 *progreſſionis primæ;* ± B *differentia quæ oritur ſubducendo* ± C *de termino proximè ſuperiori, qui ſtat è regione termini* 1 *progreſſionis primæ;* A *prædictus termini altiſſimi diviſor numeralis, &* l *litera quæ in quantitate propoſitâ eſt; & erit* A l l ± B l ± C *diviſor tentandus.*

Ut ſi quantitas propoſita ſit $x^4-x^3-5xx+12x-6$, pro x ſcribo ſucceſſivè 3, 2, 1, 0, − 1, − 2; & prodeuntes numeros, 39, 6, 1, − 6, − 21, − 26, unà cum eorum diviſoribus, è regione diſpono; addoque & ſubduco diviſores terminis progreſſionis illius quadratis ductiſque in diviſorem numeralem termini x^4 qui unitas eſt, *viz.* terminis 9,4,1,0,1,4; & ſummas differentiaſque è latere pariter diſpono. Dein progreſſiones, quæ in iiſdem obveniunt, è latere etiam ſcribo, ut ſequitur. Harum progreſſionum terminos, 2 & − 3, qui ſtant è regione termini 0 progreſſionis illius quæ in columnâ primâ eſt, uſurpo ſucceſſive pro ± *C*. Differentias, quæ oriuntur ſubducendo hos terminos de terminis ſuperioribus, 0 & 0, nempe − 2 & + 3, uſurpo reſpectivè

3	39	1.3.13.39.	9	− 30. − 4.6.8.10.12.22.48.	− 4.	6.
2	6	1.2. 3. 6.	4	− 2.1.2.3.5. 6. 7. 10.	− 2.	3.
1	1	1.	1	0. 2.	0.	0.
0	6	1.2. 3. 6.	0	− 6. − 3. − 2. − 1.1.2.3.6.	2.	− 3.
− 1	21	1.3. 7.21.	1	− 20. − 6. − 2.0.2.4. 8.22.	4.	− 6.
− 2	26	1.2.13.26.	4	− 22. − 9.2.3.5.6.17.30.	6.	− 9.

[m] Rectè quidem *plurium*. Nam ſi quantitas trium dimenſionum diviſorem aliquem duarum dimenſionum admitteret, omninò unius admitteret. Quotus enim, qui oritur ex diviſione quantitatis

tivè pro ± *B*. Unitatem item pro *A*; & *x* pro *l*. Et ſic pro *All* ± *Bl* ± *C* habeo diviſores duos tentandos, $xx + 2x - 2$, & $xx - 3x + 3$; per quorum utrumque res ſuccedit. INVENTIO DIVISORUM.

Rurſus ſi proponatur quantitas $3y^5 - 6y^4 + y^3 - 8yy - 14y + 14$, operatio erit ut ſequitur. Primò rem tento addendo & ſubducendo diviſores quadratis terminorum progreſſionis 2, 1, 0, 1 uſurpato 1 pro *A*; ſed res non ſuccedit. Quare pro *A* uſurpo 3, alterum nempe termini altiſſimi $3y^5$ diviſorem numeralem, &

3	170		27		−7. 17
2	38	1.2.19.38.	12	−26.−7.10.11.13.14.31.50.	−7. 11
1	10	1.2. 5.10.	3	− 7.−2. 1. 2. 4. 5. 8.13.	−7. 5
0	14	1.2. 7.14.	0	−14.−7.−2.−1.1.2.7.14.	−7.− 1
−1	10	1.2. 5.10.	3	− 7.−2. 1. 2. 4. 5. 8.13.	−7.− 7
−2	190		12		−7.−13

quadratis iſtis multiplicatis per 3, hoc eſt numeris 12, 3, 0, 3, addo ſubducoque diviſores; & progreſſiones in terminis reſultantibus haſce duas invenio, − 7, − 7, −7, −7; & 11, 5, − 1, − 7. Expeditionis gratiâ neglexeram diſviſores extimorum numerorum, 170 & 190. Quare, continuatis progreſſionibus, ſumo proximos earum hinc inde terminos, *viz.* −7 & 17 ſuperiùs, & −7, & −13 inferiùs; ac tento, ſi ſubductis his de numeris 27 ac 12, qui ſtant è regione in quartâ columnâ, differentiæ dividunt iſtos 170 & 190, qui ſtant è regione in columnâ ſecundâ. Et quidem differentia inter 27 & − 7, id eſt 34, dividit 170; & differentia 12 & − 7, id eſt 19, dividit 190. Item differentia inter 27 & 17, id eſt 10, dividit 170; ſed differentia inter 12 & − 13, id eſt 25, non dividit 190. Quare poſteriorem progreſſionem rejicio. Juxta priorem ± *C* eſt − 7, & ± *B* nihil; terminis progreſſionis nullam habentibus differentiam. Quare diviſor tentandus *All* ± *Bl* ± *C*, erit $3yy + 7$. Et diviſio ſuccedit, prodeunte $y^3 - 2yy - 2y + 2$.

Si nullus inveniri poteſt hoc pacto diviſor qui ſuccedit, concludendum eſt quantitatem propoſitam non admittere diviſorem duarum dimenſionum. Poſſet eadem methodus extendi ad inventionem diviſorum dimenſionum plurium; quærendo in prædictis ſummis differentiiſque progreſſiones, non arithmeticas qui-

tatis trium dimenſionum per quantitatem duarum, unius eſt dimenſionis, et quantitatem diviſam dividit.

CAPUT SEPTIMUM. dem, ſed alias quaſdem; quarum terminorum differentiæ primæ, ſecundæ, tertiæ, &c. ſunt in arithmeticâ progreſſione: at in his Tiro non eſt detinendus.

Ubi in quantitate propoſitâ duæ ſunt literæ, & omnes ejus termini ad dimenſiones æquè altas aſcendunt; pro unâ iſtarum literarum pone unitatem, dein per regulas præcedentes quære diviſorem, ac diviſoris hujus comple deficientes dimenſiones, reſtituendo literam illam pro unitate.

Ut ſi quantitas ſit $6y^4 - cy^3 - 21ccyy + 3c^3y + 20c^4$, ubi termini omnes ſunt quatuor dimenſionum; pro c pono 1: quantitas evadit $6y^4 - y^3 - 21yy + 3y + 20$, cujus diviſor, ut ſuprà, eſt $3y + 4$; & completâ deficiente dimenſione poſterioris termini per dimenſionem c, fit $3y + 4c$ diviſor quæſitus. Ita ſi quantitas ſit $x^4 - bx^3 - 5bbxx + 12b^3x - 6b^4$; poſito 1 pro b, & quantitatis reſultantis, $x^4 - x^3 - 5xx + 12x - 6$, invento diviſore $xx + 2x - 2$, compleo ejus deficientes dimenſiones per dimenſiones b, & ſic habeo diviſorem quæſitum $xx + 2bx - 2bb$.

Ubi in quantitate propoſitâ tres vel plures ſunt literæ, & ejus termini omnes ad eaſdem dimenſiones aſcendunt; poteſt diviſor per præcedentes regulas inveniri; ſed expeditiùs hoc modo: *Quære omnes diviſores terminorum omnium in quibus literarum aliqua non eſt; item terminorum omnium in quibus alia aliqua literarum non eſt; pariter & omnium in quibus tertia litera, quartaque, & quinta, non eſt, ſi tot ſunt literæ. Et ſic percurre omnes literas: et è regione literarum colloca diviſores reſpectivè. Dein vide, ſi in ſerie aliquâ diviſorum per omnes literas pergente, partes omnes, unicam tantùm literam involventes, tot vicibus reperiantur, quot ſunt literæ, unâ demptâ, in quantitate propoſitâ: et partes duas literas involventes tot vicibus quot ſunt literæ, demptis duabus, in eâdem quantitate. Si ita eſt, partes iſtæ omnes, ſub ſignis ſuis ſemel ſumptæ, erunt diviſor quæſitus.*

Ut ſi proponatur quantitas $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$; terminorum $8b^3 - 12bbc - 4bcc + 6c^3$, in quibus non eſt x, diviſores unius dimenſionis, per præcedentes regulas inventi, erunt $2b - 3c$, & $4b - 6c$; terminorum, $12x^3 + 9cxx + 8ccx$

$+8ccx+6c^3$, in quibus non eſt b, diviſor unicus $4x+3c$; ac terminorum, $12x^3-14bxx-12bbx+8b^3$, in quibus non eſt c, diviſores $2x-b$, & $4x-2b$. Hos diviſores è regione literarum x, b, c diſpono, ut hîc vides. Cùm tres ſint literæ, & diviſorum partes ſingulæ non niſi ſingulas literas involvant, in ſerie diviſorum debent partes illæ bis reperiri. At diviſorum $4b-6c$, & $2x-b$, partes $4b$, $6c$, $2x$, b non niſi ſemel occurrunt: extra diviſorem illum, cujus ſunt partes, non reperiuntur. Quare diviſores illos negligo. Reſtant tantùm tres diviſores, $2b-3c$, $4x+3c$, & $4x-2b$. Hi in ſerie ſunt per omnes literas x, b, c pergente; & eorum partes ſingulæ, $2b$, $3c$, $4x$, bis reperiuntur in ipſis, ut oportuit; idque cum ſignis iiſdem, ſi modò ſigna diviſoris $2b-3c$ mutentur, & ejus loco ſcribatur $-2b+3c$: nam ſigna diviſoris cujuſvis mutare licet. Sumo itaque horum partes omnes $2b$, $3c$, $4x$ ſemel ſub ſignis ſuis, & aggregatum, $-2b+3c+4x$, diviſor erit quem invenire oportuit. Nam ſi per hunc dividas quantitatem propoſitam, prodibit $3xx-2bx+2cc-4bb$.

x	$2b-3c.\ 4b-6c.$
b	$4x+3c.$
c	$2x-b.\ 4x-2b.$

Rurſus ſi quantitas ſit $12x^5-10ax^4-9bx^4-26aax^3+12abx^3+6bbx^3+24a^3xx-8aabxx-8abbxx-24b^3xx-4a^3bx+6aabbx-12ab^3x+18b^4x+12a^4b+32aab^3-12b^5$; diviſores terminorum, in quibus x non eſt, colloco è regione x; illos terminorum, in quibus a non eſt, è regione a; & illos terminorum quibus b non eſt, è regione b, ut hîc vides. Dein illos omnes qui ſunt unius dimenſionis rejiciendos eſſe ſentio; quia ſimplices b, $2b$, $4b$, x, $2x$, & partes compoſitorum $3x-4a$, $6x-8a$, non niſi ſemel in omnibus diviſoribus reperiantur; tres autem ſunt literæ in quantitate propoſitâ, & partes illæ unicam tantùm involvunt, atque adeo bis reperiri deberent. Similiter diviſores duarum dimenſionum, $aa+3bb$, $2aa+6bb$, $4aa+12bb$, $bb-3aa$, & $4bb-12aa$, rejicio; quia partes eorum, aa, $2aa$, $4aa$, bb & $4bb$, unicam tantùm literam, a vel b, involventes non niſi ſemel reperiuntur. Diviſoris autem $2bb-6aa$, qui ſolus reſtat è regione x, partes $2bb$ & $6aa$, quæ ſimiliter unicam tantùm literam involvunt, iterum reperiuntur; nempe pars $2bb$ in diviſore $4xx-3bx+2bb$, & pars

x	$b,\ 2b,\ 4b,\ aa+3bb,\ 2aa+6bb,\ 4aa+12bb,$ $bb-3aa,\ 2bb-6aa,\ 4bb-12aa.$
a	$4xx-3bx+2bb,\ 12xx-9bx+6bb.$
b	$x,\ 2x,\ 3x-4a,\ 6x-8a,\ 3xx-4ax,\ 6xx-8ax;$ $2xx+ax-3aa,\ 4xx+2ax-6aa.$

$6aa$

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$6aa$ in diviſore $4xx+2ax-6aa$. Quin etiam hi tres diviſores in ſerie ſunt, ſtantes è regione trium literarum x, a, b; & omnes eorum partes, $2bb$, $6aa$, $4xx$, quæ unicam tantùm literam involvunt, bis reperiuntur in ipſis, idque ſub propriis ſignis; partes verò $3bx$, $2ax$, quæ duas literas involvunt, non niſi ſemel occurrunt in ipſis. Quare horum trium diviſorum partes omnes diverſæ $2bb$, $6aa$, $4xx$, $3bx$, $2ax$ ſub ſignis ſuis connexæ, diviſorem deſideratum $2bb-6aa+4xx-3bx+2ax$ conflabunt. Per hunc itaque divido quantitatem propoſitam, & oritur $3x^3-4axx-2aab-6b^3$.

Si quantitatis alicujus termini omnes non ſunt æque alti, complendæ ſunt dimenſiones deficientes per dimenſiones literæ cujuſvis aſſumptæ; dein per præcedentes regulas invento diviſore, litera aſſumpta delenda eſt. Ut ſi quantitas ſit $12x^3-14bxx+9xx-12bbx-6bx+8x+8b^3-12bb-4b+6$; aſſume literam quamvis c, & per dimenſiones ejus comple dimenſiones quantitatis propoſitæ ad hunc modum: $12x^3-14bxx+9cxx-12bbx-6bcx+8ccx+8b^3-12bbc-4bcc+6c^3$. Dein hujus diviſore, $4x-2b+3c$, invento dele c; & habebitur diviſor deſideratus $4x-2b+3$.

Aliquando diviſores faciliùs quàm per has regulas inveniri poſſunt. Ut ſi litera aliqua in quantitate propoſitâ ſit unius tantùm dimenſionis; quærendus erit maximus communis diviſor terminorum, in quibus litera illa reperitur, & reliquorum terminorum, in quibus non reperitur; nam diviſor ille totam dividet. Et ſi nullus eſt ejuſmodi communis diviſor, nullus erit diviſor totius. Exempli gratiâ, ſi proponatur quantitas $x^4-3ax^3-8aaxx+18a^3x+cx^3-acxx-8aacx+6a^3c-8a^4$; quæratur communis diviſor terminorum, $+cx^3-acxx-8aacx+6a^3c$, in quibus c unius eſt tantùm dimenſionis, & terminorum reliquorum, $x^4-3ax^3-8aaxx+18a^3x-8a^4$; ac diviſor ille, nempe $xx+2ax-2aa$, dividet totam quantitatem.

Cæterùm maximus duorum numerorum diviſor communis, ſi primâ fronte non innoteſcit, invenitur perpetuâ ablatione minoris de majori & reliqui de ablato. Nam quæſitus erit diviſor, qui tandem nihil relinquit [n]. Sic ad inveniendum maximum communem diviſorem

[n] Euclid. Elem. Lib. 7. Prop. 2.

5 numerorum

numerorum 203 & 667, aufer ter 203 de 667; & reliquum 58 ter de 203; & reliquum 29 bis de 58; reſtabitque nihil: quod indicat 29 eſſe diviſorem quæſitum. INVENTIO DIVISORUM.

Haud ſecus in ſpeciebus communis diviſor, ubi compoſitus eſt, invenitur, ſubducendo alterutram quantitatem, aut multiplicem ejus, de alterâ: Si modò & quantitates illæ & reſiduum juxta literæ alicujus dimenſiones, ut Diviſione oſtenſum eſt, ordinentur, & quâlibet vice concinnentur, dividendo ipſas per ſuos omnes diviſores, qui aut ſimplices ſunt, aut ſingulos terminos inſtar ſimplicium dividunt. Sic ad inveniendum communem diviſorem Numeratoris ac Denominatoris fractionis hujus, $\frac{x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4}{x^3 - axx - 8aax + 6a^3}$, multiplica Denominatorem per x, ut primus ejus terminus evadat idem cum primo termino numeratoris. Dein aufer, & reſtabit $-2ax^3+12a^3x-8a^4$, quod concinnatum, dividendo per $-2a$, evadit $x^3 - 6aax + 4a^3$. Hoc aufer de Denominatore, & reſtabit $-axx-2aax+2a^3$: quod itidem per $-a$ diviſum fit $xx+2ax-2aa$. Hoc autem per x multiplica (ut ejus primus terminus evadat idem cum primo termino noviſſimi ablati $x^3 - 6aax + 4a^3$, de quo auferendum eſt) & reſtabit $-2axx - 4aax + 4a^3$; quod, per $-2a$ diviſum, fit etiam $xx+2ax-2aa$. Et hoc, cùm idem ſit ac ſuperius reſiduum, proindeque ablatum relinquat nihil, quæſitus erit diviſor per quem fractio propoſita, factâ Numeratoris ac Denominatoris diviſione, reduci poteſt ad ſimpliciorem, nempe ad $\frac{xx - 5ax + 4aa}{x - 3a}$.

Atque ita ſi habeatur fractio, $\frac{6a^5 + 15a^4b - 4a^3cc - 10aabcc}{9a^3b - 27aabc - 6abcc + 18bc^3}$; termini ejus imprimis abbreviandi ſunt, dividendo numeratorem per aa, ac Denominatorem per $3b$. Dein ablato bis $3a^3 - 9aac - 2acc + 6c^3$ de $6a^3 + 15aab - 4acc - 10bcc$, reſtabit $\begin{matrix} 15b \\ +18c \end{matrix} aa \begin{matrix} -10bcc \\ -12c^3 \end{matrix}$. Quod concinnatum, dividendo terminum utrumque per $5b + 6c$, perinde ac ſi $5b+6c$ ſimplex eſſet quantitas, evadit $3aa - 2cc$. Hoc multiplicatum per a aufer de $3a^3-9aac-2acc+6c^3$, & ſecundâ vice reſtabit $-9aac + 6c^3$: quod itidem concinnatum, per applicationem ad $-3c$, evadit etiam $3aa - 2cc$ ut antè. Quare $3aa-2cc$ quæſitus eſt diviſor. Quo invento, divide per eum partes fractionis propoſitæ, & obtinebitur $\frac{2a^3 + 5aab}{3ab - 9bc}$.

Quod

Caput Septimum.

Quòd ſi diviſor communis hoc pacto non inveniatur, certum eſt nullum omninò exiſtere; niſi forſan è terminis prodeat, per quos Numerator ac Denominator fractionis abbreviantur. Ut ſi habeatur fractio $\frac{aadd - ccdd - aacc + c^4}{4aad - 4acd - 2acc + 2c^3}$, ac termini ejus juxta dimenſiones literæ d diſponantur, ita ut Numerator evadat $\begin{matrix} aa \\ -cc \end{matrix} dd \begin{matrix} -aacc \\ +c^4 \end{matrix}$, ac Denominator $\begin{matrix} 4aa \\ -4ac \end{matrix} d \begin{matrix} -2acc \\ +2c^3 \end{matrix}$: hos imprimis oportet abbreviare, dividendo utrumque Numeratoris terminum per $aa-cc$, & utrumque Denominatoris per $2a-2c$, perinde àc ſi $aa-cc$, & $2a-2c$, eſſent ſimplices quantitates. Atque ita vice Numeratoris emerget $dd-cc$, & vice Denominatoris $2ad-cc$, ex quibus ſic præparatis nullus communis diviſor obtineri poteſt. Sed è terminis $aa-cc$ & $2a-2c$ per quos Numerator ac Denominator abbreviati ſunt, prodit ejuſmodi diviſor, nempe $a+c$, cujus ope fractio ad hanc $\frac{add + cdd - acc - c^3}{4ad - 2cc}$ reduci poteſt. Quòd ſi neque termini, $aa-cc$, & $2a-2c$, communem diviſorem habuiſſent, fractio propoſita fuiſſet irreducibilis.

Et hæc generalis eſt methodus inveniendi communes diviſores: *Sed plerumque expeditiùs inveniuntur quærendo omnes alterutrius quantitatis diviſores primos, hoc eſt, qui per alios dividi nequeunt, ac dein tentando ſiqui alteram divident abſque reſiduo.* Sic ad reducendum $\frac{a^3 - aab + abb - b^3}{aa - ab}$ ad minimos terminos, inveniendi ſunt diviſores quantitatis $aa-ab$, nempe a, & $a-b$. Dein tentandum eſt an alteruter, a, vel $a-b$, dividit etiam $a^3 - aab + abb - b^3$ abſque reſiduo.

SECT. III.

De REDUCTIONE FRACTIONUM *ad communem Denominatorem.*

FRactiones ad communem Denominatorem reducuntur multiplicando terminos utriuſque per denominatorem alterius.

Sic habitis $\frac{a}{b}$ & $\frac{c}{d}$, duc terminos unius $\frac{a}{b}$ in d, & viciſſim terminos alterius $\frac{c}{d}$ in b, & evadent $\frac{ad}{bd}$ & $\frac{bc}{bd}$, quarum communis eſt denominator

denominator bd. Atque ita a & $\frac{ab}{c}$, ſive $\frac{a}{1}$ & $\frac{ab}{c}$, evadunt $\frac{ac}{c}$ & $\frac{ab}{c}$. Ubi verò Denominatores communem habent diviſorem, ſufficit multiplicare alternè per Quotientes. Sic fractiones $\frac{a^3}{bc}$ & $\frac{a^3}{bd}$ ad haſce, $\frac{a^3d}{bcd}$ & $\frac{a^3c}{bcd}$, reducuntur, multiplicando alternè per Quotientes, c ac d, ortos diviſione denominatorum per communem diviſorem b. REDUCTIO FRACTIONUM.

Hæc autem reductio præcipuè uſui eſt in Additione & Subductione fractionum; quæ, ſi diverſos habent denominatores, ad eundem reducendæ ſunt, antequam uniri poſſunt. Sic $\frac{a}{b} + \frac{c}{d}$ per reductionem evadit $\frac{ad}{bd} + \frac{bc}{bd}$, ſive $\frac{ad + bc}{bd}$. Et $a + \frac{ab}{c}$ evadit $\frac{ac + ab}{c}$. Et $\frac{a^3}{bc} - \frac{a^3}{bd}$ evadit $\frac{a^3d - a^3c}{bcd}$, vel $\frac{d - c}{bcd} a^3$. Et $\frac{c^4 + x^4}{cc - xx} - cc - xx$ evadit $\frac{2x^4}{cc - xx}$. Atque ita $\frac{2}{3} + \frac{5}{7}$ evadit $\frac{14}{21} + \frac{15}{21}$, ſive $\frac{14 + 15}{21}$, hoc eſt $\frac{29}{21}$. Et $\frac{11}{6} - \frac{3}{4}$ evadit $\frac{22}{12} - \frac{9}{12}$, ſive $\frac{13}{12}$. Et $\frac{3}{4} - \frac{5}{12}$ evadit $\frac{9}{12} - \frac{5}{12}$, ſive $\frac{4}{12}$, hoc eſt $\frac{1}{3}$. Et $3\frac{4}{7}$ ſive $\frac{3}{1} + \frac{4}{7}$ evadit $\frac{21}{7} + \frac{4}{7}$, ſive $\frac{25}{7}$. Et $25\frac{1}{2}$ evadit $\frac{51}{2}$.

Fractiones, ubi plures ſunt, gradatim uniri debent. Sic habito $\frac{aa}{x} - a + \frac{2xx}{3a} - \frac{ax}{a - x}$; ab $\frac{aa}{x}$ aufer a, & reſtabit $\frac{aa - ax}{x}$; huic adde $\frac{2xx}{3a}$, & prodibit $\frac{3a^3 - 3aax + 2x^3}{3ax}$; unde aufer denique $\frac{ax}{a - x}$, & reſtabit $\frac{3a^4 - 6a^3x + 2ax^3 - 2x^4}{3aax - 3axx}$. Atque ita ſi habeatur $3\frac{4}{7} - \frac{2}{3}$, imprimis aggregatum $3\frac{4}{7}$ inveniendum eſt, nempe $\frac{25}{7}$; dein ab hoc auferendum $\frac{2}{3}$, & reſtabit $\frac{61}{21}$.

SECT. IV.

De REDUCTIONE RADICALIUM *ad minimos terminos.*

Radicalis, ubi totius radix extrahi nequit, plerumque concinnatur extrahendo radicem diviſoris alicujus.

Sic $\sqrt{aabc}$, extrahendo radicem diviſoris aa, fit $a\sqrt{bc}$. Et $\sqrt{48}$, extrahendo radicem diviſoris 16, fit $4\sqrt{3}$. Et $\sqrt{48aabc}$, extrahendo radicem diviſoris $16aa$, fit $4a\sqrt{3bc}$. Et $\sqrt{\frac{a^3b - 4aabb + 4ab^3}{cc}}$, extrahendo radicem diviſoris $\frac{aa - 4ab + 4bb}{cc}$, fit $\frac{a - 2b}{c}\sqrt{ab}$. Et

$\sqrt{\frac{aaoomm}{ppzz} + \frac{4aam^3}{pzz}}$, extrahendo radicem diviſoris $\frac{aamm}{ppzz}$, fit $\frac{am}{pz}\sqrt{oo+4mp}$. Et $6\sqrt{\frac{75}{98}}$, extrahendo radicem diviſoris $\frac{25}{49}$, fit $\frac{30}{7}\sqrt{\frac{3}{2}}$, ſive $\frac{30}{7}\sqrt{\frac{6}{4}}$: radicemque denominatoris adhuc extrahendo, fit $\frac{15}{7}\sqrt{6}$. Et ſic $a\sqrt{\frac{b}{a}}$, ſive $a\sqrt{\frac{ab}{aa}}$, extrahendo radicem denominatoris, fit $\sqrt{ab}$. Et $\sqrt[3]{:8a^3b+16a^4}$, extrahendo radicem cubicam diviſoris $8a^3$, fit $2a\sqrt[3]{:b+2a}$. Haud ſecus $\sqrt[4]{a^3x}$, extrahendo radicem quadraticam diviſoris aa fit, $\sqrt{a}$ in $\sqrt[4]{ax}$; vel extrahendo radicem quadrato-quadraticam diviſoris a^4, fit $a\sqrt[4]{\frac{x}{a}}$. Atque ita $\sqrt[6]{:a^7x^5}$ convertitur in $a\sqrt[6]{:ax^5}$, vel in $ax\sqrt[6]{:\frac{a}{x}}$, vel in $\sqrt{ax} \times \sqrt[3]{:aax}$.

Cæterùm hæc reductio non tantùm concinnandis radicalibus inſervit, ſed & earum Additioni & Subductioni; ſi modò ex parte radicali conveniant, ubi ad formam ſimpliciſſimam reducuntur. Tunc enim uniri poſſunt, quod aliter non fit. Sic $\sqrt{48} + \sqrt{75}$, per reductionem, evadit $4\sqrt{3} + 5\sqrt{3}$, hoc eſt $9\sqrt{3}$. Et $\sqrt{48} - \sqrt{\frac{16}{27}}$, per reductionem evadit $4\sqrt{3} - \frac{4}{9}\sqrt{3}$, hoc eſt $\frac{32}{9}\sqrt{3}$. Et ſic $\sqrt{\frac{4ab^3}{cc}} + \sqrt{\frac{a^3b - 4aabb + 4ab^3}{cc}}$, extrahendo quicquid eſt rationale, evadit $\frac{2b}{c}\sqrt{ab} + \frac{a-2b}{c}\sqrt{ab}$, hoc eſt $\frac{a}{c}\sqrt{ab}$. Et $\sqrt[3]{:8a^3b+16a^4} - \sqrt[3]{:b^4+2ab^3}$, evadit $2a\sqrt[3]{:b+2a} - b\sqrt[3]{:b+2a}$, hoc eſt $\overline{2a-b}\sqrt[3]{:b+2a}$.

SECT.

(°) Invento ſcilicet numero quem minimum indices datarum radicalium dividant, datarum unaquæque in aliam tranſmutanda eſt, quæ minimum illum numerum indicem habeat. Eò igitur res redit, ut quantitas radicalis data in aliam tranſmutetur dato indice. Detur igitur quantitas radicalis $\sqrt[n]{a^m}$, ad aliam reducenda, cujus index dato numero p æqualis ſit. Factum puta: ſit $\sqrt[p]{a^{mx}} = \sqrt[n]{a^m}$. Extrahendo igitur utrobique radicem à numero m denominatam, $\sqrt[p]{a^x} = \sqrt[n]{a}$: capiendoque utrinque poteſtates à numero n denominatas, $\overline{\sqrt[p]{a^x}}|^n = a$: atque rurſum capiendo utrinque poteſtates à numero p denominatas, $a^{xn} = a^p$. Ergo $xn = p$, & $x = \frac{p}{n}$. Numeri autem p, n, dati. Numerus igitur x datus, qui index eſt poteſtatis illius quantitatis a^m, cujus radix à numero p denominata, quantitatis a^m radici à numero n denominatæ æqualis eſt. Datam igitur radicalem $\sqrt[n]{a^m}$ ita in aliam transformaveris, quæ indicem habeat numero p æqualem, ſi ſigno radicali $\sqrt[p]{}$ quantitatem illam ſuffixeris, quæ provenerit ſi quantitatem ſigno radicali $\sqrt[n]{}$ ſuffixam,

nimirum

SECT. V.

REDUCTIO RADICALIUM.

De REDUCTIONE RADICALIUM *ad eandem denominationem.*

CUM in radicalibus diverſæ denominationis inſtituenda eſt multiplicatio vel diviſio, oportet omnes ad eandem denominationem reducere; idque præfigendo ſignum radicale, cujus index eſt minimus numerus quem earum indices dividunt abſque reſiduo, & ſuffixas quantitates toties, demptâ unâ vice, in ſe ducendo, quoties index ille jam major evaſerit. (°)

Sic enim $\surd ax$ in $\surd^3{:}aax$ evadit $\surd^6{:}a^3x^3$ in $\surd^6{:}a^4xx$, hoc eſt $\surd^6{:}a^7x^5$. Et $\surd a$ in $\surd^4{:}ax$ evadit $\surd^4{:}aa$ in $\surd^4{:}ax$, hoc eſt $\surd^4{:}a^3x$. Et $\surd 6$ in $\surd^4{:}\frac{5}{6}$ evadit $\surd^4{:}36$ in $\surd^4{:}\frac{5}{6}$, hoc eſt $\surd^4{:}30$. Eadem ratione $a\surd bc$ evadit $\surd aa$ in $\surd bc$, hoc eſt $\surd aabc$. Et $4a\surd 3bc$ evadit $\surd 16aa$ in $\surd 3bc$, hoc eſt $\surd 48aabc$. Et $2a\surd^3{:}\overline{b+2a}$ evadit $\surd^3{:}8a^3$ in $\surd^3{:}\overline{b+2a}$, hoc eſt $\surd^3{:}\overline{8a^3b+16a^4}$. Atque ita $\frac{\surd ac}{b}$ fit $\frac{\surd ac}{\surd bb}$, ſive $\sqrt{\frac{ac}{bb}}$. Et $\frac{6abb}{\surd 18ab^3}$ fit $\frac{\surd 36aab^4}{\surd 18ab^3}$, ſive $\surd 2ab$. Et ſic in aliis.

SECT. VI.

De REDUCTIONE RADICALIUM *ad ſimpliciores radicales per extractionem radicum.*

RAdices quantitatum, quæ ex integris & radicalibus quadraticis componuntur, ſic extrahe.

nimirum ſi ipſam a^m, toties demptâ unâ vice in ſe ipſam duxeris, quoties multiplicandus eſt numerus n, ut numerum numero p æqualem faciat. Atque ita tandem verum ni fallor ſenſum præcepti Newtoniani aſſecuti ſumus, quod verborum ambiguitate incautos facilè deciperet: cùm ita intelligi poſſet, ac ſi quantitas ſigno radicali ſubjecta toties in ſe ducenda eſſet, quot, unâ ſaltem demptâ, unitates fuerint in numero quo novus index p datum n exuperaverit.

Cæterùm radicalium, necnon potentialium, reductiones adhibitis indicibus fractis facillimè peragendæ ſunt, cum nihil aliud requiritur niſi ut dati indices ad denominatorem revocentur communem. Exempli gratiâ, $\overline{ax}|^{\frac{1}{2}} \times \overline{a^2x}|^{\frac{1}{3}} = a^{\frac{1}{2}} \times \overline{a}|^{\frac{2}{3}} \times x^{\frac{1}{2}} \times x^{\frac{1}{3}} = a^{\frac{7}{6}} \times x^{\frac{5}{6}} = \overline{a^7 \times x^5}|^{\frac{1}{6}}$. Rurſum $a^{\frac{1}{2}} \times \overline{ax}|^{\frac{1}{4}} = a^{\frac{1}{2}} \times a^{\frac{1}{4}} \times x^{\frac{1}{4}} = a^{\frac{3}{4}} \times x^{\frac{1}{4}} = \overline{a^3x}|^{\frac{1}{4}}$. Sic etiam $a^5 \times x^2 \times \overline{ab}|^{\frac{2}{11}} = a^5 \times a^{\frac{2}{11}} \times b^{\frac{2}{11}} \times x^2 = a^{\frac{57}{11}} \times b^{\frac{2}{11}} \times x^{\frac{22}{11}} = \overline{a^{57}b^2x^{22}}|^{\frac{1}{11}}$. Rurſum $\overline{ax}|^{\frac{1}{2}} \times \overline{a^2x}|^{-\frac{1}{3}} = a^{\frac{1}{2}} \times a^{-\frac{2}{3}} \times x^{\frac{1}{2}} \times x^{-\frac{1}{3}} = a^{-\frac{1}{6}} \times x^{\frac{1}{6}} = \overline{\frac{x}{a}}|^{\frac{1}{6}}$.

Deſignet A *quantitatis alicujus partem majorem,* B *partem minorem: Et erit* $\frac{A+\sqrt{AA-BB}}{2}$ *quadratum majoris partis radicis;* & $\frac{A-\sqrt{AA-BB}}{2}$ *quadratum partis minoris, quæ quidem majori adnectenda eſt cum ſigno ipſius* B.

Ut ſi quantitas ſit $3+\sqrt{8}$, ſcribendo 3 pro A, & $\sqrt{8}$ pro B, erit $\sqrt{AA-BB}=1$; indeque quadratum majoris partis radicis, $\frac{3+1}{2}$, id eſt 2; & quadratum minoris partis $\frac{3-1}{2}$, id eſt 1. Ergo radix eſt $1+\sqrt{2}$. Rurſus ſi ex $\sqrt{32}-\sqrt{24}$ radix extrahenda ſit, ponendo $\sqrt{32}$ pro A, & $\sqrt{24}$ pro B, erit $\sqrt{AA-BB}=\sqrt{8}$; & inde $\frac{\sqrt{32}+\sqrt{8}}{2}$, & $\frac{\sqrt{32}-\sqrt{8}}{2}$, hoc eſt $3\sqrt{2}$, & $\sqrt{2}$, quadrata partium radicis. Radix itaque eſt $\sqrt[4]{18}-\sqrt[4]{2}$. Eodem modo; ſi de $aa+2x\sqrt{aa-xx}$ radix extrahi debet, pro A ſcribe aa; & pro B, $2x\sqrt{aa-xx}$, & erit $AA-BB=a^4-4aaxx+4x^4$. Cujus radix eſt $aa-2xx$. Unde quadratum unius partis radicis erit $aa-xx$, illud alterius xx; adeoque radix, $x+\sqrt{aa-xx}$. Rurſus ſi habeatur $aa+5ax-2a\sqrt{ax+4xx}$, ſcribendo $aa+5ax$ pro A, & $2a\sqrt{ax+4xx}$ pro B, fiet $AA-BB=a^4+6a^3x+9aaxx$; cujus radix eſt $aa+3ax$. Unde quadratum majoris partis radicis erit $aa+4ax$, illud minoris ax; & radix, $\sqrt{aa+4ax}-\sqrt{ax}$. Denique ſi habeatur $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$, ponendo $6+\sqrt{8}=A$, & $-\sqrt{12}-\sqrt{24}=B$, fiet $AA-BB=8$. Unde radicis pars major $\sqrt{3+\sqrt{8}}$; hoc eſt (ut ſuprà) $1+\sqrt{2}$; & pars minor $\sqrt{3}$; atque adeo radix ipſa $1+\sqrt{2}-\sqrt{3}$. Cæterùm ubi plures ſunt hujuſmodi termini radicales, poſſunt partes radicis citiùs inveniri, dividendo factum quarumvis duarum radicalium per tertiam aliquam radicalem, quæ producit quotam rationalem & integrum. Nam dupli quoti iſtius radix erit duplum partis radicis quæſitæ. Ut in exemplo noviſſimo $\frac{\sqrt{8}\times\sqrt{12}}{\sqrt{24}}=2$, $\frac{\sqrt{8}\times\sqrt{24}}{\sqrt{12}}=4$, $\frac{\sqrt{12}\times\sqrt{24}}{\sqrt{8}}=6$. Ergo partes radicis ſunt 1, $\sqrt{2}$, $\sqrt{3}$ ut ſuprà.

Eſt & regula extrahendi altiores radices ex quantitatibus numeralibus duarum potentiâ commenſurabilium partium.

Sit

Sit quantitas A ± B. *Ejus pars major* A. *Index radicis extrahendæ c. Quære minimum numerum n, cujus poteſtas* n^c *dividitur per* $AA - BB$ *ſine reſiduo, & ſit quotus* Q. *Computa* $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$ *in numeris integris proximis. Sit illud r. Divide* $A\sqrt{Q}$ *per maximum diviſorem rationalem: Sit quotus s, ſitque* $\frac{r+\frac{n}{r}}{2s}$ *in numeris integris proximis t. Et erit* $\frac{ts \pm \sqrt{ttss-n}}{\sqrt[2c]{Q}}$ *radix quæſita, ſi modò radix extrahi poteſt.* Reductio Radicalium.

Ut ſi radix cubica extrahenda ſit ex $\sqrt{968} + 25$; erit $AA - BB = 343$; ejus diviſores 7, 7, 7; ergo $n = 7$, & $Q = 1$. Porro $\overline{A+B} \times \sqrt{Q}$, ſeu $\sqrt{968} + 25$, extractâ prioris partis radice, fit paulo major quam 56; ejus radix cubica in numeris proximis eſt 4. Ergo $r = 4$. Inſuper $A\sqrt{Q}$, ſeu $\sqrt{968}$, extrahendo quicquid rationale eſt, fit $22\sqrt{2}$. Ergo $\sqrt{2}$, ejus pars radicalis, eſt s; & $\frac{r+\frac{n}{r}}{2s}$, ſeu $\frac{5\frac{3}{4}}{2\sqrt{2}}$, in numeris integris proximis eſt 2. Ergo $t = 2$. Denique ts eſt $2\sqrt{2}$, $\sqrt{ttss-n}$ eſt 1, & $\sqrt[2c]{Q}$, ſeu $\sqrt[6]{1}$, eſt 1. Ergo $2\sqrt{2} + 1$ eſt radix quæſita, ſi modò radix extrahi queat. Tento itaque per multiplicationem ſi cubus ipſius $2\sqrt{2} + 1$ ſit $\sqrt{968} + 25$, & res ſuccedit.

Rurſus ſi radix cubica extrahenda ſit ex $68 - \sqrt{4374}$; erit $AA - BB = 250$, cujus diviſores ſunt 5, 5, 5, 2. Ergo $n = 5 \times 2 = 10$, & $Q = 4$. Et $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$, ſeu $\sqrt[c]{\overline{68+\sqrt{4374}} \times 2}$, in numeris proximis integris, eſt $7 = r$. Inſuper $A\sqrt{Q}$, ſeu $68\sqrt{4}$, extrahendo quicquid rationale eſt, fit $136\sqrt{1}$. Ergo $s = 1$, & $\frac{r+\frac{n}{r}}{2s}$, ſeu $\frac{7+\frac{10}{7}}{2}$, in numeris integris proximis eſt $4 = t$: ergo $ts = 4$, $\sqrt{ttss-n} = \sqrt{6}$, & $\sqrt[2c]{Q} = \sqrt[6]{4}$, ſeu $\sqrt[3]{2}$; atque adeo radix tentanda $\frac{4-\sqrt{6}}{\sqrt[3]{2}}$.

Iterum ſi radix quadrato-cubica extrahenda ſit ex $29\sqrt{6} + 41\sqrt{3}$; erit $AA - BB = 3$, adeoque $n = 3$, $Q = 81$, $r = 5$, $s = \sqrt{6}$, $t = 1$, $ts = \sqrt{6}$, $\sqrt{ttss-n} = \sqrt{3}$ & $\sqrt[2c]{Q} = \sqrt[10]{81}$ ſeu $\sqrt[5]{9}$; atque adeo radix tentanda $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

Cæterùm

Cæterùm in hujuſmodi operationibus ſi quantitas fractio ſit, vel partes ejus communem habent diviſorem; radices denominatoris & factorum ſeorſim extrahe. Ut ſi ex $\sqrt{242} - 12\frac{1}{2}$ radix cubica extrahenda ſit; hoc, reductis partibus ad communem denominatorem, fiet $\frac{\sqrt{968} - 25}{2}$. Dein, extractâ ſeorſim numeratoris ac denominatoris radice cubicâ, orietur $\frac{2\sqrt{2} - 1}{\sqrt[3]{2}}$. Rurſus ſi ex $\sqrt[3]{3993} + \sqrt[6]{17578125}$ radix aliqua extrahenda ſit; divide partes per communem diviſorem $\sqrt[3]{3}$, & emerget $\overline{11 + \sqrt{125}}$. Unde quantitas propoſita valet $\sqrt[3]{3}$ in $\overline{11 + \sqrt{125}}$; cujus radix invenietur, extrahendo ſeorſim radicem factoris utriuſque, $\sqrt[3]{3}$, & $11 + \sqrt{125}$.

CAPUT OCTAVUM.

De formâ ÆQUATIONIS.

Æquationes, quæ ſunt quantitatum aut ſibi mutuò æqualium, aut ſimul nihilo æquipollentium, congeries, duobus præcipuè modis conſiderandæ veniunt; vel ut ultimæ concluſiones ad quas in Problematis ſolvendis deventum eſt, vel ut media, quorum ope finales æquationes acquirendæ ſunt. Prioris generis æquatio ex unicâ tantum incognitâ quantitate cognitis involutâ conflatur, modò Problema ſit definitum, & aliquid certi quærendum innuat. Sed eæ poſterioris generis involvunt plures quantitates incognitas, quæ ideo debent inter ſe comparari, & ita connecti, ut ex omnibus una tandem emerget æquatio nova, cui ineſt unica, quam quærimus, incognita quantitas admiſta cognitis. Quæ quantitas ut exinde faciliùs eliciatur, æquatio iſta variis plerumque modis transformanda eſt, donec evadat ea ſimpliciſſima quæ poteſt, atque etiam ſimilis alicui ex ſequentibus earum gradibus; in quibus x deſignat quantitatem quæſitam, ad cujus dimenſiones termini, ut vides, ordinantur; & p, q, r, s, alias quaſcunque quantitates, ex quibus determinatis & cognitis etiam x determinatur, &, per methodos explicandas, inveſtigari poteſt.

$x = p.$		$x - p = 0.$
$xx = px + q$	Vel	$xx - px - q = 0.$
$x^3 = pxx + qx + r.$		$x^3 - pxx - qx - r = 0.$
$x^4 = px^3 + qxx + rx + s.$		$x^4 - px^3 - qxx - rx - s = 0.$
&c.		&c.

Ad

Ad horum normam itaque termini æquationum, ſecundùm dimenſiones incognitæ quantitatis, in ordinem ſemper redigendi ſunt; ita ut primum locum occupent in quibus incognita quantitas eſt plurimarum dimenſionum, inſtar x, xx, x^3, x^4; & ſecundum locum, in quibus ea eſt unâ dimenſione minor, inſtar p, px, pxx, px^3, & ſic præterea. Et quod ſigna terminorum attinet, poſſunt ea omnibus modis ſe habere: Imò & unus vel plures ex intermediis terminis aliquando deeſſe. Sic $x^3 \quad * - bbx + b^3 = 0$ vel $x^3 = bbx - b^3$, eſt æquatio tertii gradûs; $Z^4 \begin{smallmatrix}+a\\-b\end{smallmatrix} Z^3 \quad * \begin{smallmatrix}+ab^3\\-b^4\end{smallmatrix} = 0$ æquatio quarti. Nam gradus æquationum æſtimantur ex maximâ dimenſione quantitatis incognitæ, nullo reſpectu ad quantitates cognitas habito, nec ad intermedios terminos. Attamen ex defectu intermediorum terminorum æquatio plerumque fit multò ſimplicior, & nonnunquam ad gradum inferiorum quodammodo deprimitur. Sic enim $x^4 = qxx + s$ æquatio ſecundi gradûs cenſenda eſt, ſiquidem ea in duas ſecundi gradûs æquationes reſolvi poteſt. Nam ſuppoſito $xx = y$, & y pro xx in æquatione illâ perinde ſcripto, ejus vice prodibit $yy = qy + s$, æquatio ſecundi gradûs; cujus ope, cum y inventa fuerit, æquatio $xx = y$, ſecundi etiam gradûs, dabit x.

De Forma Æquationis.

Atque hæ ſunt concluſiones ad quas Problemata deduci debent. Sed antequam eorum reſolutionem aggrediar, opus erit ut modos transformandi & in ordinem redigendi æquationes, & ex mediis eliciendi finales æquationes abſtractè doceam. Æquationis autem ſolitariæ reductionem in ſequentibus regulis complectar.

CAPUT NONUM.

De concinnandâ Æquatione ſolitariâ.

REGULA PRIMA.

SIquæ ſunt quantitates quæ ſe mutuo deſtruere, vel per Additionem aut Subductionem coaleſcere poſſunt, termini perinde minuendi ſunt.

Veluti ſi habeatur $5b - 3a + 2x = 5a + 3x$, aufer utrinque $2x$, & adde $3a$, proditque $5b = 8a + x$. Atque ita $\frac{2ab + bx}{a} - b = a + b$, delendo æquipollentes $\frac{2ab}{a} - b = b$, evadit $\frac{bx}{a} = a$.

Ad-

CAPUT NONUM.

Ad hanc Regulam referri debet etiam ordinatio terminorum æquationis, quæ fieri ſolet per tranſlationem ad contrarias partes cum ſigno contrario. Ut ſi habitâ æquatione $5b = 8a + x$ deſideretur x; aufer utrinque $8a$, vel, quod eodem recidit, transfer $8a$ ad contrarias partes cum ſigno mutato, & prodibit $5b - 8a = x$. Eodem modo ſi habeatur $aa - 3ay = ab - bb + by$, ac deſideretur y, tranſpone $-3ay$, & $ab - bb$, eò ut ex unâ parte conſiſtant termini multiplicati per y, & ex alterâ reliqui termini, & prodibit $aa - ab + bb = 3ay + by$; unde y elicietur per Reg. 5. ſequentem, dividendo ſcilicet utramque partem per $3a + b$, prodibit enim $\frac{aa - ab + bb}{3a + b} = y$. Atque ita æquatio $abx + a^3 - aax = abb - 2abx - x^3$, per debitam tranſpoſitionem & ordinationem, evadit $x^3 = \begin{array}{c} aa \\ -3ab \end{array} x \begin{array}{c} -a^3 \\ +abb \end{array}$, vel $x^3 \begin{array}{c} -aa \\ +3ab \end{array} x \begin{array}{c} +a^3 \\ -abb \end{array} = 0$.

REGULA SECUNDA.

Siqua compareat quantitas per quam omnes æquationis termini multiplicantur, debent omnes per illam quantitatem dividi; vel ſi per eandem quantitatem omnes dividantur, debent omnes per illam multiplicari.

Sic habito $15bb = 24ab + 3bx$, divide terminos omnes per b, & fit $15b = 24a + 3x$: deinde per 3, & fit $5b = 8a + x$. Vel habito $\frac{b^3}{ac} - \frac{bbx}{cc} = \frac{xx}{c}$, multiplica omnes per c, & prodit $\frac{b^3}{a} - \frac{bbx}{c} = xx$.

REG.

(p) Hanc quidem operationem Franciſcus Vieta *Symmetricam Climactiſmum* vocavit: eandem Juniores *involutionem* dixerunt. Haud verò in omni caſu ſufficiet ad irrationalium expurgationem, ne quidem ſi omnia ſubquadratica ſint, quod genus eſt ſimpliciſſimum. Puta enim æquationem proponi ex quatuor ſubquadraticis compoſitam, $\sqrt{a} - \sqrt{b} - \sqrt{c} - \sqrt{d} = 0$. Hujuſce æquationis nomina ſi ita diſtribuas, ut ex utrâque parte bina ſint, veluti hoc modo $\sqrt{a} - \sqrt{b} = \sqrt{c} + \sqrt{d}$, partibus quadratis, veniet utrinque nomen unum irrationale; hinc enim orietur $a - 2\sqrt{ab} + b$, illinc $c + 2\sqrt{cd} + d$. Novæ hujus æquationis nomina ſic diſponito, ut ab unâ parte nomina omnia rationalia ſint, ab alterâ irrationalia, nempe $a + b - c - d = 2\sqrt{ab} + 2\sqrt{cd}$. Pro nominum rationalium ſummâ ponatur litera g, ut ſit $g = 2\sqrt{ab} + 2\sqrt{cd}$. Partibus rurſum quadratis, à binomio irrationali veniet unum tantùm nomen irrationale; è multiplicatione enim quadraticâ fiet $g^2 = 4 \times \overline{ab + 2\sqrt{abcd} + cd}$: collectiſque nominibus quæ utrinque rationalia ſunt, $g^2 - 4ab - 4cd = 8\sqrt{abcd}$. Pro $\frac{g^2 - 4ab - 4cd}{8}$ ponatur litera h, & ex tertiâ demum multiplicatione quadraticâ veniet æquatio irrationalibus libera, $h^2 = abcd$. Atque idem ſemper fiet, quoties æquationis propoſitæ nomina non plura ſint numero quàm quatuor; tolletur ſanè id omne quod irrationale eſt per ſymmetricam climactiſmum Vietæ, idque etiamſi omnia nomina irrationalia ſint, dummodo ſubquadratici tantùm generis. Pro uno autem nomine irrationali in hoc negotio haberi volo

REGULA TERTIA.

De Æquatione concinnanda.

Siqua ſit fractio irreducibilis, in cujus denominatore reperiatur litera illa ad cujus dimenſiones æquatio ordinanda eſt, omnes æquationis termini per iſtum denominatorem, aut per aliquem diviſorem ejus, multiplicandi ſunt.

Ut ſi æquatio $\frac{ax}{a-x}+b=x$ ſecundum x ordinanda ſit, multiplicentur omnes ejus termini per $a-x$, denominatorem fractionis $\frac{ax}{a-x}$, ſiquidem x inibi reperiatur; & prodit $ax+ab-bx=ax-xx$, ſeu $ab-bx=-xx$, &, factâ utriuſque partis tranſlatione, $xx=bx-ab$. Atque ita ſi habeatur $\frac{a^3-abb}{2cy-cc}=y-c$, terminique juxta y ordinandi ſint, multiplicentur per denominatorem $2cy-cc$, vel ſaltem per diviſorem $2y-c$, quo y tollatur è denominatore, & exurget $\frac{a^3-abb}{c}=2yy-3cy+cc$; & ordinando $\frac{a^3-abb}{c}-cc+3cy=2yy$. Ad eundem modum $\frac{aa}{x}-a=x$, multiplicando per x, evadit $aa-ax=xx$; & $\frac{aabb}{cxx}=\frac{xx}{a+b-x}$, multiplicando primò per xx, dein per $a+b-x$, evadit $\frac{a^3bb+aab^3-aabbx}{c}=x^4$.

REGULA QUARTA.

Sicut ſurdæ quantitati irreducibili litera illa involvatur, ad cujus dimenſiones æquatio ordinanda eſt, cæteri omnes termini ad contrarias partes cum ſignis mutatis transferendi ſunt, & utraque pars æquationis in ſe ſemel *multiplicanda, ſi radix* quadratica *ſit, vel* bis *ſi ſit* cubica, *&c.* P

quicquid ſimul ſigno radicali ſubſit, licet ex pluribus conflatum ſit. Exempli gratiâ, æquationis $\sqrt{a^2-bx}+\sqrt{cx-gh+d^2}=\sqrt{gd}$, tria ſunt nomina irrationalia, quorum unum eſt $\sqrt{gd}$, alterum $\sqrt{a^2-bx}$, tertium $\sqrt{cx-gh+d^2}$. At ſi æquationis propoſitæ nominibus quatuor ſubquadraticè irrationalibus nomen quintum accedat, ſive rationale, ſive ſubquadraticè etiam irrationale, nomina quidem irrationalia multiplicatione quadraticâ nunquam ſuſtuleris. Quum enim quinque nomina aliter diſtribui nequeant, quin vel ex alterâ parte trinomium conſiſtat, binomium ex alterâ; vel ex alterâ quidem parte quadrinomium, ex alterâ nomen ſingulare; vel denique ex unâ parte quinquenomium; quadratica verò trimonii multiplicatio tria nomina irrationalia ſemper pariat, binomii nomen irrationale unum, quadrinomii autem ſex, quinquenomii decem, non ex ullâ quidem dati quinquenomii diſtributione obtinebis, quin nomina irrationalia, ex quadraticis partium multiplicationibus facta, vel plura ſint numero quàm quatuor, vel ſaltem non pauciora, accedente etiam nomine rationali. Factâque quovis modo partium tranſlatione, partibuſque iterum quadratis, iterum provenient nomina irrationalia numero haud pauciora quàm quatuor, quibus etiam accedet nomen unum rationale: idque toties fiet quoties multiplicatio quadratica iterata fuerit. Quoties igitur æquationis propoſitæ, ex quinque pluribuſve nominibus conflatæ, quatuor plurave nomina irrationalia ſint, Climactiſmi opera ad irrationalitatem tollendam, licet illa in infimo quidem ſubſiſtat ſubquadraticorum gradu, nihil prorſus juvabit: ſed confugiendum erit ad viam generalem Fermatii, Newtono ſub fine hujuſce capitis traditam, quæ ſanè nullam non aſymmetriam exulare cogit.

Sic ad ordinandum juxta x æquationem $\sqrt{aa-xx}+a=x$, transferatur a ad alteras partes, fitque $\sqrt{aa-ax}=x-a$; & quadratis partibus, $aa-ax=xx-2ax+aa$, ſeu $0=xx-ax$; hoc eſt $x=a$. Sic etiam $\sqrt{3:aax+2axx-x^3}-a+x=0$, tranſponendo $-a+x$, evadit $\sqrt{3:aax+2axx-x^3}=a-x$; & partibus cubicè multiplicatis, $aax+2axx-x^3=a^3-3aax+3axx-x^3$; ſeu $xx=4ax-aa$. Et ſic $y=\sqrt{ay+yy-a\sqrt{ay-yy}}$, quadratis partibus, evadit $yy=ay+yy-a\sqrt{ay-yy}$: &, terminis debitè tranſpoſitis, $ay=a\sqrt{ay-yy}$, ſeu $y=\sqrt{ay-yy}$; & partibus iterum quadratis, $yy=ay-yy$; & tranſponendo denuo, $2yy=ay$, ſive $2y=a$.

Regula Quinta.

Terminis ſecundùm Dimenſiones literæ alicujus, ope præcedentium regularum, diſpoſitis, ſi maxima ejuſdem literæ dimenſio per cognitam quamlibet quantitatem multiplicetur, debet tota æquatio per eandem dividi.

Sic $2y=a$, dividendo per 2, evadit $y=\frac{1}{2}a$. Et $\frac{bx}{a}=a$, dividendo per $\frac{b}{a}$, evadat $x=\frac{aa}{b}$. Et $\begin{smallmatrix}2ac\\-cc\end{smallmatrix}x\begin{smallmatrix}+a^3\\+aac\end{smallmatrix}xx\begin{smallmatrix}-2a^3c\\+aacc\end{smallmatrix}x-a^3cc=0$, dividendo per $2ac-cc$, evadit $x^3+\frac{\begin{smallmatrix}a^3\\aac\end{smallmatrix}xx\begin{smallmatrix}-2a^3c\\+aacc\end{smallmatrix}x-a^3cc}{2ac-cc}=0$, ſive $x^3+\frac{a^3+aac}{2ac-cc}xx-aax-\frac{a^3c}{2a-c}=0$.

Regula Sexta.

Aliquando reductio inſtitui poteſt dividendo æquationem per compoſitam aliquam quantitatem.

Sic enim $y^3=\begin{smallmatrix}-\\+\end{smallmatrix}\frac{2c}{b}yy+3bcy-bbc$, ad hanc, $yy=-2cy+bc$, reducitur, transferendo terminos omnes ad eaſdem partes, hoc modo, $y^3\begin{smallmatrix}+\\-\end{smallmatrix}\frac{2c}{b}yy-3bcy+bbc=0$, & dividendo per $y-b$; ut in capite de diviſione oſtenſum eſt: prodibit enim $yy+2cy-bc=0$. Aſt hujuſmodi diviſorum inventio difficilis eſt, & eam priùs docuimus.

Regula Septima.

Aliquandò etiam reductio per extractionem radicis ex utrâque æquationis parte inſtituitur.

Quemadmodum ſi habeatur $xx=\frac{1}{4}aa-bb$, extractâ utrobique radice, prodit $x=\sqrt{\frac{1}{4}aa-bb}$. Quòd ſi habeatur $xx+aa=2ax+bb$, transfer

transfer $2ax$, & exurget $xx-2ax+aa=bb$; extractiſque partium radicibus, $x-a=+$ vel $-b$, ſeu $x=a\pm b$. Sic etiam habito $xx=ax-bb$, adde utrinque $-ax+\frac{1}{4}aa$, & prodit $xx-ax+\frac{1}{4}aa=\frac{1}{4}aa-bb$; & extractâ utrobique radice $x-\frac{1}{2}a=\pm\sqrt{\frac{1}{4}aa-bb}$, ſeu $x=\frac{1}{2}a\pm\sqrt{\frac{1}{4}aa-bb}$. De Æquatione Concinnanda.

Et ſic univerſaliter: Si ſit $xx=.px.q$, erit $x=.\frac{1}{2}p\pm\sqrt{\frac{1}{4}pp.q}$. Ubi $\frac{1}{2}p$ & q iiſdem ſignis, ac p & q in æquatione priori, afficienda ſunt; ſed $\frac{1}{4}pp$ ſemper affirmativè ponendum. Eſtque hoc exemplum Regula, ad cujus ſimilitudinem æquationes omnes quadraticæ ad formam ſimplicium reduci poſſunt. *E. g.* Propoſitâ æquatione $yy=\frac{2xxy}{a}+xx$, ad extrahendam radicem y, confer $\frac{2xx}{a}$ cum p, & xx cum q; hoc eſt, ſcribe $\frac{xx}{a}$ pro $\frac{1}{2}p$, & $\frac{x^4}{aa}+xx$ pro $\frac{1}{4}pp.q$, atque orietur $y=\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}$, vel $y=\frac{xx}{a}-\sqrt{\frac{x^4}{aa}+xx}$. Eodem modo æquatio $yy=ay-2cy+aa-cc$, conferendo $a-2c$ cum p, & $aa-cc$ cum q, dabit $y=\frac{1}{2}a-c+\sqrt{\frac{5}{4}aa-ac}$. Quinetiam æquatio quadrato-quadratica $x^4=-aaxx+ab^3$, cujus termini impares deſunt, ope hujus regulæ evadit $xx=-\frac{1}{2}aa+\sqrt{\frac{1}{4}a^4+ab^3}$, & extractâ iterum radice, $x=\sqrt{-\frac{1}{2}aa+\sqrt{\frac{1}{4}a^4+ab^3}}$. Et ſic in aliis.

Suntque hæ regulæ pro concinnandâ æquatione ſolitariâ; quarum uſum cùm Analyſta ſatis perſpexerit, ita ut æquationem quamcunque propoſitam ſecundum quamlibet literarum in eâ complexarum diſponere noverit, & ejuſdem literæ, ſi ea unius ſit dimenſionis, aut maximæ poteſtatis ejus ſi plurium, valorem elicere; haud difficilem ſentiet comparationem plurium æquationum inter ſe: quam pergo jam docere.

CAPUT DECIMUM.

De duabus pluribuſve æquationibus in unum transformandis, ut incognitæ quantitates exterminentur.

CUM in alicujus problematis ſolutionem plures habentur æquationes ſtatum quæſtionis comprehendentes, quarum unicuique plures etiam incognitæ quantitates involvuntur; æquationes iſtæ (duæ per vices ſi modò ſint plures duabus) ſunt ita connectendæ, ut una ex incognitis quantitatibus per ſingulas operationes tollatur, & emergat æquatio nova. Sic habitis æquationibus $2x = y + 5$, & $x = y + 2$, demendo æqualia ex æqualibus, prodibit $x = 3$. Et ſciendum eſt, quòd per quamlibet æquationem una quantitas incognita poteſt tolli; atque adeo cùm tot ſunt æquationes quot quantitates incognitæ, omnes poſſunt ad unam denique reduci, in quâ unica manebit quantitas incognita. Sin quantitates incognitæ ſint unâ plures quàm æquationes habentur, tum in æquatione ultimò reſultante duæ manebunt quantitates incognitæ; & ſi ſint duabus plures quàm æquationes habentur, tum in æquatione ultimò reſultante manebunt tres, & ſic præterea.

Poſſunt etiam duæ vel plures quantitates incognitæ per duas tantùm æquationes fortaſſe tolli. Ut ſi ſit $ax - by = ab - az$, & $bx + by = bb + az$: tum, æqualibus ad æqualia additis, prodibit $ax + bx = ab + bb$, exterminatis utriſque y & z. Sed ejuſmodi caſus vel arguunt vitium aliquod in ſtatu quæſtionis latere, vel calculum erroneum eſſe, aut non ſatis artificioſum. Modus autem quo una quantitas incognita per ſingulas æquationes tollatur, ex ſequentibus patebit.

SECT. II.

Exterminatio quantitatis incognitæ per æqualitatem valorum ejus.

CUM quantitas tollenda unius eſt tantùm dimenſionis in utrâque æquatione, valor ejus uterque per regulas jam ante traditas quærendus eſt, & alter valor ſtatuendus æqualis alteri.

Sic

Sic poſitis $a+x=b+y$, & $2x+y=3b$; ut exterminetur y, æquatio prima dabit $a+x-b=y$; & ſecunda dabit $3b-2x=y$. Eſt ergo $a+x-b=3b-2x$, ſive ordinando $x=\frac{4b-a}{3}$. De Quantitatibus Exterminandis.

Atque ita $2x=y$, & $5+x=y$ dant $2x=5+x$, ſeu $x=5$.

Et $ax-2by=ab$, & $xy=bb$ dant $\frac{ax-ab}{2b}$ $(=y)$ $=\frac{bb}{x}$; ſive ordinando, $xx-bx-\frac{2b^3}{a}=0$.

Item $\frac{bbx-aby}{a}=ab+xy$, & $bx+\frac{ayy}{c}=2aa$, tollendo x, dant $\frac{aby+aab}{bb-ay}$ $(=x)=\frac{2aac-ayy}{bc}$: Et reducendo, $y^3-\frac{bb}{a}yy-\frac{2aac-bbc}{a}y+bbc=0$.

Denique $x+y-z=0$, & $ay=xz$, tollendo z, dant $x+y$ $(=z)$ $=\frac{ay}{x}$, ſive $xx+xy=ay$.

Hoc idem quoque perficitur, ſubducendo alterutrum valorem quantitatis incognitæ ab altero, & ponendo reſiduum æquale nihilo. Sic in exemplorum primo, tolle $3b-2x$ ab $a+x-b$, & manebit $a+3x-4b=0$, ſive $x=\frac{4b-a}{3}$.

SECT. III.

Exterminatio quantitatis incognitæ ſubſtituendo pro eâ valorem ſuum.

CUM in alterâ ſaltem æquatione, tollenda quantitas unius tantùm dimenſionis exiſtit, valor ejus in eâ quærendus eſt; & pro ſe in æquationem alteram ſubſtituendus. Sic propoſitis $xyy=b^3$, & $xx+yy=by-ax$; ut exterminetur x, prima dabit $\frac{b^3}{yy}=x$: Quare in ſecundam ſubſtituo $\frac{b^3}{yy}$ pro x, & prodit $\frac{b^6}{y^4}+yy=by-\frac{ab^3}{yy}$, ac reducendo $y^6-by^5+ab^3yy+b^6=0$.

Propoſitis autem $ayy+aay=z^3$; & $yz-ay=az$, ut y tollatur, ſecunda dabit $y=\frac{az}{z-a}$. Quare pro y ſubſtituo $\frac{az}{z-a}$ in primam, proditque $\frac{a^3zz}{zz-2az+aa}+\frac{a^2z}{z-a}=z^3$. Et reducendo, $z^4-2az^3+aazz-2a^3z+a^4=0$.

Pari modo propoſitis $\frac{xy}{c}=z$, & $cy+zx=cc$, ad z tollendum, pro eo ſubſtituo $\frac{xy}{c}$ in æquationem ſecundam, & prodit $cy+\frac{xxy}{c}=cc$.

Cæterùm

Cæterùm qui in hujuſmodi computationibus exercitatus fuerit, ſæpenumero contractiores modos percipiet, quibus incognita quantitas exterminari poſſit. Sic habitis $ax = \frac{bbx - b^3}{z}$, & $x = \frac{az}{x - b}$, ſi æqualia multiplicentur æqualibus, prodibunt æqualia $axx = abb$, ſive $x = b$. Sed caſus ejuſmodi particulares ſtudioſis proprio marte, cùm res tulerit, inveſtigandos linquo.

SECT. IV.

Exterminatio quantitatis incognitæ quæ plurium in utrâque æquatione dimenſionum exiſtit.

CUM in neutrâ æquatione tollenda quantitas unius tantùm dimenſionis exiſtit, valor maximæ poteſtatis ejus in utrâque quærendus eſt; deinde, ſi poteſtates iſtæ non ſint eædem, æquatio poteſtatis minoris multiplicanda eſt per tollendam quantitatem, aut per ejus quadratum aut cubum, *&c.* ut ea evadat ejuſdem poteſtatis cum æquatione alterâ. Tum valores illarum poteſtatum ponendæ ſunt æquales, & æquatio nova prodibit, ubi maxima poteſtas, ſive dimenſio, tollendæ quantitatis diminuitur. Et hanc operationem iterando quantitas illa tandem auferetur.

Quemadmodum ſit $xx + 5x = 3yy$, & $2xy - 3xx = 4$; ut x tollatur, prima dabit $xx = -5x + 3yy$, & ſecunda $xx = \frac{2xy - 4}{3}$. Pono itaque $3yy - 5x = \frac{2xy - 4}{3}$, & ſic x ad unicam tantùm dimenſionem reducitur, adeoque tolli poteſt per ea quæ paulo ante oſtendi. Scilicet æquationem noviſſimam debitè reducendo prodit $9yy - 15x = 2xy - 4$, ſive $x = \frac{9yy + 4}{2y + 15}$. Hunc itaque valorem pro x in aliquam ex æquationibus primo propoſitis (velut in $xx + 5x = 3yy$) ſubſtituo, & oritur $\frac{81y^4 + 72yy + 16}{4yy + 60y + 225} + \frac{45yy + 20}{2y + 15} = 3yy$. Quam, ut in ordinem redigatur, multiplico per $4yy + 60y + 225$, & prodit $81y^4 + 72yy + 16 + 90y^3 + 40y + 675yy + 300 = 12y^4 + 180y^3 + 675yy$; ſive $69y^4 - 90y^3 + 72yy + 40y + 316 = 0$.

Præterea ſi ſit $y^3 = xyy + 3x$, & $yy = xx - xy - 3$; ut y tollatur, multiplico poſteriorem æquationem per y, & ſit $y^3 = xxy - xyy - 3y$,

totidem dimenſionum quot prior. Jam ponendo valores ipſius y ſibimet æquales, habeo $xyy+3x=xxy-xyy-3y$, ubi y deprimitur ad duas dimenſiones. Per hanc itaque, & ſimpliciorem ex æquationibus primo propoſitis, $yy=xx-xy-3$, quantitas y prorſus tolli poteſt, inſiſtendo veſtigiis prioris exempli. DE QUANTITATIBUS EXTERMINANDIS.

Sunt & alii modi quibus hæc eadem abſolvi poſſunt; idque ſæpenumero contractiùs. Quemadmodum ex $yy=\frac{2xxy}{a}+xx$, & $yy=2xy+\frac{x^4}{aa}$; ut y deleatur, extrahe in utrâque radicem y, ſicut in Reg. 7. oſtenſum eſt; & prodibunt $y=\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}$, & $=x+\sqrt{\frac{x^4}{aa}+xx}$. Jam hos ipſius y valores ponendo æquales, habebitur $\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}=x+\sqrt{\frac{x^4}{aa}+xx}$, & rejiciendo æqualia $\sqrt{\frac{x^4}{aa}+xx}$, reſtabit $\frac{xx}{a}=x$, vel $xx=ax$, & $x=a$.

Porro ut ex æquationibus $x+y+\frac{yy}{x}=20$, & $xx+yy+\frac{y^4}{xx}=140$, tollatur x; aufer y de partibus æquationis primæ, & reſtabit $x+\frac{yy}{x}=20-y$; & partibus quadratis fit $xx+2yy+\frac{y^4}{xx}=400-40y+yy$; tollendoque utrinque yy, reſtat $xx+yy+\frac{y^4}{xx}=400-40y$. Quare cum $400-40y$ & 140 iiſdem quantitatibus æquentur, erit $400-40y=140$, ſive $y=6\frac{1}{2}$. Et ſic opus in pleriſque aliis æquationibus contrahere liceat.

Cæterùm cùm quantitas exterminanda multarum dimenſionum exiſtit, ad eam ex æquationibus tollendam calculus maximè laborioſus nonnunquam requiritur: Sed labor tunc plurimum minuetur per exempla ſequentia tanquam regulas adhibita.

REGULA PRIMA.

Ex $axx+bx+c=0$, & $fxx+gx+h=0$.

Exterminato x prodit

$$\overline{ah-bg-2cf}\times ah:+\overline{bh-cg}\times bf:+\overline{agg+cff}\times c=0.$$

REGULA

CAPUT DECIMUM.

REGULA SECUNDA.

Ex $ax^3 + bxx + cx + d = 0$, & $fxx + gx + h = 0$,

Exterminato x prodit

$$\overline{ah - bg - 2cf} \times ahh : + \overline{bh - cg - 2df} \times bfh : + \overline{ch - dg} \times \overline{agg + cff} : + \overline{3agh + bgg + dff} \times df = 0.$$

REGULA TERTIA.

Ex $ax^4 + bx^3 + cxx + dx + e = 0$, & $fxx + gx + h = 0$,

Exterminato x prodit

$$\overline{ah - bg - 2cf} \times ah^3 : + \overline{bh - cg - 2df} \times bfhh : + \overline{agg + cff} \times \overline{chh - dgh + egg - 2efh} + \overline{3agh + bgg + dff} \times dfh : + \overline{2ahh + 3bgh - dfg + eff} \times eff : - \overline{bg - 2ah} \times efgg = 0.$$

REGULA QUARTA.

Ex $ax^3 + bxx + cx + d = 0$, & $fx^3 + gxx + hx + k = 0$,

Exterminato x prodit

$$\overline{ah - bg - 2cf} \times \overline{adhh - achk} : + \overline{ak + bh - cg - 2df} \times bdfh : - \overline{ak + bh + 2cg + 3df} \times aakk : + \overline{cdh - ddg - cck + 2bdk} \times \overline{agg + cff} : + \overline{3agh + bgg + dff - 3afk} \times ddf : - \overline{3ak - bh + cg + df} \times bcfk : + \overline{bk - 2dg} \times \overline{bhfk - bbk - 3adh - cdf} \times agk = 0.$$

Verbi gratiâ, ut ex æquationibus $xx + 5x - 3yy = 0$, & $3xx - 2xy + 4 = 0$, exterminetur x : in regulam primam pro a, b, c ; f, g, & h reſpectivè ſubſtituo 1, 5, $-3yy$; 3, $-2y$, & 4. Et ſignis + & − probè obſervatis, oritur $\overline{4 + 10y + 18yy} \times 4 : + \overline{20 - 6y^3} \times 15 : + \overline{4yy - 27yy} \times -3yy = 0$. Sive $16 + 40y + 72yy + 300 - 90y^3 + 69y^4 = 0$.

Simili ratione, ut y deleatur ex æquationibus $y^3 - xyy - 3x = 0$ & $yy + xy - xx + 3 = 0$, in regulam ſecundam pro a, b, c, d; f, g, h, & x ſubſtituo, 1, $-x$, 0, $-3x$; 1, x, $-xx + 3$, & y, reſpectivè; proditque $\overline{3 - xx + xx} \times \overline{9 - 6xx + x^4} : - \overline{3x + x^3 + 6x} \times \overline{-3x + x^3} : + 3xx \times xx : + \overline{9x - 3x^3 - x^3 - 3x} \times -3x = 0$. Tum delendo ſuperflua & multiplicando, fit $27 - 18xx + 3x^4$, $-9xx + x^6$, $+3x^4 - 18x^2 + 12x^4 = 0$. Et ordinando, $x^6 + 18x^4 - 45xx + 27 = 0$.

Hactenus

Hactenus de unicâ incognitâ quantitate è duabus æquationibus tollendâ. Quòd ſi plures è pluribus tollendæ ſunt, opus per gradus peragetur. Ex æquationibus $ax=yz$, $x+y=z$, & $5x=y+3z$, ſi quantitas y elicienda ſit, imprimis tolle alteram quantitatem x, aut z, puta x; ſubſtituendo pro eâ valorem ejus $\frac{yz}{a}$ (per æquationem primam inventum) in æquationem ſecundam ac tertiam. Quo pacto obtinebuntur $\frac{yz}{a}+y=z$, & $\frac{5yz}{a}=y+3z$: è quibus deinde tolle z ut ſuprà.

De Quantitatibus exterminandis.

SECT. IV.

De modo tollendi quantitates quotcunque ſurdas ex æq[illegible]

HUC referre lic[illegible] adarum extermina[illegible], fingendo eas lit[illegible] [illegible]bet æquales [q]. Quemadmodum ſi ſit $\sqrt{ay}-\sqrt{aa-a[illegible]}$ [illegible] $+\sqrt{^3:ayy}$, ſcribendo t, pro $\sqrt{ay}$, v pro $\sqrt{aa-ay}$, & x [illegible] $:ayy$, habebuntur æquationes $t-v=2a+x$, $tt=ay$, $vv=aa-a[illegible]$, & $x^3=ayy$: ex quibus tollendo gradatim t, v, & x, reſult[illegible] tandem æquatio libera ab omni Aſymmetriâ.

CAPUT UNDECIMUM.

Quomodo Quæſtio aliqua ad æquationem redigatur.

POſtquam Tiro in æquationibus pro arbitrio transformandis & concinnandis aliquamdiu exercitatus fuerit, ordo exigit, ut ingenii vires in quæſtionibus ad æquationem redigendis tentet. Propoſitâ autem aliquâ Queſtione, Artificis ingenium in eo præſertim requiritur, ut omnes ejus conditiones totidem æquationibus deſignet. Ad quod faciendum perpendet imprimis, an propoſitiones ſive ſententiæ, quibus enunciatur, ſint omnes aptæ quæ terminis algebraicis deſignari poſſint, haud ſecùs quàm conceptus noſtri characteribus græcis vel latinis. Et ſi ita (ut ſolet in quæſtionibus quæ circa numeros vel abſtractas quantitates verſantur),

[q] Vide Fermatii diſſertationem de Novo ſecundarum, et ulterioris ordinis, radicum in analyticis uſu, inter opera ejus mathematica, p. 58.

CAPUT XI. ſantur), tunc nomina quantitatibus ignotis, atque etiam notis, ſi opus fuerit, imponat; & ſenſum quæſtionis ſermone, ut ita loquar, analytico deſignet. Et conditiones ejus, ad algebraicos terminos ſic tranſlatæ, tot dabunt æquationes, quot ei ſolvendæ ſufficiunt.

Quemadmodum ſi quærantur tres numeri continuè proportionales, quorum ſumma ſit 20, & quadratorum ſumma 140; poſitis x, y, & z nominibus numerorum trium quæſitorum, quæſtio è latinis literis in algebraicas vertetur, ut ſequitur.

Quæſtio Latinè enunciata.	Eadem Algebraicè.
Quæruntur tres numeri his conditionibus,	$x.\ y.\ z$?
Ut ſint continue proportionales,	$x.y::y.z.$ ſive $xz=yy$
Ut omnium ſumma ſit 20,	$x+y+z=20.$
Et ut quadratorum ſumma ſit 140.	$xx+yy+zz=140.$

Atque ita quæſtio deducitur ad æquationes $xz=yy$; $x+y+z=20$; & $xx+yy+zz=140$; quarum ope x, y & z per regulas ſuprà traditas inveſtigandi ſunt.

Cæterùm notandum eſt ſolutiones quæſtionum eo magis expeditas & artificioſas ut plurimùm evadere, quo pauciores incognitæ quantitates ſub initio ponuntur. Sic in hâc quæſtione poſito x pro primo numero & y pro ſecundo, erit $\frac{yy}{x}$ tertius continuè proportionalis; quem proinde ponens pro tertio numero quæſtionem ad æquationes ſic reduco.

Quæſtio Latinè enunciata.	Eadem Algebraicè.
Quæruntur tres numeri continue proportionales,	$x.\ y.\ \frac{yy}{x}$?
Quorum ſumma ſit 20,	$x+y+\frac{yy}{x}=20.$
Et quadratorum ſumma 140.	$xx+yy+\frac{y^4}{xx}=140.$

Habentur itaque æquationes $x+y+\frac{yy}{x}=20$; & $xx+yy+\frac{y^4}{xx}=140$; quarum reductione x & y determinandi ſunt.

I Aliud

Aliud exemplum accipe. Mercator quidam nummos ejus triente quotannis adauget, demptis 100 lb. quas annuatim impendit in familiam; & poſt tres annos fit duplo ditior. Quæruntur nummi.

De Quæst. ad Æquat. Redigend.

Ad hoc autem reſolvendum ſciendum eſt, quòd plures latent propoſitiones, quæ omnes ſic eruuntur & enunciantur.

Latinè.	Algebraicè.
Mercator habet nummos quoſdam.	x.
Ex quibus anno primo expendit 100 lb.	$x-100$.
Et reliquum adauget triente.	$x-100+\frac{x-100}{3}$, ſive $\frac{4x-400}{3}$.
Annoque ſecundo expendit 100 lb.	$\frac{4x-400}{3}-100$, ſive $\frac{4x-700}{3}$.
Et reliquum adauget triente.	$\frac{4x-700}{3}+\frac{4x-700}{9}$, ſive $\frac{16x-2800}{9}$.
Et ſic anno tertio expendit 100 lb.	$\frac{16x-2800}{9}-100$, ſive $\frac{16x-3700}{9}$.
Et reliquo trientem ſimiliter lucratus eſt.	$\frac{16x-3700}{9}+\frac{16x-3700}{27}$, ſive $\frac{64x-14800}{27}$.
Fitque duplo ditior quam ſub initio.	$\frac{64x-14800}{27}=2x$.

Quæſtio itaque ad æquationem $\frac{64x-14800}{27}=2x$ redigitur; cujus reductione eruendus eſt x. Nempe duc eam in 27, & fit $64x-14800=54x$: ſubduc $54x$, & reſtat $10x-14800=0$, ſeu $10x=14800$, & dividendo per 10 fit $x=1480$. Quare 1480 lb. ſunt nummi ſub initio, ut & lucrum.

Vides itaque quòd ad ſolutiones quæſtionum, quæ circa numeros vel abſtractas quantitatum relationes ſolummodo verſantur, nihil aliud fere requiritur, quàm ut è ſermone Latino, vel alio quovis in quo Problema proponitur, tranſlatio fiat in ſermonem (ſi ita loquar) Algebraicum; hoc eſt, in Characteres qui apti ſunt ut noſtros de quantitatum relationibus conceptus deſignent. Nonnunquam verò poteſt accidere, quòd ſermo, quo cum ſtatus quæſtionis exprimitur, ineptus videatur qui in Algebraicum poſſit verti; ſed paucis mutationibus adhibitis, & ad ſenſum potiùs quàm verborum ſonos attendendo, verſio reddetur facilis. Sic

 enim

CAPUT XII. enim quælibet apud Gentes loquendi formæ propria habent Idiomata: Quæ ubi obvenerint, tranſlatio ex unis in alias non verbo tenus inſtituenda eſt, ſed ex ſenſu determinanda. Cæterùm ut hujuſmodi problemata hâc methodo ad æquationes redigendi familiaritatem convincam & illuſtrem, & cùm Artes exemplis faciliùs quàm præceptis addiſcantur, placuit ſequentium problematum ſolutiones adjungere.

CAPUT DUODECIMUM.

PROB. I.

Datâ duorum numerorum ſummâ a, & differentia quadratorum b, invenire numeros?

Sit eorum minor x, & erit alter $a-x$, eorumque quadrata xx & $aa-2ax+xx$: Quorum differentia $aa-2ax$ ſupponitur b. Eſt itaque $aa-2ax=b$, indeque per reductionem $aa-b=2ax$ ſeu $\frac{aa-b}{2a}$ $(=\frac{1}{2}a-\frac{b}{2a})=x$.

Exempli Gr. Si ſumma numerorum, ſeu a, ſit 8, & quadratorum differentia ſeu b 16; erit $\frac{1}{2}a-\frac{b}{2a}$ $(=4-1)=3=x$ & $a-x=5$. Quare numeri ſunt 3 & 5.

PROB. II.

Invenire tres quantitates, x, y & z, quarum paris cujuſque ſumma datur.

Si ſumma paris x & y ſit a; paris x & z, b; ac paris y & z, c: Pro determinandis tribus quæſitis x, y & z, tres habebuntur æquationes, $x+y=a$; $x+z=b$; & $y+z=c$. Jam ut incognitarum duæ, puta y & z, exterminentur, aufer x utrinque in primâ & ſecundâ æquatione, & emergent $y=a-x$, & $z=b-x$; quos valores pro y & z ſubſtitue in tertiâ, & orietur $a-x+b-x=c$; & per reductionem $x=\frac{a+b-c}{2}$. Invento x, æquationes ſuperiores, $y=a-x$, & $z=b-x$, dabunt y & z.

EXEMP. Si ſumma paris x & y ſit 9; paris x & z, 10; & paris y & z, 13; tum in valoribus x, y & z ſcribe 9 pro a, 10 pro b, & 13 pro c; & evadet $a+b-c=6$, adeoque $x\ (=\frac{a+b-c}{2}) = 3$, $y(=a-x)=6$, & $z(=b-x)=7$. QUÆSTIONES ARITHMETICÆ.

PROB. III.

Quantitatem datam ita in partes quotcunque dividere, ut majores partes ſuperent minimam per datas differentias.

Sit a quantitas in quatuor ejuſmodi partes dividenda; ejuſque prima atque minima pars, x; & ſuper hanc exceſſus ſecundæ partis, b; tertia partis, c; & quartæ partis, d; & erit $x+b$ ſecunda pars, $x+c$ tertia pars, & $x+d$ quarta pars; quarum omnium aggregatum, $4x+b+c+d$, æquatur toti lineæ a. Aufer jam utrinque $b+c+d$, & reſtat $4x=a-b-c-d$, ſive $x=\frac{a-b-c-d}{4}$.

EXEMP. Proponatur linea 20 pedum, ſic in 4 partes diſtribuenda, ut ſuper primam partem exceſſus ſecundæ ſit 2 pedum, tertiæ 3 ped. & quartæ 7 ped. Et quatuor partes erunt $x\ (=\frac{a-b-c-d}{4}$ ſive $\frac{20-2-3-7}{4}) = 2$, $x+b=4$, $x+c=5$, & $x+d=9$.

Eodem modo quantitas in plures partes iiſdem conditionibus dividitur.

PROB. IV.

Viro cuidam nummos inter mendicantes diſtribuere volenti, deſunt octo denarii quo minùs det ſingulis tres denarios. Dat itaque ſingulis duos denarios, & tres denarii ſuperſunt. Quæritur numerus mendicantium.

Eſto numerus mendicantium x, & deerunt 8 denarii quo minùs det omnibus $3x$ denarios; habet itaque $3x-8$ denarios. Ex his autem dat $2x$ denarios, & reliqui denarii $x-8$ ſunt tres. Hoc eſt $x-8=3$, ſeu $x=11$.

PROB.

Caput XII.

PROB. V.

Si Tabellarii duo, A & B, 59 milliaribus diſtantes, tempore matutino obviam eant, quorum A conficit 7 milliaria in 2 horis, & B 8 mill. in 3 horis, ac B unâ horâ ſeriùs iter inſtituit quàm A: Quæritur longitudo itineris quod A conficiet antequam conveniet B.

Dic longitudinem illam x; & erit $59-x$ longitudo itineris B: Et cùm A pertranſeat 7 mill. in 2 hor. pertranſibit ſpatium x in $\frac{2x}{7}$ horis, eo quòd ſit 7 mill. x mill. : : 2 hor. . $\frac{2x}{7}$ hor. Atque ita cùm B pertranſeat 8 mill. in 3 hor. pertranſibit ſpatium ſuum $59-x$ in $\frac{177-3x}{8}$ horis. Jam cùm horum temporum differentia ſit 1 hor.; ut evadant æqualia, adde differentiam illam breviori tempori, nempe tempori $\frac{177-3x}{8}$, & emerget $1+\frac{177-3x}{8}=\frac{2x}{7}$. Et per reductionem $35=x$. Nam, multiplicando per 8, fit $185-3x=\frac{16x}{7}$. Dein, multiplicando etiam per 7, fit $1295-21x=16x$, ſeu $1295=37x$. Et, dividendo denique per 37, exoritur $35=x$. Sunt itaque 35 mill. iter quod A conficiet antequam conveniet B.

Idem generaliùs.

Datis duorum mobilium A & B eodem curſu pergentium celeritatibus, unà cum intervallo locorum ac temporum à quibus incipiunt moveri: Determinare metam in quâ convenient.

Pone mobilis A eam eſſe celeritatem quâ ſpatium c pertranſire poſſit in tempore f, & mobilis B eam eſſe quâ ſpacium d pertranſire poſſit in tempore g; & locorum intervallum eſſe e; ac h, temporum in quibus moveri incipiunt.

CASUS I.

Deinde ſi ambo ad eaſdem plagas tendant, & A ſit mobile quod ſub initio motûs longius diſtat à metâ: Pone diſtantiam illam eſſe x, indeque aufer intervallum e, & reſtabit $x-e$ pro diſtantiâ B à metâ. Et cùm A pertranſeat ſpatium c in tempore f, tempus in quo pertranſibit ſpatium x erit $\frac{fx}{c}$, eo quòd ſit ſpatium c ad ſpatium

tium x, ut tempus f ad tempus $\frac{fx}{c}$. Atque ita cùm B pertranſeat ſpatium d in g, tempus, in quo pertranſibit ſpatium $x-e$, erit $\frac{gx-ge}{d}$. Jam cùm horum temporum differentia ſupponatur b, ut ea evadant æqualia, adde b breviori tempori, nempe tempori $\frac{fx}{c}$ (ſi modò B priùs incipiat moveri) & evadet $\frac{fx}{c}+b=\frac{gx-ge}{d}$. Et per reductionem $\frac{cge+cdb}{cg-df}$, vel $\frac{ge+\frac{db}{d}}{g-\frac{d}{c}f}$, $=x$. Sin A priùs moveri incipiat, adde b tempori $\frac{gx-ge}{d}$, & evadet $\frac{fx}{c}=b+\frac{gx-ge}{d}$, & per reductionem $\frac{cge-cdb}{cg-df}=x$. QUÆSTIONES ARITHMETICÆ.

CASUS II.

Quòd ſi mobilia obviam eant, & x ut antè ponatur initialis diſtantia mobilis A à metâ, tum $e-x$ erit initialis diſtantia ipſius B ab eâdem metâ; & $\frac{fx}{c}$ tempus in quo A conficiet diſtantiam x, atque $\frac{ge-gx}{d}$ tempus in quo B conficiet diſtantiam ſuam $e-x$. Quorum temporum minori, ut ſuprà, adde differentiam b, nempe tempori $\frac{fx}{c}$, ſi B priùs incipiat moveri, & ſic habebitur $\frac{fx}{c}+b=\frac{ge-gx}{d}$, & per reductionem $\frac{cge-cdb}{cg+df}=x$. Sin A priùs incipiat moveri, adde b tempori $\frac{ge-gx}{d}$, & evadet $\frac{fx}{c}=b+\frac{ge-gx}{d}$, & per reductionem $\frac{cge+cdb}{cg+df}=x$.

EXEMPL. I. Si quotidie Sol unum gradum conficit, & Luna tredecim; & ad tempus aliquod, Sol ſit in principio Cancri, atque poſt tres dies Luna in principio Arietis: Quæritur locus conjunctionis proximè futuræ. Reſp. in $10\frac{3}{4}$ gr. Cancri. Nam cùm ambo ad eaſdem plagas eant, & ſerior ſit Epocha motûs Lunæ, quæ longius diſtat à metâ: erit A Luna, B Sol, & $\frac{cge+cdb}{cg-df}$ longitudo itineris lunaris; quæ, ſi ſcribatur 13 pro c, 1 pro f, d, ac g, 90 pro e, & 3 pro b, evadet $\frac{13\times1\times90+13\times1\times3}{13\times1-1\times1}$; hoc eſt $\frac{1209}{12}$, ſive $100\frac{3}{4}$. Hos itaque gradus adjice principio Arietis, & prodibit $10\frac{3}{4}$ gr. Cancri.

EXEMPL.

CAPUT XII. EXEMPL. II. Si Tabellarii duo A & B, 59 milliaribus diſtantes tempore matutino obviam eant, quorum A conficit 7 milliaria in 2 horis, & B, 8 milliaria in 3 horis, & B unâ horâ feriùs iter inſtituit quàm A: Quæritur iter quod A conficiet antequam conveniat B. Reſp. 35 mill. Nam cùm obviam eant, & A primò inſtituat iter, erit $\frac{cge+cdh}{cg+df}$ iter quæſitum. Et hoc, ſi ſcribatur 7 pro c, 2 pro f, 8 pro d, 3 pro g, 59 pro e, & 1 pro h, evadet $\frac{7\times3\times59+7\times8\times1}{7\times3+8\times2}$; hoc eſt $\frac{1295}{37}$, ſive 35.

PROB. VI.

Datâ agentis alicujus poteſtate, invenire quot ejuſmodi agentes datum effectum a in dato tempore b producent.

Sit ea agentis poteſtas quâ effectum c producere poteſt in tempore d; & erit ut tempus d ad tempus b, ita effectus c, quem agens iſte producere poteſt in tempore d, ad effectum quem poteſt producere in tempore b, qui proinde erit $\frac{bc}{d}$. Deinde ut unius agentis effectus, $\frac{bc}{d}$, ad omnium effectum, a, ita agens iſte unicus ad omnes agentes: adeoque agentium numerus erit $\frac{ad}{bc}$.

EXEMPL. Si ſcriba in 8 diebus 15 folia deſcribere poteſt, quot ejuſmodi ſcribæ requiruntur ad deſcribendum 405 folia in 9 diebus? Reſp. 24. Nam ſi ſubſtituantur 8 pro d, 15 pro c, 405 pro a, & 9 pro b, numerus $\frac{ad}{bc}$ evadet $\frac{405\times8}{9\times15}$, hoc eſt $\frac{3240}{135}$, ſive 24.

PROB. VII.

Datis plurium agentium viribus, tempus x determinare in quo datum effectum d conjunctim producent.

Agentium A, B, C, vires ponantur, quæ in temporibus e, f, g, producant effectus a, b, c reſpectivè; & hæ in tempore x producent effectus $\frac{ax}{e}$, $\frac{bx}{f}$, $\frac{cx}{g}$. Quare eſt $\frac{ax}{e}+\frac{bx}{f}+\frac{cx}{g}=d$, & per reductionem $x=\frac{d}{\frac{a}{e}+\frac{b}{f}+\frac{c}{g}}$.

EXEMPL.

EXEMPL. Tres mercenarii opus aliquod certis temporibus perficere poſſunt, *viz.* A ſemel in tribus ſeptimanis, B ter in octo ſeptimanis, & C quinquies in duodecim ſeptimanis. Quæritur quanto tempore ſimul abſolvent? Sunt itaque Agentium A, B, C, vires, quæ temporibus 3, 8, 12 producant effectus 1, 3, 5 reſpective: et quæritur tempus quo abſolvent effectum 1. Quare pro a, b, c; d; e, f, g; ſcribe 1, 3, 5; 1; 3, 8, 12, & proveniet $x = \frac{1}{\frac{1}{3}+\frac{3}{8}+\frac{5}{12}}$, ſive $\frac{8}{9}$ ſept. hoc eſt 6 dies $5\frac{1}{3}$ horæ, tempus quo ſimul abſolvent. QUÆSTIONES ARITHMETICÆ.

PROB. VIII.

Diſſimiles duarum pluriumve rerum miſturas ita componere, ut res illæ commiſtæ datam inter ſe rationem acquirant.

Sit unius miſturæ data quantitas $d\text{A} + e\text{B} + f\text{C}$, alterius eadem quantitas $g\text{A} + h\text{B} + k\text{C}$, & eadem tertiæ $l\text{A} + m\text{B} + n\text{C}$ ubi A, B, & C denotent res miſtas, & d, e, f, g, h, &c. Proportiones earundem in miſturis. Et ſit $p\text{A} + q\text{B} + r\text{C}$ miſtura, quam ex his tribus oportet componere; fingeque x, y & z numeros eſſe, per quos ſi tres datæ miſturæ reſpective multiplicentur, earum ſumma evadet $p\text{A} + q\text{B} + r\text{C}$.

$$\text{Eſt itaque} \left.\begin{matrix} dx\text{A} + ex\text{B} + fx\text{C} \\ + gy\text{A} + hy\text{B} + ky\text{C} \\ + lx\text{A} + mz\text{B} + nz\text{C} \end{matrix}\right\} = p\text{A} + q\text{B} + r\text{C},$$

Adeoque, collatis terminis, $dx + gy + lz = p$; $ex + hy + mz = q$; & $fx + ky + nz = r$; & per reductionem $x = \frac{p - gy - lz}{d} = \frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$. Et rurſus æquationes $\frac{p - gy - lz}{d} = \frac{q - hy - mz}{e}$, & $\frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$ per reductionem dant $\frac{ep - dq + dmz - elz}{eg - dh} \ (= y) = \frac{fq - er + erz - fmz}{fh - ek}$: Quæ, ſi abbrevietur ſcribendo α pro $ep - dq$, β pro $dm - el$, γ pro $eg - dh$, δ pro $fq - er$, ζ pro $en - fm$, & θ pro $fh - ek$, evadet $\frac{\alpha + \beta z}{\gamma} = \frac{\delta + \zeta z}{\theta}$; & per reductionem $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = x$. Invento z, pone $\frac{\alpha + \beta z}{\gamma} = y$, & $\frac{p - gy - lz}{d} = x$.

CAPUT XII. EXEMPL. Si tres ſint metallorum colliquefactorum miſturæ, quarum primæ pondo continet argenti ℥ 12, æris ℥ 1, & ſtanni ℥ 3, ſecundæ pondo continet argenti ℥ 1, æris ℥ 12, & ſtanni ℥ 3, & tertiæ pondo continet æris ℥ 14, ſtanni ℥ 2, & argenti nihil; ſintque hæ miſturæ ita componendæ, ut pondo compoſitionis contineat argenti ℥ 4, æris ℥ 9, & ſtanni ℥ 3: pro d, e, f; g, h, k; l, m, n; p, q, r ſcribe 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3 reſpectivè, & erit α $(=ep-dq=1\times 4-12\times 9)=-104$, & β $(=dm-el=12\times 14-1\times 0)=168$, & ſic $\gamma=-143$, $\delta=24$, $\zeta=-40$, & $\theta=33$. Adeoque z $\left(=\frac{\theta\alpha-\gamma\delta}{\gamma\zeta-\beta\theta}=\frac{-3432+3432}{5720-5544}\right)=0$, y $\left(=\frac{\alpha+\beta z}{\gamma}=\frac{-104+0}{-143}\right)=\frac{8}{11}$, & x $\left(=\frac{p-gy-lz}{d}=\frac{4-\frac{8}{11}}{12}\right)=\frac{3}{11}$. Quare ſi miſceantur $\frac{8}{11}$ partes pondo miſturæ ſecundæ, $\frac{3}{11}$ partes pondo primæ, & nihil tertiæ, aggregatum erit pondo continens quatuor uncias argenti, novem æris, & tres ſtanni.

PROB. IX.

Datis plurium ex iiſdem rebus miſturarum pretiis, & proportionibus miſtorum inter ſe, pretium cujuſvis è miſtis determinare.

Cujuſvis rerum A, B, C, miſturæ, $dA+gB+lC$, pretium eſto p; miſturæ $eA+hB+mC$ pretium q; & miſturæ $fA+kB+nC$ pretium r; & rerum illarum A, B, C quærantur pretia x, y & z. Utpote pro rebus A, B, & C ſubſtitue earum pretia x, y & z, & exurgent æquationes $dx+gy+lz=p$, $ex+hy+mz=q$, & $fx+ky+nz=r$, ex quibus pergendo ut in præcedente Problemate, elicientur itidem $\frac{\theta\alpha-\gamma\delta}{\gamma\zeta-\beta\theta}=z$; $\frac{\alpha+\beta z}{\gamma}=y$; & $\frac{p-gy-lz}{d}=x$.

EXEMPL. Emit quidam 40 modios tritici, 24 modios hordei, & 20 modios avenæ ſimul 15 libris 12 ſolidis; Deinde conſimilis grani emit 26 modios tritici, 30 modios hordei, & 50 modios avenæ ſimul 16 libris: ac tertiò conſimilis etiam grani emit 24 modios tritici, 120 modios hordei, & 100 modios avenæ ſimul 34 lib. Quæritur quanti æſtimandus ſit modius cujuſque grani? Reſp. Modius tritici 5 ſolidis, hordei 3 ſolidis, & avenæ 2 ſolidis. Nam pro d, g, l; e, h, m; f, k, n; p, q, & r ſcribendo reſpectivè 40, 24, 20; 26, 30, 50; 24, 120, 100; $15\frac{3}{5}$, 16, & 34; prodit

prodit $\alpha\,(=ep-dq=26\times 15\frac{3}{5}-40\times 16)=-234\frac{2}{5}$; & $\beta\,(=dm-el$ *Quæstiones Arithmeticæ.* $=40\times 50-26\times 20)=1480$. Atque ita $\gamma=-576$, $\delta=-500$, $\zeta=1400$, & $\theta=-2400$. Adeoque $z\,(=\frac{\theta\alpha-\gamma\delta}{\gamma\zeta-\beta\theta}=\frac{562560-288000}{-806400+3552000}=\frac{274560}{2745600})=\frac{1}{10}$, $y\,(=\frac{\alpha+\beta z}{\gamma}=\frac{-234\frac{2}{5}+148}{-567})=\frac{3}{20}$. Et $x\,(=\frac{p-gy-lz}{d}=\frac{15\frac{3}{5}-\frac{18}{5}-2}{40})=\frac{1}{4}$. Conſtitit itaque modius tritici $\frac{1}{4}$ lb ſeu 5 ſolidis, modius hordei $\frac{3}{20}$ lb ſeu 3 ſolidis, & modius avenæ $\frac{1}{10}$ lb ſeu 2 ſolidis.

PROB. X.

Datis & miſturæ & miſtorum gravitatibus ſpecificis invenire proportionem miſtorum inter ſe.

Sit e gravitas ſpecifica miſturæ A + B, cujus A gravitas ſpecifica eſt a, & B gravitas b; & cùm gravitas abſoluta, ſeu pondus, componatur ex mole corporis & gravitate ſpecificâ, erit a A pondus ipſius A, b B pondus ipſius B, & e A + e B pondus aggregati A + B; adeoque aA + bB = eA + eB, indeque aA − eA = eB − bB, ſeu $e-b.\ a-e$ :: A. B.

Exempl. Sit auri gravitas ut 19, argenti ut $10\frac{1}{3}$, & Coronæ Hieronis ut 17; eritque 10. 3 (:: $e-b.\ a-e$ :: A. B) :: moles auri in coronâ, ad molem argenti: vel 190. 31 (:: 19×10. $10\frac{1}{3}\times 3$ :: $a\times\overline{e-b}.\ b\times\overline{a-e}$) :: pondus auri in coronâ, ad pondus argenti: & 221. 31 :: pondus coronæ, ad pondus argenti.

PROB. XI.

Si boves a depaſcant pratum b in tempore c; & boves d depaſcant pratum æquè bonum e in tempore f, & gramen uniformiter creſcat: Quæritur quot boves depaſcent pratum ſimile g in tempore h.

Si boves a in tempore c depaſcant pratum b; tum, per analogiam, boves $\frac{e}{b}a$ in eodem tempore c, vel boves $\frac{ec}{bf}a$ in tempore f, vel boves $\frac{ec}{bh}a$ in tempore h, depaſcent pratum e: puta ſi

 gramen

CAPUT XII. gramen poſt tempus c non creſceret. Sed cùm propter graminis incrementum boves d in tempore f, depaſcant ſolummodo pratum e, ideo graminis in prato e incrementum illud per tempus $f-c$ tantum erit, quantum per ſe ſufficit paſcendis bobus $d - \frac{eca}{bf}$ per tempus f, hoc eſt, quantum ſufficit paſcendis bobus $\frac{df}{h} - \frac{eca}{bh}$ per tempus h. Et in tempore $h-c$, per analogiam, tantum erit incrementum, quantum per ſe ſufficit paſcendis bobus $\frac{h-c}{f-c}$ in $\frac{df}{h} - \frac{eca}{bh}$, ſive $\frac{bdfh - ecah - bdcf + aecc}{bfh - bch}$. Hoc incrementum adjice bobus $\frac{aec}{bh}$, & prodibit $\frac{bdfh - ecah - bdcf + ecfa}{bfh - bch}$ numerus boum quibus paſcendis ſufficit pratum e per tempus h. Adeoque per analogiam pratum g bobus $\frac{bdfgh - ecagh - bdcgf + ecfga}{befh - bceh}$ per idem tempus h paſcendis ſufficiet.

EXEMPL. Si 12 boves depaſcant $3\frac{1}{3}$ jugera prati in 4 ſeptimanis; & 21 boves depaſcant 10 jugera conſimilis prati in 9 ſeptimanis; quæritur quot boves depaſcant 24 jugera in 18 ſeptimanis? Reſp. 36. Iſte enim numerus invenietur ſubſtituendo in $\frac{bdfgh - ecagh - bdcgf + ecfga}{befh - bceh}$ numeros 12, $3\frac{1}{3}$, 4, 21, 10, 9, 24, & 18 pro literis a, b, c, d, e, f, g & h reſpectivè. Sed ſolutio fortè haud minùs expedita erit, ſi è primis principiis ad formam ſolutionis præcedentis literalis eruatur. Utpote ſi 12 boves in 4 ſeptimanis depaſcant $3\frac{1}{3}$ jugera, tum, per analogiam, 36 boves in 4 ſeptimanis, vel 16 boves in 9 ſeptimanis, vel 8 boves in 18 ſeptimanis depaſcent 10 jugera: puta ſi gramen non creſceret. Sed cùm propter graminis incrementum 21 boves in 9 ſeptimanis depaſcant ſolummodo 10 jugera, illud graminis in 10 jugeris per poſteriores 5 ſeptimanas incrementum tantum erit, quantum per ſe ſufficit exceſſui boum 21 ſupra 16, hoc eſt 5 bobus per 9 ſeptimanas, vel quod perinde eſt $\frac{5}{2}$ bobus per 18 ſeptimanas paſcendis. Et in 14 ſeptimanis (exceſſu 18 ſupra 4 primas) incrementum illud graminis, per analogiam, tantum erit quantum ſufficiat 7 bobus per 18 ſeptimanas paſcendis; eſt enim 5 ſept. 14 ſept. $\frac{5}{2}$ boves 7 boves. Quare 8 bobus, quos 10 jugera ſine incremento graminis paſcere poſſunt per 18 ſeptimanas,

timanas, adde hoſce 7 boves, quibus paſcendis ſolum incrementum graminis ſufficit, & ſumma erit 15 boves. Ac denique ſi 10 jugera 15 bobus per 18 ſeptimanas paſcendis ſufficiant, tum, per analogiam, 24 jugera per idem tempus ſufficient 36 bobus.

QUÆSTIONES ARITHMETICÆ.

PROB. XII.

Datis ſphæricorum corporum in eâdem rectâ motorum, ſibique occurrentium, magnitudinibus & motibus, determinare motus eorundem poſt reflexionem.

Hujus reſolutio ex his dependet conditionibus, ut corpus utrumque tantum reactione patiatur quantum agit in alterum, & ut eâdem celeritate poſt reflexionem recedant ab invicem, quâ ante accedebant. His poſitis ſint corporum A & B celeritates a & b reſpectivè; & motus (ſiquidem componantur ex mole & celeritate corporum) erunt aA & bB. Et ſi corpora ad eaſdem plagas tendant, & A celerius movens inſequatur B, pone x decrementum motûs aA, & incrementum motûs bB, percuſſione exortum; & poſt reflectionem motus erunt $aA-x$, & $bB+x$; & celeritates $\frac{aA-x}{A}$ ac $\frac{bB+x}{B}$; quarum differentia æquatur $a-b$ differentiæ celeritatum ante reflexionem. Habetur itaque æquatio $\frac{bB+x}{B}-\frac{aA-x}{A}=a-b$, & inde per reductionem fit $x=\frac{2aAB-2bAB}{A+B}$; quo pro x in celeritatibus, $\frac{aA-x}{A}$, & $\frac{bB+x}{B}$, ſubſtituto, prodeunt $\frac{aA-aB+2bB}{A+B}$ celeritas ipſius A, & $\frac{2aA-bA+bB}{A+B}$ celeritas ipſius B, poſt reflexionem.

Quòd ſi corpora obviam eant, tum ſigno ipſius b ubique mutato, celeritates poſt reflectionem erunt $\frac{aA-aB-2bB}{A+B}$ & $\frac{2aA+bA-bB}{A+B}$: Quarum alterutra ſi fortè negativa obvenerit, id arguit motum illum poſt reflexionem ad plagam dirigi ei contrariam, ad quam A tendebat ante reflexionem. Id quod etiam de motu ipſius A in caſu priori intelligendum eſt.

EXEMPL.

EXEMPL. Si corpora homogenea A trium librarum cum celeritatis gradibus 8, & B novem librarum cum celeritatis gradibus 2 ad eaſdem plagas tendant: tunc pro A, a, B & b ſcribe 3, 8, 9 & 2; & $\left(\frac{aA-aB+2bB}{A+B}\right)$ evadit -1, ac $\left(\frac{2aA-bA+bB}{A+B}\right)$ 5. Recedet itaque A cum uno gradu celeritatis poſt reflexionem, & B cum quinque gradibus progredietur.

PROB. XIII.

Invenire tres numeros continuè proportionales quorum ſumma ſit 20, *&* *quadratorum ſumma* 140.

Pone numerorum primum x, & ſecundum y; eritque tertius $\frac{yy}{x}$, adeoque $x+y+\frac{yy}{x}=20$; & $xx+yy+\frac{y^4}{xx}=140$. Et per reductionem $xx\,{}^{+y}_{-20}\,x+yy=0$, & $x^4\,{}^{+yy}_{-140}\,xx+y^4=0$. Jam ut exterminetur x, pro a, b, c, d, e, f, g & h, in Reg. 3. ſubſtitue reſpectivè $1,\ 0,\ yy-140,\ 0,\ y^4;\ 1,\ y-20,$ & yy; et emerget $\overline{-yy+280}\times y^6:+\overline{2yy-40y+260}\times\overline{260y^4-40y^5}:+3y^4\times y^4:-2yy\times\overline{y^6-40y^5+400y^4}:=0$. Et, per multiplicationem, $1600y^6+20800y^5-67600y^4=0$. Ac reducendo, $4yy-52y+169=0$. Sive (radice extractâ) $2y-13=0$, ſeu $y=6\frac{1}{2}$. Id quod etiam breviùs aliâ methodo ſed minùs obviâ ſuprà inventum eſt. Porro, ut inveniatur x, ſubſtitue $6\frac{1}{2}$ pro y in æquatione $xx\,{}^{+y}_{-20}\,x+yy=0$. Et exurget $xx-13\frac{1}{2}x+42\frac{1}{4}=0$, ſeu $xx=13\frac{1}{2}x+42\frac{1}{4}$. Et, extractâ radice, $x=6\frac{3}{4}+$ vel $-\sqrt{3\frac{5}{16}}$. Nempe $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$ eſt maximus quæſitorum trium numerorum, & $6\frac{3}{4}-\sqrt{3\frac{5}{16}}$ minimus. Nam x alterutrum extremorum numerorum ambiguè deſignat, indeque gemini prodeunt valores, quorum alteruter poteſt eſſe x, exiſtente altero $\frac{yy}{x}$.

Idem aliter. Poſitis numeris x, y & $\frac{yy}{x}$, ut antè, erit $x+y+\frac{yy}{x}=20$, ſeu $xx={}^{20}_{-y}\,x-yy$; & extractâ radice, $x=10-\frac{1}{2}y+\sqrt{100-10y-\frac{3}{4}yy}$ primus numerus: Hunc & y aufer de 20, & reſtat $\frac{yy}{x}=10-\frac{1}{2}y-\sqrt{100-10y-\frac{3}{4}yy}$ tertius numerus. Eſtque ſumma quadratorum a tribus hiſce numeris $400-40y$, adeoque $400-40y$

I =

$= 140$, ſive $y = 6\frac{1}{2}$. Invento medio numero $6\frac{1}{2}$, ſubſtitue eum pro y in primo ac tertio numero ſuprà invento; & evadet primus $6\frac{3}{4} + \sqrt{3\frac{5}{16}}$, ac tertius $6\frac{3}{4} - \sqrt{3\frac{5}{16}}$, ut antè. QUÆSTIONES ARITHMETICÆ.

PROB. XIV.

Invenire quatuor numeros continuè proportionales quorum duo medii ſimul conſtituant 12, *& duo extremi* 20.

Sit x ſecundus numerus; & erit $12 - x$ tertius; $\frac{xx}{12-x}$ primus; & $\frac{144-24x+xx}{x}$ quartus; adeoque $\frac{xx}{12-x} + \frac{144-24x+xx}{x} = 20$. Et, per reductionem, $xx = 12x - 30\frac{6}{7}$, ſeu $x = 6 + \sqrt{5\frac{1}{7}}$. Quo invento cæteri numeri è ſuperioribus dantur.

PROB. XV.

Invenire quatuor numeros continuè proportionales, quorum datur ſumma a, & ſumma quadratorum b.

Etſi deſideratas quantitates ut plurimùm immediatè quærere ſolemus, ſiquando tamen duæ obvenerint ambiguæ, hoc eſt, quæ conditionibus omnino ſimilibus præditæ ſunt, (ut hîc duo medii & duo extremi numerorum quatuor proportionalium) præſtat alias quantitates non ambiguas quærere, per quas hæ determinantur: quemadmodum harum ſummam, vel differentiam, vel rectangulum. Ponamus ergo ſummam duorum mediorum numerorum eſſe s, & rectangulum r; & erit ſumma extremorum $a - s$, & rectangulum etiam r, propter proportionalitatem. Jam ut ex his eruantur quatuor illi numeri, pone x primum & y ſecundum; eritque $s - y$ tertius; & $a - s - x$ quartus; & rectangulum ſub mediis $sy - yy = r$; indeque medii, $y = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - r}$, & $s - y = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - r}$. Item rectangulum ſub extremis $ax - sx - xx = r$; indeq; extremi, $x = \frac{a-s}{2} + \sqrt{\frac{ss - 2as + aa}{4} - r}$, & $a - s - x = \frac{a-s}{2} - \sqrt{\frac{ss-2as+aa}{4} - r}$.

Summa

CAPUT XII. Summa quadratorum ex hiſce quatuor numeris eſt $2ss-2as+aa-4r$, quæ eſt $=b$. Ergo $r=\frac{1}{2}ss-\frac{1}{2}as+\frac{1}{4}aa-\frac{1}{4}b$, quo ſubſtituto pro r prodeunt quatuor numeri ut ſequitur.

Duo medii $\begin{cases}\frac{1}{2}s+\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}.\\ \frac{1}{2}s-\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}.\end{cases}$

Duo extremi $\begin{cases}\frac{a-s}{2}+\sqrt{\frac{1}{4}b-\frac{1}{4}ss}.\\ \frac{a-s}{2}-\sqrt{\frac{1}{4}b-\frac{1}{4}ss}.\end{cases}$

Reſtat tamen etiamnum inquirendus valor ipſius s. Quare ad abbreviandos terminos pro numeris hiſce ſubſtitue.

$\frac{1}{2}s+p.$ & $\frac{a-s}{2}+q.$

$\frac{1}{2}s-p.$ & $\frac{a-s}{2}-q.$

Et pone rectangulum ſub ſecundo & quarto æquale quadrato tertii, ſiquidem hæc problematis conditio nondum impleatur; eritque $\frac{as-ss}{4}-\frac{1}{2}qs+\frac{pa-ps}{2}-pq=\frac{1}{4}ss-ps+pp$. Pone etiam rectangulum ſub primo & tertio æquale quadrato ſecundi; & erit $\frac{as-ss}{4}+\frac{1}{2}qs\frac{-pa+ps}{2}-pq=\frac{1}{4}ss+ps+pp$. Harum æquationum priorem aufer è poſteriori, & reſtabit $qs-pa+ps=2ps$, ſeu $qs=pa+ps$. Reſtitue jam $\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$ in locum p, & $\sqrt{\frac{1}{4}b-\frac{1}{4}ss}$ in locum q, & habebitur $s\sqrt{\frac{1}{4}b-\frac{1}{4}ss}=\overline{a+s}\times\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$. Et quadrando $ss=-\frac{b}{a}s+\frac{1}{2}aa-\frac{1}{2}b$, ſeu $s=-\frac{b}{2a}+\sqrt{\frac{bb}{4aa}+\frac{1}{2}aa-\frac{1}{2}b}$; quo invento, dantur quatuor numeri quæſiti è ſuperioribus.

PROB. XVI.

Si penſio annua librarum a per quinque annos proximè ſequentes ſolvenda, ematur paratâ pecuniâ c, quæritur quanti æſtimanda ſit uſura uſuræ centum librarum per annum.

Pone $1-x$ uſuram uſuræ pecuniæ x in anno, hoc eſt quod pecunia 1 poſt annum ſolvenda valeat x paratæ pecuniæ; &, per analogiam, pecunia a poſt annum ſolvenda valebit ax paratæ pecuniæ,

niæ, poſt duos annos, axx; poſt tres, ax^3; poſt quatuor, ax^4; & poſt quinque, ax^5. Adde jam hos quinque terminos, & erit $ax^5 + ax^4 + ax^3 + axx + ax = c$, ſeu $x^5 + x^4 + x^3 + xx + x = \frac{c}{a}$, æquatio quinque dimenſionum; cu*jus ope cùm x, per regulas poſt docendas, inventum fuerit*, pone $x . 1 :: 100 . y$. Et erit $y - 100$ uſura uſuræ centum librarum per annum.

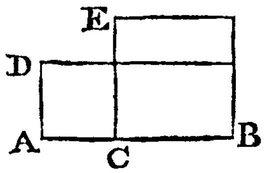
*Nempe inveniendo figuras primas radicis per conſtructionem quamvis mechanicam & reliquas per methodum Vietæ.

Atque has, in quæſtionibus ubi ſolæ quantitatum proportiones abſque poſitionibus linearum conſiderandæ veniunt, inſtantias dediſſe ſufficiat: Pergamus jam ad Problematum Geometricorum ſolutiones.

CAPUT DECIMUM TERTIUM.

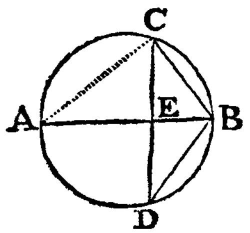
Quomodo Quæſtiones Geometricæ ad æquationem redigantur.

Quæſtiones *Geometricæ* eâdem facilitate iiſdemque legibus ad æquationes nonnunquam redigi poſſunt, ac quæ de abſtractis quantitatibus proponuntur. Ut ſi recta AB in extremâ & mediâ proportione ſecanda ſit in C, hoc eſt, ita ut BE, quadratum maximæ partis, ſit æquale rectangulo, BD, ſub totâ & minore parte contento: poſito $AB = a$, & $BC = x$, erit $AC = a - x$, & $xx = a$ in $a - x$; æquatio quæ per reductionem dat $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}aa}$.



Sed in rebus Geometricis quæ frequentiùs occurrunt, à variis linearum poſitionibus & relationibus complexis ita dependere ſolent, ut egeant ulteriori inventione & artificio, quo ad Algebraicos terminos deduci poſſint. Et licet in hujuſmodi caſibus difficile ſit aliquid præſcribere, & cujuſque ingenium ſibi debeat eſſe operandi norma: conabor tamen diſcentibus viam præſternere. Sciendum eſt itaque, quòd quæſtiones circa eaſdem lineas, definito quolibet modo ſibi invicem relatas, poſſint variè proponi, ponendo alias atque alias quærendas eſſe ex aliis atque aliis datis. Sed de quibuſcunque tamen datis vel quæſitis inſtituitur quæſtio, ſolutio ejus eâdem planè methodo ex Analyſeos ſerie perficietur, nullâ omninò circumſtantiâ variatâ præter fictas linearum ſpecies,

CAPUT XIII. ſive nomina, quibus datas à quæſitis ſolemus diſtinguere. Quemadmodum ſi quæſtio ſit de Iſoſcele CBD in circulum inſcripto,



cujus latera BC, BD, & baſis CD cum diametro circuli AB conferenda ſunt: ea vel proponi poteſt de inveſtigatione *diametri* ex datis lateribus & baſi, vel de inveſtigatione *baſis* ex datis lateribus & diametro, vel denique de inveſtigatione *laterum* ex datis baſi & diametro: Sed utcunque proponitur, redigetur ad æquationem per eandem ſeriem Analyſeos. Nempe ſi quæratur *Diameter* pono AB $= x$, CD $= a$, & BC vel BD $= b$. Tum, ductâ AC, propter ſimilia triangula ABC & CBE, eſt AB. BC : : BC. BE, ſive $x . b :: b .$ BE. Quare BE $= \frac{bb}{x}$. Eſt & CE $= \frac{1}{2}$CD, ſive $\frac{1}{2}a$: et, propter angulum CEB rectum, CE$q+$ BE$q=$BCq; hoc eſt, $\frac{1}{4}aa + \frac{b^4}{xx} = bb$. Quæ æquatio, per reductionem, dabit quæſitum x.

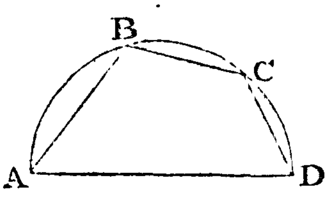
Sin quæratur *Baſis*, pono AB $= c$, CD $= x$, & BC vel BD $= b$. Tum (ductâ AC) propter ſimilia triangula ABC & CBE, eſt AB. BC : : BC. BE, ſive $c . b :: b .$BE. Quare BE $= \frac{bb}{c}$. Eſt & CE $= \frac{1}{2}$CD, ſive $\frac{1}{2}x$: &, propter angulum CEB rectum, CE$q+$BE$q=$BCq; hoc eſt, $\frac{1}{4}xx + \frac{b^4}{cc} = bb$; æquatio quæ per reductionem dabit quæſitum x.

Atque ita ſi *Latus* BC vel BD quæratur, pono AB $= c$, CD $= a$, & BC vel BD $= x$. Et (AC ut antè ductâ) propter ſimilia triangula ABC & CBE, eſt AB. BC : : BC. BE; ſive $c . x :: x .$ BE. Quare BE $= \frac{xx}{c}$. Eſt & CE $= \frac{1}{2}$ CD, ſive $\frac{1}{2}a$; &, propter angulum CEB rectum, eſt CE$q+$BE$q=$BCq; hoc eſt, $\frac{1}{4}aa + \frac{x^4}{cc} = xx$; æquatio quæ per reductionem dabit quæſitum x.

Vides itaque quòd in unoquoque caſu calculus, quo perveniatur ad æquationem, per omnia ſimilis ſit, & eandem æquationem pariat, excepto tantùm quòd lineas aliis atque aliis literis deſignavi, prout datæ vel quæſitæ ponuntur. Ex diverſis quidem datis & quæſitis oritur diverſitas in reductione æquationis inventæ: Nam æquationis, $\frac{1}{4}aa + \frac{b^4}{xx} = bb$, alia eſt reductio, ut obtineatur $x = \frac{2bb}{\sqrt{4bb-aa}}$, valor de AB; & æquationis, $\frac{1}{4}xx + \frac{b^4}{cc} = bb$, alia reductio,

ductio, ut obtineatur $x = \frac{2b}{c}\sqrt{cc-bb}$, valor de CD; & æqua- De Quæst. Geometr.
tionis, $\frac{1}{4}aa + \frac{a^4}{cc} = xx$, reductio longè alia, ut obtineatur $x = \sqrt{\frac{1}{2}cc \pm \frac{1}{2}c\sqrt{cc-aa}}$, valor de BC vel BD: perinde ut hæc $\frac{1}{4}aa + \frac{b^4}{cc} = bb$, ad eliciendum c, a, vel b diverſis modis reduci debet: ſed in harum æquationum inventione nulla fuit diverſitas. Et hinc eſt quod jubent, ut nullum inter datas & quæſitas quantitates habeatur diſcrimen. Nam cùm eadem computatio cuique caſui datorum & quæſitorum competat, convenit, ut ſine diſcrimine concipiantur & conferantur, quo rectiùs judicetur de modis computandi: vel potiùs convenit, ut fingas quæſtionem de ejuſmodi datis & quæſitis propoſitam eſſe, per quas arbitreris te poſſe ad æquationem facillimè pervenire.

Propoſito igitur aliquo Problemate, quantitates quas involvit confer, & nullo inter datas & quæſitas habito diſcrimine, perpende quomodo aliæ ex aliis dependeant, ut cognoſcas, quænam, ſi aſſumantur, ſyntheticè gradiendo dabunt cæteras. Ad quod faciendum, non opus eſt, ut primâ fronte de modo cogites, quo aliæ ex aliis per calculum Algebraicum deduci poſſint; ſed ſufficit animadverſio generalis, quòd poſſint directo nexu quomodocunque deduci. Verbi gratiâ; ſi quæſtio ſit de circuli diametro AD, tribuſque lineis AB, BC, & CD in ſemicirculo inſcriptis, & ex reliquis datis quæratur BC; primo intuitu manifeſtum eſt diametrum AD determinare ſemicirculum; dein lineas AB & CD per inſcriptionem determinare puncta B & C, atque adeo quæſitum BC, idque nexu maximè directo; & quo pacto tamen BC ex his datis



per Analyſin eruatur, non ita manifeſtum eſt. Hoc idem quoque de AB vel CD, ſi ex reliquis datis quærerentur, intelligendum eſt. Quòd ſi AD ex datis AB, BC & CD quæreretur, æquè patet id non fieri poſſe Syntheticè; ſiquidem punctorum A ac D diſtantia dependet ex angulis B & C; & illi anguli ex circulo cui datæ lineæ ſunt inſcribendæ; & ille circulus non datur, ignotâ AD diametro. Rei igitur naturâ poſtulat, ut AD non Syntheticè, ſed ex ejus aſſumptione, quæratur, ut ad data fiat regreſſus.

CAPUT XIII. Cùm varios ordines, quibus termini quæſtionis ſic evolvi poſſint, perſpexeris, *E ſyntheticis quoſlibet adhibe, aſſumendo lineas tanquam datas à quibus ad alias facillimus videtur progreſſus, & ad ipſas viciſſim difficillimus.* Nam computatio, ut per varia media poſſit incedere, tamen ab iſtis lineis initium ſumet; ac promptiùs perficietur, fingendo quæſtionem ejuſmodi eſſe, ac ſi de iſtis datis & quæſito aliquo, ab iſtis facillimè prodituro, inſtitueretur, quàm de quæſtione, prout reverâ, proponitur cogitando. Sic in exemplo jam allato, ſi ex reliquis datis quæritur AD; cùm id ſynthetice fieri non poſſe percipiam, ſed ab ipſo tamen, ſi modò daretur, diſcurſum ad alia directo nexu incedere, aſſumo AD tanquam datum; & abinde computationem non ſecùs incipio, quàm ſi reverâ daretur, & aliqua ex datis, AB, BC & CD, quæreretur. Atque hâc methodo, computationem ab aſſumptis ad cæteras quantitates eo more promovendo quo linearum relationes dirigunt, æquatio tandem inter duos ejuſdem alicujus quantitatis valores ſemper obtinebitur; ſive ex valoribus unus ſit litera, ſub initio operis, quantitati pro nomine impoſita, & alter per computationem inventus; ſive uterque per computationem diverſimodè inſtitutam inveniatur.

Cæterùm ubi terminos quæſtionis ſic in genere comparaveris, plus artis & inventionis in eo requiritur, ut advertas particulares iſtos nexus, ſive linearum relationes, quæ computationi accommodantur. Nam quæ laxiùs perpendenti videantur immediatè & relatione proximâ connecti, cùm illam relationem algebraicè deſignare volumus, circuitum plerumque, quoad conſtructiones Schematum de novo moliendas & computationem per gradus promovendam, exigunt: Quemadmodum de BC ex AD, AB & CD colligendo conſtare poteſt. Per ejuſmodi enim propoſitiones vel enunciationes ſolummodò gradiendum eſt, quæ aptæ ſunt ut terminis algebraicis deſignentur, quales præſertim ab Axiom. 19, Prop. 4, lib. 6, & Prop. 47, lib. 1. Elem. proveniunt.

Imprimis itaque promovetur calculus per additionem vel ſubductionem linearum, eò ut ex valoribus partium obtineatur valor totius, vel ex valoribus totius & unius partis obtineatur valor alterius.

Secundò promovetur ex linearum proportionalitate: ponimus enim,

enim, ut ſuprà, factum à mediis terminis diviſum per alterutrum extremorum eſſe valorem alterius. Vel quod perinde eſt, ſi valores omnium quatuor proportionalium priùs habeantur, ponimus æqualitatem inter factos extremorum & factos mediorum. Linearum verò proportionalitas ex triangulorum ſimilitudine maximè ſe prodit; quæ cùm ex æqualitate angulorum dignoſcatur, in iis comparandis Analyſta debet eſſe perſpicax, atque adeo non ignorabit Prop. 5, 13, 15, 29, & 32. lib. 1. Prop. 4, 5, 6, 7, & 8, lib. 6. Et Prop. 20, 21, 22, 27 ac 31, lib. 3. Elementorum. Quibus etiam referri poteſt Prop. 3, lib. 6, ubi ex proportionalitate linearum colligitur angulorum æqualitas, & contrà. Atque idem aliquandò præſtant Prop. 35 & 36, lib. 3. De Quæst. Geometr.

Tertiò promovetur per additionem vel ſubductionem quadratorum. In triangulis nempe rectangulis addimus quadrata minorum laterum, ut obtineatur quadratum maximi; vel à quadrato maximi lateris ſubducimus quadratum unius è minoribus, ut obtineatur quadratum alterius.

Atque his paucis fundamentis (ſi adnumeretur Prop. 1, lib. 6, Elem. cùm de ſuperficiebus agitur, ut & aliquæ propoſitiones ex lib. 11 & 12, deſumptæ cùm agitur de ſolidis) tota Ars Analytica, quoad Geometriam rectilineam, innititur. Quin etiam ad ſolas linearum ex partibus compoſitiones & ſimilitudines triangulorum poſſunt omnes Problematum difficultates reduci; adeo ut non opus ſit alia Theoremata adhibere: quippe quæ omnia in hæc duo reſolvi poſſunt, & proinde ſolutiones etiam quæ ex iſtis depromuntur. Inque hujus rei inſtantiam ſubjunxi Problema de perpendiculo in baſem obliquanguli trianguli demittendo ſine adjumento Prop. 47. lib. 1. ſolutum. Etſi verò juvet ſimpliciſſima principia à quibus problematum ſolutiones dependet non ignoràſſe, & iſtis ſolis adhibitis poſſe quælibet ſolvere; expeditionis tamen gratiâ convenit, ut non ſolùm Prop. 47. lib. 1. Elem. cujus uſus eſt frequentiſſimus; ſed & alia etiam *Theoremata* nonnunquam adhibeantur.

Quemadmodum ſi, perpendiculo in baſem obliquanguli trianguli demiſſo, de ſegmentis baſis ad calculum promovendum agatur; ex uſu erit ſcire, quòd Differentia quadratorum è lateribus æquetur duplo rectangulo ſub baſi & diſtantiâ perpendiculi à medio baſis.

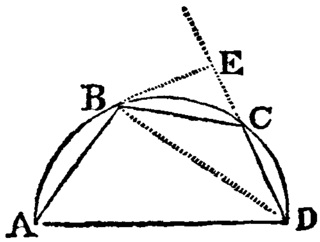
 Si trianguli alicujus verticalis angulus biſecetur, computationi non ſolum inſerviet, quòd baſis ſecetur in ratione laterum, ſed etiam quòd differentia factorum à lateribus & à ſegmentis baſis æquetur quadrato lineæ biſecantis angulum.

Si de figuris in circulo inſcriptis res eſt, Theorema non raró ſubveniet, quòd Inſcripti cujuſlibet quadrilateri factus à diagoniis æquetur ſummæ factorum à lateribus oppoſitis.

Et hujuſmodi plura inter exercendum obſervet Analyſta, & in penum fortè reſervet; ſed parciùs utatur, ſi pari facilitate, aut non multo difficiliùs, poſſit ſolutionem è ſimplicioribus computandi principiis extruere. Quamobrem ad tria primò propoſita tanquam notiora, ſimpliciora, magis generalia, pauca, & omnibus tamen ſufficientia animum præſertim advertat, & omnes difficultates ad ea præ cæteris reducere conetur.

Sed ut hujuſmodi Theoremata ad ſolvenda Problemata accommodari poſſint, *Schemata* plerumquè ſunt ultrà *conſtruenda*, idque ſæpiſſimè producendo aliquas ex lineis donec ſecent alias, aut ſint aſſignatæ longitudinis; vel ab inſigniori quolibet puncto ducendo lineas aliis parallelas aut perpendiculares, vel inſigniora puncta conjungendo, ut & aliter nonnunquam conſtruendo, prout exigunt ſtatus Problematis, & Theoremata quæ ad ejus ſolutionem adhibentur. Quemadmodum ſi duæ non concurrentes lineæ datos angulos cum tertiâ quàdam efficiant, producimus fortè, ut concurrentes conſtituant triangulum, cujus anguli, & proinde laterum rationes, dantur. Vel ſi quilibet angulus detur, aut ſit alicui æqualis, in triangulum ſæpè complemus ſpecie datum, aut alicui ſimile, idque vel producendo aliquas ex lineis in ſchemate, vel ſubtenſam aliter ducendo. Si triangulum ſit obliquangulum, in duo rectangula ſæpè reſolvimus, demittendo perpendiculum. Si de figuris multilateris agatur, reſolvimus in triangula, ducendo lineas diagonales: et ſic in cæteris; ad hanc metam ſemper collimando, *ut ſchema in triangula vel data, vel ſimilia, vel rectangula, reſolvatur*. Sic in exemplo propoſito duco diagonium BD, ut Trapezium ABCD in duo triangula, ABD rectangulum, & BDC obliquangulum, reſolvatur. Deinde reſolvo triangulum obliquangulum in duo rectangula, demittendo perpendiculum à quolibet

ejus



ejus angulo, B, C, vel D, in latus oppoſitum : quemadmodum à B in CD, productam ad E, ut huic perpendiculo BE occurrat. Interea verò cùm anguli BAD & BCD duos rectos (per 22. 3. Elem.) perindè ac BCE & BCD conſtituant; percipio angulos BAD & BCE æquales eſſe, adeoque triangula BCE ac DAB ſimilia. Atque ita video computationem, aſſumendo AD, AB & BC tanquam ſi CD quæreretur, ad hunc modum inſtitui poſſe; viz. AD & AB (propter tri. ABD rect.) dant BD. AD, AB, BD & BC, propter ſim. tri. ABD & CEB, dant BE & CE. BD & BE, propter triang. BED rect. dant ED; & ED − EC dat CD. Unde continebitur æquatio inter valorem de CD ſic inventum & literam pro eâ ſuffectam. Poſſumus etiam (& maximam partem ſatiùs eſt quàm opus in ſerie continuatâ nimis proſequi,) à diverſis principiis computationem incipere, aut ſaltem diverſis modis ad eandem quamlibet concluſionem promovere; ut duo tandem obtineantur ejuſdem cujuſvis quantitatis valores, qui æquales ponantur. Sic AD, AB & BC dant BD, BE & CE, ut priùs; deinde CD + CE dat ED; ac denique BD & ED, propter triang. rect. BED, dant BE. Poteſt etiam computatio hâc lege optimè inſtitui, ut valores quantitatum inveſtigentur, quibus alia quæpiam relatio cognita intercedit, & illa deinde relatio æquationem dabit. Sic cùm relatio inter lineas BD, DC, BC & CE ex Prop. 12, lib. 2, Elem. conſtet; nempe quòd ſit $BDq - BCq - CDq = 2CD \times CE$: Quæro BDq ex aſſumptis AD & AE; ac CE ex aſſumptis AD, AB & BC. Et, aſſumendo denique CD, facio $BDq - BCq - CDq = 2CD \times CE$. Ad hos modos & hujuſmodi conſiliis ductus, de ſerie Analyſeos, deque ſchemate propter eam conſtruendo, ſemper debes unà proſpicere.

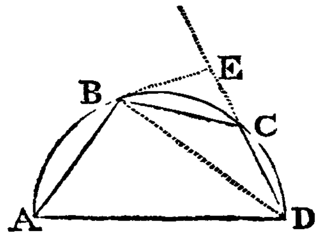
Ex his, credo, manifeſtum eſt, quid ſibi velint Geometræ, cùm jubent, putes factum eſſe quod quæris. Nullo enim inter cognitas & incognitas quantitates habito diſcrimine, quaſlibet ad ineundum calculum aſſumere potes, quaſi omnes ex prævià ſolutione fuiſſent notæ, & non ampliùs de ſolutione Problematis, ſed de probatione ſolutionis ageretur. Sic in primo ex tribus jam deſcriptis computandi modis, etſi fortè AD reverà quæratur, fingo tamen CD quærendum eſſe, quaſi vellem probare an valor ejus ab

AD

CAPUT XIII. AD derivatus quadret cum ejus quantitate prius cognitâ. Sic etiam in duobus poſterioribus modis, pro metâ non propono quantitatem aliquam quærendam eſſe, ſed æquationem è relationibus linearum utcunque eruendam: et in ejus rei gratiam aſſumo omnes, AD, AB, BC & CD, tanquam notas; perinde ac ſi, quæſtione priùs ſolutâ, de tentamine jam ageretur, an conditionibus ejus hæ probè ſatisfaciant, quadrando cum quibuſlibet æquationibus quas linearum relationes produnt. Opus quidem hâc ratione & conſiliis primâ fronte aggreſſus ſum; ſed, cùm ad æquationem deventum eſt, ſententiam muto, & quantitatem deſideratam per iſtius æquationis reductionem & ſolutionem quæro. Sic denique plures quantitates tanquam cognitas ſæpenumerò aſſumimus, quàm in ſtatu quæſtionis exprimuntur. Hujuſque rei inſignem in 55° ſequentium problematum inſtantiam videre eſt, ubi a, b & c in æquatione $aa+bx+cxx=yy$, pro determinatione Sectionis Conicæ aſſumpſi, ut & alias etiam lineas r, s, t, v de quibus Problema, prout proponitur, nihil innuit. Nam quaſlibet quantitates aſſumere licet, quarum ope poſſibile ſit ad æquationes pervenire: hoc ſolum cavendo, ut ex illis tot æquationes obtineri poſſint, quot aſſumptæ ſunt quantitates reverâ incognitæ.

Poſtquam de computandi methodo conſtat, & ornatur ſchema, quantitatibus, quæ computationem ingrediuntur (hoc eſt ex quibus aſſumptis aliarum valores derivandi ſunt, donec tandem ad æquationem perveniatur) nomina impone; delegendo quæ problematis omnes conditiones involvunt, & operi præ cæteris accommodatæ videntur, & concluſionem, quantum poſſis conjicere, ſimpliciorem reddent, ſed non plures tamen quàm propoſito ſufficiunt. Itaque pro quantitatibus, quæ ex aliarum vocabulis facilè deduci poſſint, propria vocabula vix tribuas. Sic ex totâ lineâ & ejus partibus, ex tribus lateribus trianguli rectanguli, & ex tribus vel quatuor proportionalibus, unum aliquod minimùm ſine nomine permittere ſolemus; eo quòd valor ejus è reliquorum nominibus facilè derivari poſſit. Quemadmodum in exemplo jam allato, ſi dicam $AD=x$ & $AB=a$, ipſum BD nullâ literâ deſigno; quòd ſit tertium latus trianguli rectanguli ABD, & proinde valeat $\sqrt{xx-aa}$. Dein ſi dicam $BC=b$, cùm triangula DAB & BCE ſint ſimilia, & inde lineæ AD, AB, BC, CE proportionales, quarum tribus, AD,

AB,



AB, & BC, impoſita ſunt nomina; ea propter quartam CE ſine nomine permitto, & ejus vice valorem $\frac{ab}{x}$, ex hâc proportionalitate deтectum, uſurpo. Atque ita ſi DC vocetur c, ipſi DE nomen non aſſigno; quòd ex partibus ejus DC & CE, ſive c & $\frac{ab}{x}$, valor $c + \frac{ab}{x}$ prodeat. De Quæst. Geometr.

Cæterùm dum de his moneo, Problema ad æquationem penè redactum eſt. Nam poſtquam literæ pro ſpeciebus principalium linearum præſcriptæ ſunt, nihil aliud agendum reſtat, quàm ut ex iſtis ſpeciebus valores aliarum linearum, juxta methodum præconceptam, eruantur, donec modo quovis proviſo in æquationem coeant. Et in hoc caſu nihil reſtare video, niſi ut per triangula rectangula BCE & BDE dupliciter eliciam BE. Nempe eſt BCq − CEq (ſive $bb - \frac{aabb}{xx}$) = BEq, ut & BDq − DEq) ſive $xx - aa - cc - \frac{2abc}{x} - \frac{aabb}{xx}$) = BE$q$. Et hinc, utrobique deleto $\frac{aabb}{xx}$, æquationem habebo $bb = xx - aa - cc - \frac{2abc}{x}$: quæ reducta fit

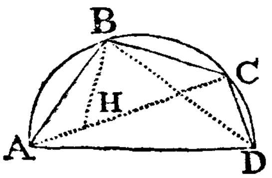
$$x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix}\, x + 2abc.$$

Cùm verò de ſolutione Problematis hujus plures modos, etſi non multum diſſimiles, in præcedentibus recenſuerim, quorum iſte, de Prop. 12, Lib. 2, Elem. deſumptus, ſit cæteris quodammodo concinnior; eundem placet etiam ſubjungere. Sit itaque AD = x, AB = a, BC = b, & CD = c, eritque BD$q = xx - aa$, & CE = $\frac{ab}{x}$, ut priùs. Hiſce dein ſpeciebus in Theorema BDq − BCq − CDq = 2CD × CE ſubſtitutis, orietur $xx - aa - bb - cc = \frac{2abc}{x}$; & factâ reductione, $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix}\, x + 2abc$; ut antè.

Sed ut pateat quanta ſit in ſolutionum inventione varietas, & proinde quòd in eas incidere prudenti Geometræ non ſit admodum difficile: viſum fuit plures adhuc modos hoc idem perficiendi docere. Atque equidem, ducto diagonio BD, ſi, vice perpendiculi BE à puncto B in latus DC ſuprà demiſſi, demittatur per-

CAPUT XIII. pendiculum à punćto D in latus BC, vel à punćto C in latus BD, quo obliquangulum triangulum BCD in duo rećtangula utcunque reſolvatur, iiſdem fermè, quas jam deſcripſi, methodis ad æquationem pervenire licet. Sunt & alii modi ab iſtis ſatis differentes.

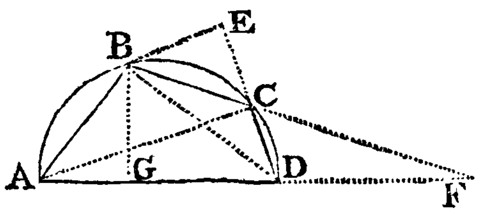
Quemadmodum ſi diagonii duo AC & BD ducantur, dabitur BD ex aſſumptis AD & AB; ut & AC ex aſſumptis AD & CD; deinde, per notum Theorema de figuris quadrilateris in circulo inſcriptis, nempe quòd ſit $AD \times BC + AB \times CD = AC \times BD$ obtinebitur æquatio. Stantibus itaque linearum AD, AB, BC, CD vocabulis x, a, b, c; erit $BD = \sqrt{xx - aa}$, & $AC = \sqrt{xx - cc}$ per 47. I. Elem. Et his linearum ſpeciebus in Theorema jam recenſitum ſubſtitutis, exibit $xb + ac = \sqrt{xx - cc} \times \sqrt{xx - aa}$. Cujus æquationis partibus denique quadratis & redućtis, obtinebitur



$$\text{iterum } x^3 = \begin{matrix} + aa \\ + bb \\ + cc \end{matrix} x + 2abc.$$

Cæterùm ut pateat etiam quo paćto ſolutiones, ex iſto Theoremate petitæ, poſſint inde ad ſolas triangulorum ſimilitudines redigi; erigatur BH ipſi BC perpendicularis & occurrens AC in H, & fient triangula BCH, BDA ſimilia, propter angulos ad B rećtos, & ad C ac D (per 21. 3. Elem.) æquales; ut & triangula BCD, BHA ſimilia, propter æquales angulos tum ad B (ut pateat demendo communem angulum DBH à duobus rećtis,) tum ad D ac A (per 21. 3. Elem.) Videre eſt itaque, quòd ex proportionalitate, BD. AD : : BC. HC, detur HC; ut & AH ex proportionalitate BD. CD : : AB. AH. Unde cùm ſit $AH + AC = AC$, habebitur æquatio. Stantibus ergo præfatis linearum vocabulis x, a, b, c, nec non ipſarum AC & BD valoribus, $\sqrt{xx - cc}$, & $\sqrt{xx - aa}$; prima proportionalitas dabit $HC = \frac{bx}{\sqrt{xx - aa}}$; & ſecunda dabit $AH = \frac{ac}{\sqrt{xx - aa}}$. Unde propter $AA + HC = AC$ erit $\frac{bx + ac}{\sqrt{xx - aa}} = \sqrt{xx - cc}$; æquatio quæ, multiplicando per $\sqrt{xx - aa}$, & quadrando, reducetur ad formam in præcedentibus ſæpiùs deſcriptam.

Adhæc ut magis pateat quanta ſit ſolvendi copia; producantur BC & AD donec conveniant in F, & fient triangula ABF & CDF ſimilia,



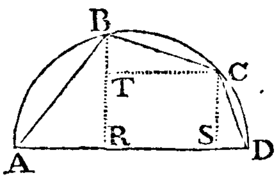
ſimilia, quippe quorum angulus ad F communis eſt, & anguli ABF & CDF (dum complent ang. CDA ad duos rectos per 13. 1 & 22. 3. Elem.) æquales. Quamobrem, ſi præter quatuor terminos de quibus inſtituitur quæſtio, daretur AF, proportio, AB. AF : : CD. CF, daret CF. Item AF − AD daret DF; & proportio, CD. DF : : AB. BF, daret BF; unde, cùm ſit BF − CF = BC, emergeret æquatio. Sed cùm duæ quantitates incognitæ, AD ac DF, tanquam datæ aſſumantur, reſtat alia æquatio invenienda. Demitto ergo BG in AF ad rectos angulos, & proportio, AD. AB : : AB. AG, dabit AG; quo habito, Theorema è 13. 2. Elem. petitum, nempe quòd ſit $BFq + 2FAG = ABq + AFq$, dabit æquationem alteram. Stantibus ergo a, b, c, x ut priùs, & dicto AF $= y$: erit, inſiſtendo veſtigiis Theoriæ jam excogitatæ, $\frac{cy}{a} =$ CF. $y - x =$ DF. $\frac{\overline{y-x} \times a}{c} =$ BF. Indeque $\frac{\overline{y-x} \times a}{c} - \frac{cy}{a} = b$, æquatio prima. Erit etiam $\frac{aa}{x} =$ AG, adeoque $\frac{aayy - 2aaxy + aaxx}{cc} + \frac{2aay}{x} = aa + yy$, æquatio ſecunda. Quæ duæ, per reductionem, dabunt æquationem deſideratam. Nempe valor ipſius y, per æquationem priorem, inventus eſt $\frac{abc + aax}{aa - cc}$; qui in ſecundam ſubſtitutus, dabit æquationem ex quâ rectè diſpoſitâ fiet

$$x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc,$$ ut antè.

Atque ita ſi AB ac DC producantur donec ſibi mutuò occurrant, ſolutio haud aliter ſe habebit, niſi fortè futura ſit paulò facilior. Quare aliud hujus rei ſpecimen, è fonte multum diſſimili petitum, potiùs ſubjungam, quærendo nempe aream quadrilateri propoſiti, idque dupliciter. Duco igitur diagonium BD, ut in duo triangula quadrilaterum reſolvatur. Dein, uſurpatis linearum vocabulis x, a, b, c ut antè, invenio BD $= \sqrt{xx - aa}$; indeque $\frac{1}{2}a\sqrt{xx-aa}$ ($= \frac{1}{2}$AB × BD) aream trianguli ABD. Porro, demiſſo BE perpendiculariter in CD, erit (propter ſimilia triangula ABD, BCE) AD. BD : : BC. BE, & proinde BE $= \frac{b}{x}\sqrt{xx-aa}$. Quare

CAPUT XIII. etiam $\frac{bc}{2x}\sqrt{xx-aa}$ ($=\frac{1}{2}$ CD $\times$ BE) erit area trianguli BCD. Haſce jam areas addendo, orietur $\frac{ax+bc}{2x}\sqrt{xx-aa}$ area totius quadrilateri. Non ſecùs, ducendo diagonium AC, & quærendo areas triangulorum ACD & ACB, eaſque addendo, rurſus obtinebitur area quadrilateri $\frac{cx+ba}{2x}\sqrt{xx-cc}$. Quare, ponendo haſce areas æquales & utraſque multiplicando per $2x$, habebitur $\overline{ax+bc}\sqrt{xx-aa} = \overline{cx+ba}\times\sqrt{xx-cc}$; æquatio quæ, quadrando ac dividendo per $aax-ccx$, redigetur ad formam ſæpiùs inventam $x^3 = \begin{matrix}+aa\\+bb\\+cc\end{matrix}.x + 2abc$.

Ex his conſtare poteſt quanta ſit ſolvendi copia; & obiter quòd alii modi ſint aliis multo concinniores. Quapropter ſi in primas de ſolutione Problematis alicujus cogitationes modus computationi malè accommodatus inciderit, relationes linearum iterùm evolvendæ ſunt, donec modum quàm poteris idoneum & elegantem machinatus fueris. Nam quæ leviori curæ ſe offerunt laborem ſatis moleſtum plerumque parient, ſi ad opus adhibeantur. Sic in Problemate de quo agitur, nil difficilius foret in ſequentem modum quàm in aliquem è præcedentibus incidere.

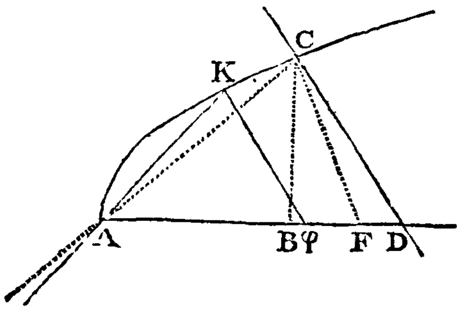


Demiſſis nempe BR & CS ad AD nomalibus, ut & CT ad BR, figura reſolvetur in triangula rectangula. Et videre eſt, quòd AD & AB dant AR; AD & CD dant SD; AD − AR − SD dat RS, vel TC. Item AB & AR dant BR; CD & SD dant CS, vel TR; & BR − TR dat BT. Denique BT ac TC dant BC, unde obtinebitur æquatio. Siquis autem hoc modo computationem aggreſſus fuerit, is in terminos Algebraicos profuſiores, quàm ſunt ulli præcedentium, incidet, & ad finalem æquationem ægriùs reducibiles.

Et hæc de ſolutione problematum in rectilineâ Geometriâ; niſi fortè operæ pretium fuerit annotâſſe prætérea, quòd cùm anguli, ſive poſitiones linearum per angulos expreſſæ, ſtatum quæſtionis ingrediuntur, angulorum vice debent adhiberi lineæ aut linearum proportiones; tales nempe quæ ab angulis datis poſſunt, per calculum Trigonometricum, derivari; aut à quibus inventis anguli quæſiti, per eundem calculum, prodeunt; hoc eſt, quæ ſe mutuò

mutuò determinant: cujus rei plures inſtantias videre eſt in ſequentibus. De Quæst. Geometr.

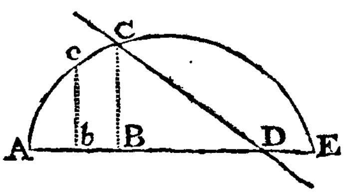
Quod ad Geometriam circa lineas curvas attinet, illæ deſignari ſolent vel deſcribendo eas per motum localem rectarum, vel adhibendo æquationes indefinitè exprimentes relationem rectarum certâ aliquâ lege diſpoſitarum, & ad curvas deſinentium. Idem fecerunt Veteres per ſectiones Solidorum, ſed minùs commodè. Computationes verò, quæ curvas primo modo deſcriptas reſpiciunt, haud secùs quàm in præcedentibus peraguntur. Quemadmodum ſi AKC ſit curva linea deſcripta per K, verticale punctum normæ AKϕ, cujus unum crus, AK, per punctum A, poſitione datum, liberè dilabitur, dum alterum, Kϕ, datæ longitudinis, ſuper rectam AD poſitione datam promovetur; & quæratur punctum C, in quo recta quævis CD, poſitione data, hanc curvam ſecabit: duco



rectas ACF, quæ normam in poſitione quæſitâ referant, & relatione linearum (ſine aliquo dati & quæſiti diſcrimine aut reſpectu ad curvam) conſideratâ, percipio dependentiam cæterarum à CF, et quâlibet harum quatuor BC, BF, AF & AC, Syntheticam eſſe; quarum duas itaque, ut CF $= a$ & CB $= x$, aſſumo; & inde computum ordiendo ſtatim lucratus ſum BF $= \sqrt{aa - xx}$, & AB $= \frac{xx}{\sqrt{aa - xx}}$; propter ang. rectum CBF, lineaſque BF, BC, BC, AB continuè proportionales. Porro ex datâ poſitione CD, datur AD; quam itaque dico b; datur etiam ratio BC ad BD, quam pono d ad e; & ſit BD $= \frac{ex}{d}$, & AB$=b-\frac{ex}{d}$. Eſt ergo $b - \frac{ex}{d} = \frac{xx}{\sqrt{xx - aa}}$; æquatio quæ, quadrando partes & multiplicando per $aa-xx$ &c. reducetur ad hanc formam $x^4 = \frac{2bdex^3 \begin{smallmatrix} -bbdd \\ +aaee \end{smallmatrix} xx - 2aaddex + aabbdd}{dd + ee}$; unde demum, è datis a, b, d & e, erui debet x, per regulas poſt tradendas; & intervallo iſto x, ſive BC, acta ipſi AD parallela recta ſecabit CD in quæſito puncto C.

Quòd ſi non deſcriptiones Geometricæ, ſed æquationes, pro curvis lineis deſignandis adhibeantur, computationes eo pacto faciliores & breviores evadent, in quantum ejuſmodi æquationes ipſis lucro

CAPUT XIII. lucro cedunt. Quemadmodum ſi datæ Ellipſeos, ACE, interſectio C cum rectâ CD poſitione datâ quæratur; pro Ellipſi deſignandâ ſumo notam aliquam æquationem ei propriam, ut $rx - \frac{r}{q}xx = yy$, ubi x indefinitè ponitur pro quâlibet axis parte Ab vel AB, & y pro perpendiculo bc, vel BC, ad curvam terminato; r vero & q dantur ex datâ ſpecie Ellipſis. Cùm itaque CD poſitione detur, dabitur & AD, quam dic a, & erit BD $a - x$; dabitur etiam angulus ADC, & inde ratio BD ad BC, quam dic 1 ad e; & erit BC (y) $= ea - ex$; cujus quadratum $eeaa - 2eeax + eexx$ æquabitur $rx - \frac{r}{q}xx$. Indeque, per reductionem, orietur $xx = \frac{2aeex + rx - aaee}{ee + \frac{r}{q}}$, ſeu $x = \frac{aee + \frac{1}{2}r \pm e\sqrt{ar + \frac{rr}{4ee} - \frac{aar}{q}}}{ee + \frac{r}{q}}$.



Quinetiam etſi Curva per deſcriptionem Geometricam, vel per ſectionem ſolidi, deſignetur, poteſt tamen inde æquatio obtineri, quæ naturam Curvæ definiet, adeoque huc omnes Problematum, quæ circa eam proponuntur, difficultates reduci.

Sic in exemplo priori, ſi AB dicatur x, & BC, y, tertia proportionalis BF erit $\frac{yy}{x}$; cujus quadratum unà cum quadrato BC æquatur CFq, hoc eſt $\frac{y^4}{xx} + yy = aa$; ſive $y^4 + xxyy = aaxx$. Eſtque hæc æquatio, quâ Curvæ AKC unumquodque punctum C unicuique baſis longitudini AB congruens, adeoque ipſa Curva, definitur; & è quâ proinde ſolutiones Problematum, quæ de hâc curvâ proponuntur, petere liceat.

Ad eundem ferè modum, cùm curva non datur ſpecie, ſed determinanda proponitur, poſſis pro arbitrio æquationem fingere, quæ naturam ejus generaliter contineat; & hanc pro eâ deſignandâ, tanquam ſi daretur, aſſumere; ut ex ejus aſſumptione quomodocunque preveniatur ad æquationes, ex quibus aſſumpta tandem determinentur: Cujus rei exempla habes in nonnullis ſequentium problematum, quæ, in pleniorem illuſtrationem hujus doctrinæ & exercitium diſcentium, congeſſi, quæque jam pergo tradere.

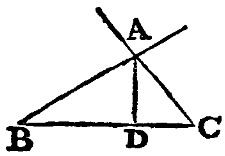
CAPUT

De Quæst. Geometr.

CAPUT DECIMUM QUARTUM.

PROB. I.

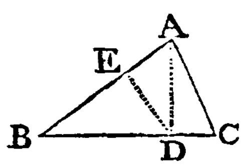
Datâ rectâ terminatâ, BC, *a cujus extremitatibus duæ rectæ*, BA, CA, *ducuntur in datis angulis*, ABC, ACB: *Invenire* AD *altitudinem concurſûs* A *ſupra datam* BC.



SIT BC $= a$, & AD $= y$; & cùm angulus ABD detur, dabitur, ex tabulâ ſinuum vel tangentium, ratio inter lineas AD & BD, quam pone ut d ad e. Eſt ergo $d.e::$ AD (y) BD. Quare BD $= \frac{ey}{d}$. Similiter, propter datum angulum ACD, dabitur ratio inter AD ac DC, quam pone ut d ad f, & erit DC $= \frac{fy}{d}$. At BD+DC = BC, hoc eſt $\frac{ey}{d} + \frac{fy}{d} = a$. Quæ reducta, multiplicando utramque partem æquationis per d, ac dividendo per $e + f$, evadit $y = \frac{ad}{e+f}$.

PROB. II.

Cujuſlibet Trianguli, ABC, *datis lateribus*, AB, AC, *&* *Baſi*, BC, *quam perpendiculum*, AD, *ab angulo verticali*, *ſecat in* D: *Invenire ſegmenta* BD *ac* DC.



SIT AB $= a$, AC $= b$, BC $= c$, & BD $= x$, eritque DC $= c - x$. Jam cùm ABq−BDq $(aa - xx)$ = ADq; & ACq − DCq $(bb - cc + 2cx - xx)$ = ADq: Erit $aa - xx = bb - cc + 2cx - xx$; quæ per reductionem fit $\frac{aa - bb + cc}{2c} = x$.

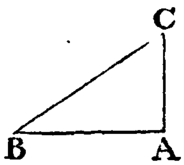
Cæterùm ut pateat omnes omnium Problematum difficultates, per ſolam linearum proportionalitatem, ſine adminiculo Prop. 47. primi Elementorum, licet non abſque circuitu, enodari poſſe; placuit ſequentem hujus ſolutionem, ex abundanti, ſubjungere. A puncto D in latus AB demitte DE normalem, &, ſtantibus jam

poſitis

CAPUT XIV. poſitis linearum nominibus, erit AB . BD :: BD . BE. $a . x :: x . \frac{xx}{a}$. Et BA − BE $(a - \frac{xx}{a})$ = EA. Nec non EA . AD :: AD . AB, adeoque EA × AB $(aa - xx)$ = ADq. Et ſic ratiocinando circa triangulum ACD, invenietur iterùm AD$q = bb - cc + 2cx - xx$. Unde obtinebitur, ut antè, $x = \frac{aa - bb + cc}{2c}$.

PROB. III.

Trianguli rectanguli, ABC, *perimetro & areâ datis, invenire hypotenuſam* BC.



ESTO perimeter a, area bb, BC $= x$, & AC $= y$; eritque AB $= \sqrt{xx - yy}$; unde rurſus perimeter (BC + AC + AB) eſt $x + y + \sqrt{xx - yy}$, & area ($\frac{1}{2}$ AC × AB) eſt $\frac{1}{2} y \sqrt{xx - yy}$. Adeoque $x + y + \sqrt{xx - yy} = a$, & $\frac{1}{2} y \sqrt{xx - yy} = bb$.

Harum æquationum poſterior dat $\sqrt{xx - yy} = \frac{2bb}{y}$; quare ſcribo $\frac{2bb}{y}$ pro $\sqrt{xx - yy}$ in æquatione priori, ut aſſymmetria tollatur; & prodit $x + y + \frac{2bb}{y} = a$; ſive multiplicando per y, & ordinando, $\frac{y}{yy} = ay - xy - 2bb$. Porro ex partibus æquationis prioris aufero $x + y$, & reſtat $\sqrt{xx - yy} = a - x - y$, cujus partes quadrando, ut aſſymmetria rurſus tollatur, prodit $xx - yy = aa - 2ax - 2ay + xx + 2xy + yy$, quæ in ordinem redacta & per 2 diviſa fit $yy = ay - xy + ax - \frac{1}{2}aa$. Denique ponendo æqualitatem inter duos valores ipſius yy, habeo $ay - xy - 2bb = ay - xy + ax - \frac{1}{2}aa$, quæ reducta fit $\frac{1}{2}a - \frac{2bb}{a} = x$.

Idem aliter.

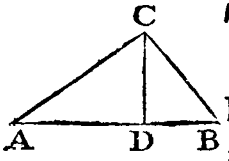
Eſto $\frac{1}{2}$ perimeter $= a$, area $= bb$, & BC $= x$, eritque AC + AB $= 2a - x$. Jam cùm ſit xx (BCq) = ACq + ABq, & $4bb$ = 2AC × AB, erit $xx + 4bb$ = ACq + ABq + 2AC × AB = quadrato ex AC + AB = quadrato ex $2a - x = 4aa - 4ax + xx$. Hoc eſt $xx + 4bb = 4aa - 4ax + xx$; quæ reducta ſit $a - \frac{bb}{a} = x$.

7 PROB.

PROBLEMATA GEOMETRICA.

PROB. IV.

Dato trianguli rectanguli perimetro & perpendiculo, invenire triangulum.



TRianguli ABC ſit C rectus angulus, & CD perpendiculum inde ad baſem AB demiſſum. Detur $AB+BC+AC=a$, & $CD=b$. Pone baſem $AB=x$, & erit laterum ſumma $a-x$. Pone laterum differentiam y, & erit majus latus $AC=\frac{a-x+y}{2}$; minus $BC=\frac{a-x-y}{2}$. Jam ex naturâ trianguli rectanguli eſt $ACq+BCq=ABq$, hoc eſt $\frac{aa-2ax+xx+yy}{2}=xx$. Eſt & $AB.AC::BC.DC$, adeoque $AB\times DC=AC\times BC$, hoc eſt $bx=\frac{aa-2ax+xx-yy}{4}$. Per priorem æquationem eſt $yy=xx+2ax-aa$. Per poſteriorem $yy=xx-2ax+aa-4bx$. Adeoque $xx+2ax-aa=xx-2ax+aa-4bx$. Et per reductionem $4ax+4bx=2aa$, ſive $x=\frac{aa}{2a+2b}$.

Geometricè ſic. In omni triangulo rectangulo, ut eſt ſumma perimetri & perpendiculi ad perimetrum, ita dimidium perimetri ad baſem.

Aufer $2x$ de a, & reſtabit $\frac{ab}{a+b}$ exceſſus laterum ſuper baſem. Unde rurſus, Ut in omni triangulo rectangulo ſumma perimetri & perpendiculi ad perimetrum, ita perpendiculum ad exceſſum laterum ſuper baſem.

PROB. V.

Datis trianguli rectanguli baſi, AB, *& ſummâ perpendiculi & laterum,* $CA+CB+CD$, *invenire triangulum.*

ESTO $CA+CB+CD=a$, $AB=b$, $CD=x$, & erit $AC+CB=a-x$. Pone $AC-CB=y$, & erit $AC=\frac{a-x+y}{2}$, & $CB=\frac{a-x-y}{2}$. Eſt autem $ACq+CBq=ABq$, hoc eſt $\frac{aa-2ax+xx+yy}{2}=bb$. Eſt & $AC\times CB=AB\times CD$, hoc eſt $\frac{aa-2ax+xx-yy}{4}=bx$. Quibus comparatis, fit $2bb-aa+2ax-xx=yy=aa-2ax+xx-4bx$. Et, per reductionem, $xx=2ax+2bx-aa+bb$, & $x=a+b-\sqrt{2ab+2bb}$.

 Geometricè

CAPUT XIV. *Geometricè ſic.* In omni triangulo rectangulo de ſummâ perimetri & perpendiculi aufer mediam proportionalem inter eandem ſummam & duplum baſis, & reſtabit perpendiculum.

Idem aliter.

Sit CA+CB+CD$=a$, AB$=b$, & AC$=x$, & erit BC$=\sqrt{bb-xx}$; CD$=\frac{x\sqrt{bb-xx}}{b}$. Et $x+$CB$+$CD$=a$, ſive CB$+$CD$=a-x$, atque adeo $\frac{b+x}{b}\sqrt{bb-xx}=a-x$. Et, quadratis partibus atque multiplicatis per bb, fiet $-x^4-2bx^3+2b^3x+b^4=aabb-2abbx+bbxx$. Quâ æquatione per tranſpoſitionem partium ad hunc modum ordinatâ, $x^4+2bx^3 \begin{smallmatrix}+3bb\\+2ab\end{smallmatrix} xx \begin{smallmatrix}+2abb\\+2b^3\end{smallmatrix} x \begin{smallmatrix}+b^4\\+2ab^3\\+aabb\end{smallmatrix} = \begin{smallmatrix}+2ab\\2bb\end{smallmatrix} xx \begin{smallmatrix}+4abb\\+4b^3\end{smallmatrix} x \begin{smallmatrix}+2ab^3\\+2b^4\end{smallmatrix}$, & extractâ utrobique radice, orietur $xx+bx+bb+ab=\overline{x+b}\sqrt{2ab+2bb}$. Et extractâ iterum radice $x=-\frac{1}{2}b+\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}\pm\sqrt{b\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}-\frac{1}{4}bb-\frac{1}{2}ab}$.

Conſtructio Geometrica.



Cape igitur AB $=\frac{1}{2}b$, BC $=\frac{1}{2}a$, CD $=\frac{1}{2}$AB, AE mediam proportionalem inter b & AC, & EF hinc inde mediam proportionalem inter b & DE, & erunt BF, BF duo latera trianguli.

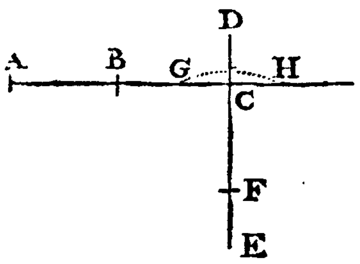
P R O B. VI.

Datis in triangulo rectangulo, ABC, *ſummâ laterum,* AC+BC, *& perpendiculo* CD, *invenire triangulum.* (*Vide fig.* Prob. VII.)

SIT AC + BC $=a$, CD $=b$, AC $=x$, & erit BC $=a-x$, AB $=\sqrt{aa-2ax+2xx}$. Eſt & CD.AC :: BC.AB. Ergo rurſus AB $=\frac{ax-xx}{b}$. Quare $ax-xx=b\sqrt{aa-2ax+2xx}$; & partibus quadratis & ordinatis, $x^4-2ax^3 \begin{smallmatrix}+aa\\-2bb\end{smallmatrix} xx+2abbx-aabb=0$. Adde ad utramque partem $aabb+b^4$, & fiet $x^4-2ax^3 \begin{smallmatrix}+aa\\-2bb\end{smallmatrix} xx+2abbx+b^4=aabb+b^4$. Et, extractâ utrobique radice, $xx-ax-bb=-b\times\sqrt{aa+bb}$; &, radice iterum extractâ, $x=\frac{1}{2}a\pm\sqrt{\frac{1}{4}aa+bb-b\sqrt{aa+bb}}$.

Conſtructio

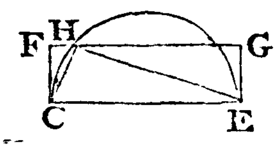
Conſtructio Geometrica. PROBLEMATA GEOMETRICA.



Cape AB = BC = $\frac{1}{2}a$. Ad C erige perpendiculum CD $= b$. Produc DC ad E ut ſit DE = DA. Et inter CD & CE cape medium proportionale CF. Centroque F, radio BC deſcriptus circulus GH ſecet rectam BC in G & H, & erunt BG & BH latera duo trianguli.

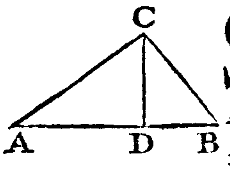
Idem aliter.

Sit AC+BC$=a$, AC − BC $=y$, AB $=x$, ac DC $=b$, & erit $\frac{a+y}{2}$ = AC; $\frac{a-y}{2}$ = BC; $\frac{aa+yy}{2}$ = ACq + BCq = ABq $= xx$. $\frac{aa-yy}{4b} = \frac{\text{AC} \times \text{BC}}{\text{DC}}$ = AB $= x$. Ergo $2xx - aa = yy = aa - 4bx$, & $xx = aa - 2bx$; &, extractâ radice, $x = -b + \sqrt{bb+aa}$. Unde in ſuperiori conſtructione eſt CE Hypotenuſa trianguli quæſiti. Datâ autem baſi & perpendiculo, tam in hoc quàm in ſuperiore Problemate, triangulum ſic expeditè conſtruitur. Fac parallelogrammum, CG, cujus latus, CE, erit baſis trianguli, latus alterum, CF, perpendiculum. Et ſuper CE deſcribe ſemicirculum ſecantem latus oppoſitum, FG, in H. Age CH, EH, & erit CHE triangulum quæſitum.



PROB. VII.

In triangulo rectangulo, datis ſummâ laterum, & ſummâ perpendiculi & baſis, invenire triangulum.



SIT laterum AC & BC ſumma a; baſis AB & perpendiculi CD ſumma b; latus AC $= x$; baſis AB $= y$; & erit BC $= a - x$; CD $= b - y$; $aa - 2ax + 2xx$ = ACq + BCq = ABq $= yy$; $ax - xx$ = AC × BC = AB × CD $= by - yy = by - aa + 2ax - 2xx$; & $by = aa - ax + xx$. Hujus quadratum $a^4 - 2a^3x + 3aaxx - 2ax^3 + x^4$, pone æquale yy in bb, hoc eſt æquale $aabb - 2abbx + 2bbxx$. Et, ordinatâ æquatione, fiet $x^4 - 2ax^3 \begin{smallmatrix} +3aa \\ -2bb \end{smallmatrix} xx \begin{smallmatrix} -2a^3 \\ +2abb \end{smallmatrix} x \begin{smallmatrix} +a^4 \\ -aabb \end{smallmatrix} = 0$. Ad utramque partem æquationis adde $b^4 - aabb$, & fiet $x^4 - 2ax^3 \begin{smallmatrix} +3aa \\ -2bb \end{smallmatrix} xx \begin{smallmatrix} -2a^3 \\ +2abb \end{smallmatrix} x \begin{smallmatrix} +a^4 \\ -2aabb \\ +b^4 \end{smallmatrix} =$

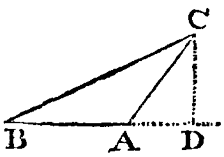
 $b^4 - aabb$. Et, extractâ utrobique radice, $xx - ax + aa - bb = -b\sqrt{bb - aa}$; &, radice iterum extractâ, $x = \frac{1}{2}a \pm \sqrt{bb - \frac{3}{4}aa - b\sqrt{bb - aa}}$.

Conſtructio Geometrica.

Cape R mediam proportionalem inter $b + a$ & $b - a$; & S mediam proportionalem inter R & $b - $R; & T mediam proportionalem inter $\frac{1}{2}a + $S & $\frac{1}{2}a - $S; & erunt $\frac{1}{2}a + $T & $\frac{1}{2}a - $T, latera trianguli.

PROB. VIII.

Trianguli cujuſcunque, ABC, *datis areâ, perimetro, & uno angulorum,* A, *cætera determinare.*



ESTO perimeter $= a$, & area $= bb$, & ab ignotorum angulorum alterutro, C, ad latus oppoſitum AB demitte perpendiculum CD; & propter angulum A datum, erit AC ad CD in datâ ratione, puta d ad e. Dic ergo AC $= x$, & erit CD $= \frac{ex}{d}$; per quam divide duplam aream, & prodibit $\frac{2bbd}{ex} = $ AB. Adde AD (nempe $\sqrt{ACq - CDq}$, ſive $\frac{x}{d}\sqrt{dd - ec}$) & emerget BD $= \frac{2bbd}{ex} + \frac{x}{d}\sqrt{dd - ee}$; cujus quadrato adde CD$q$, & orietur $\frac{4b^4dd}{eexx} + xx + \frac{4bb}{e} \times \sqrt{dd - ee} = $ BCq. Adhæc à perimetro aufer AC & AB, & reſtabit $a - x - \frac{2bbd}{ex} = $ BC; cujus quadratum $aa - 2ax + xx - \frac{4abbd}{ex} + \frac{4bbd}{e} + \frac{4b^4dd}{eexx}$ pone æquale quadrato prius invento; &, neglectis æquipollentibus, erit $\frac{4bb}{e}\sqrt{dd - ee} = aa - 2ax - \frac{4abbd}{ex} + \frac{4bbd}{e}$. Et hæc, aſſumendo $4af$ pro datis terminis $aa + \frac{4bbd}{e} - \frac{4bb}{e} \times \sqrt{dd - ee}$, & reducendo, evadit $xx = 2fx - \frac{2bbd}{e}$, ſive $x = f \pm \sqrt{ff - \frac{2bbd}{e}}$.

Eadem æquatio prodiiſſet etiam quærendo crus AB; nam crura AB & AC ſimiliter ſe habent ad omnes conditiones problematis. Quare ſi AC ponatur $f - \sqrt{ff - \frac{2bbd}{e}}$, erit AB $= f + \sqrt{ff - \frac{2bbd}{e}}$, & viciſſim; atque horum ſumma $2f$ ſubducta de perimetro relinquit tertium latus BC $= a - 2f$.

PROB.

PROB. IX.

Problemata Geometrica.

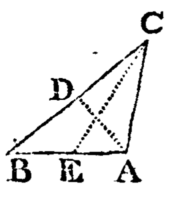
Datis altitudine, baſi, & ſummâ laterum invenire triangulum.

SIT altitudo CD $= a$, baſis AB dimidium $= b$, laterum ſemiſumma $= c$, & ſemidifferentia $= z$; eritque majus latus, puta BC $= c + z$, & minus AC $= c - z$. Subduc CDq de BCq & ACq, & exibit hinc BD $= \sqrt{cc + 2cz + zz - aa}$, & inde AD $= \sqrt{cc - 2cz + zz - aa}$. Subduc etiam AB de BD, & exibit iterum AD $= \sqrt{cc + 2cz + zz - aa} - 2b$. Quadratis jam valoribus AD, & ordinatis terminis, orietur $bb + cz = b\sqrt{cc + 2cz + zz - aa}$. Rurſuſque quadrando & redigendo in ordinem, obtinebitur $cczz - bbzz = bbcc - bbaa - b^4$. Et $z = b\sqrt{1 - \frac{aa}{cc - bb}}$. Unde dantur latera.

PROB. X.

Datis baſi, AB, *ſummâ laterum,* AC + BC, *& angulo verticali* C, *determinare latera.*

SIT baſis $= a$, ſemiſumma laterum $= b$, & ſemidifferentia $= x$, eritque majus latus BC $= b + x$, & minus AC $= b - x$.



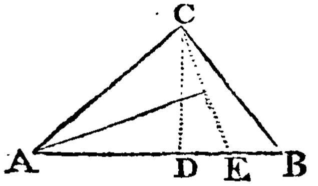
Ab alterutro ignotorum angulorum A ad latus oppoſitum BC demitte perpendiculum AD; &, propter angulum C datum, dabitur ratio AC ad CD, puta d ad e, & proinde erit CD $= \frac{eb - ex}{d}$. Eſt etiam, per 13. II. Elementorum, $\frac{ACq - ABq + BCq}{2BC}$, hoc eſt $\frac{2bb + 2xx - aa}{2b + 2x}$, $=$ CD; adeoque habetur æquatio inter valores CD. Et hæc reducta fit $x = \sqrt{\frac{eaa + 2ebb - 2dbb}{2d + 2e}}$. Unde dantur latera.

Si anguli ad baſin quærerentur, concluſio foret concinnior; utpote ducatur EC datum angulum biſecans & baſi occurrens in E; & erit AB. AC + BC (: : AE. AC) : : ſin. ang. ACE. ſin. ang. AEC. Et ab angulo AEC, ejuſque complemento BEC, ſi ſubducatur dimidium anguli C, relinquentur anguli ABC, & BAC.

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PROB. XI.

Datis Trianguli lateribus invenire angulos.



DEntur latera AB $= a$, AC $= b$, BC $= c$, quæratur angulus A. Demiſſo ad AB perpendiculo CD, quod angulo iſti opponitur, erit imprimis $bb - cc = \mathrm{AC}q - \mathrm{BC}q = \mathrm{AD}q - \mathrm{BD}q = \overline{\mathrm{AD}+\mathrm{BD}} \times \overline{\mathrm{AD}-\mathrm{BD}} = \mathrm{AB} \times \overline{2\mathrm{AD} - \mathrm{AB}} = 2\mathrm{AD} \times a - aa$. Adeoque $\frac{1}{2}a + \frac{bb-cc}{2a} = \mathrm{AD}$. Unde prodit hocce *primum Theorema.*

I. Ut AB, ad AC + BC, ita AC − BC, ad quartam proportionalem N. $\frac{\mathrm{AB}+\mathrm{N}}{2} = \mathrm{AD}$. Ut AC ad AD, ia radius ad Coſinum anguli A.

Adhæc $\mathrm{DC}q = \mathrm{AC}q - \mathrm{AD}q = \frac{2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4}{4aa} = \frac{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{-a+b+c}}{4aa}$. Unde, multiplicatis numeratoris & denominatoris radicibus per b, conflatur hocce *Theorema ſecundum.*

II. Ut $2ab$ ad medium proportionale inter $\overline{a+b+c} \times \overline{a+b-c}$, & $\overline{a-b+c} \times \overline{-a+b+c}$, ita radius ad ſinum anguli A.

Inſuper in AB cape AE = AC, & age CE, & erit angulus ECD æqualis dimidio anguli A. Aufer AD de AE, & reſtabit $\mathrm{DE} = b - \frac{1}{2}a - \frac{bb-cc}{2a} = \frac{cc - aa + 2ab - bb}{2a} = \frac{\overline{c+a-b} \times \overline{c-a+b}}{2a}$. Unde $\mathrm{DE}q = \frac{\overline{c+a-b} \times \overline{c+a-b} \times \overline{c-a+b} \times \overline{c-a+b}}{4aa}$. Et hinc conſit *Theorema tertium quartumque*, viz.

III. Ut $2ab$ ad $\overline{c+a-b} \times \overline{c-a+b}$ (ita AC ad DE) ita radius ad ſinum verſum anguli A

IV. Et, ut medium proportionale inter $a + b + c$, & $a + b - c$ ad medium proportionale inter $c + a - b$, & $c - a + b$ (ita CD ad DE) ita radius ad tangentem dimidii anguli A, vel dimidii cotangens ad radium.

Præterea eſt $\mathrm{CE}q = \mathrm{CD}q + \mathrm{DE}q = \frac{2abb + bcc - baa - b^3}{a} = \frac{b}{a} \times \overline{c+a-b} \times \overline{c-a+b}$. Unde *Theorema quintum & ſextum.*

V. Ut

V. Ut medium proportionale inter $2a$ & $2b$ ad medium proportionale inter $c+a-b$, & $c-a+b$, vel ut 1 ad medi. proportionale inter $\frac{c+a-b}{2a}$, & $\frac{c-a+b}{2b}$ (ita AC ad $\frac{1}{2}$CE vel CE ad DE) ita radius ad ſinum dimidii anguli A.

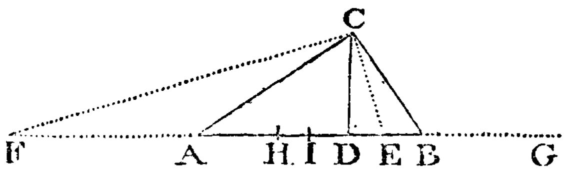
VI. Et ut medium proportionale inter $2a$ & $2b$ ad medium proportionale inter $a+b+c$, & $a+b-c$ (ita CE ad CD) ita radius ad coſinum dimidii anguli A.

Si præter angulos deſideretur etiam area trianguli, duc CDq in $\frac{1}{4}$ABq, & radix *viz.* $\frac{1}{4}\sqrt{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{-a+b+c}}$, erit area illa quæſita.

PROB. XII.

Trianguli cujuſvis rectilinei datis lateribus & baſi, invenire ſegmenta baſis, perpendiculum, aream & angulos.

TRianguli ABC dentur latera AC, BC & baſis AB. Biſeca AB in I, & in eâ utrinque productâ cape AF & AE æquales AC, atque BG & BH æquales BC. Junge CE, CF; & à C ad baſem demitte perpendiculum CD.



Et erit $\text{AC}q - \text{BC}q = \text{AD}q + \text{CD}q - \text{CD}q - \text{BD}q = \text{AD}q - \text{BD}q = \overline{\text{AD}+\text{BD}} \times \overline{\text{AD}-\text{BD}} = \text{AB} \times 2\text{DI}$. Ergo $\frac{\text{AC}q - \text{BC}q}{2\text{AB}} = \text{DI}$. Et $2\text{AB}.\ \text{AC}+\text{BC} :: \text{AC}-\text{BC}.\ \text{DI}$. Quod eſt Theorema pro determinandis ſegmentis baſis.

De IE, hoc eſt de $\text{AC} - \frac{1}{2}\text{AB}$, aufer DI, & reſtabit $\text{DE} = \frac{\text{BC}q - \text{AC}q + 2\text{AC} \times \text{AB} - \text{AB}q}{2\text{AB}}$, hoc eſt $= \frac{\overline{\text{BC}+\text{AC}-\text{AB}} \times \overline{\text{BC}-\text{AC}+\text{AB}}}{2\text{AB}}$, ſive $= \frac{\text{HE} \times \text{EG}}{2\text{AB}}$. Aufer DE de FE ſive 2AC, & reſtabit $\text{FD} = \frac{\text{AC}q + 2\text{AC} \times \text{AB} + \text{AB}q - \text{BC}q}{2\text{AB}}$, hoc eſt $= \frac{\overline{\text{AC}+\text{AB}+\text{BC}} \times \overline{\text{AC}+\text{AB}-\text{BC}}}{2\text{AB}}$, ſive $= \frac{\text{FG} \times \text{FH}}{2\text{AB}}$. Et cùm ſit CD medium proportionale inter DE ac DF, CE medium proportionale inter DE et EF, ac CF medium proportionale inter DF & EF: erit $\text{CD} = \frac{\sqrt{\text{FG} \times \text{FH} \times \text{HE} \times \text{EG}}}{2\text{AB}}$, $\text{CE} = \sqrt{\frac{\text{AC} \times \text{HE} \times \text{EG}}{\text{AB}}}$; & $\text{CF} = \sqrt{\frac{\text{AC} \times \text{FG} \times \text{FH}}{\text{AB}}}$. Duc CD in $\frac{1}{2}$AB, & habebitur area =

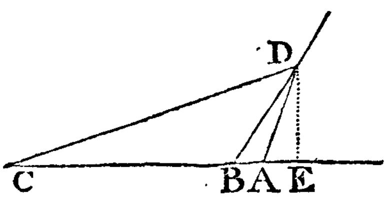
$\frac{1}{4}\sqrt{}$

$\frac{1}{4}\sqrt{FG \times FH \times HE \times EG}$. Pro angulo verò A determinando prodeunt Theoremata multiplicia, *viz.*

1. $2AB \times AC : HE \times EG$ (: : AC. DE) : : radius ad ſinum verſum anguli A.
2. $2AB \times AC.\ FG \times FG$ (: : AC. FD) : : radius ad coſin verſ. A.
3. $2AB \times AC.\ \sqrt{FG \times FH \times HE \times EG}$ (: : AC. CD) : : rad. ad ſin. A.
4. $\sqrt{FG \times FH}.\ \sqrt{HE \times EG}$ (: : CF. CE) : : rad. ad tang. $\frac{1}{2}$ A.
5. $\sqrt{HE \times EG}.\ \sqrt{FG \times FH}$ (: : CE. FC) : : rad. ad cotang. $\frac{1}{2}$ A.
6. $2\sqrt{AB \times AC}.\ \sqrt{HE \times EG}$ (: : FE. CE) : : rad. ad ſin. $\frac{1}{2}$ A.
7. $2\sqrt{AB \times AC}.\ \sqrt{FG \times FH}$ (: : FE. FC) : : rad. ad coſin. $\frac{1}{2}$ A.

PROB. XIII.

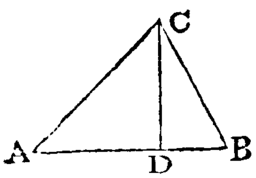
Datum angulum, CBD, *rectâ datâ*, CD, *ſubtendere; ita ut ſi à termino iſtius rectæ*, D, *ad punctum* A *in rectâ* CB *productâ datum agatur* AD, *fuerit angulus* ADC *æqualis angulo* ABD.



DIcatur $CD = a$, $AB = b$, $BD = x$, & erit BD. BA : : CD. DA $= \frac{ab}{x}$. Demitte perpendiculum DE, erit BE $= \frac{BDq - ADq + BAq}{2BA} = \frac{xx - \frac{aabb}{xx} + bb}{2b}$. Ob datum angulum DBA, pone BD. BE :: $b.\ e$, & habebitur iterum BE $= \frac{ex}{b}$: ergo $xx - \frac{aabb}{xx} + bb = 2ex$. Et $x^4 - 2ex^3 + bbxx - aabb = 0$.

PROB. XIV.

Invenire Triangulum, ABC, *cujus tria latera*, AB, AC, BC, *& perpendiculum* DC, *ſunt in Arithmeticâ progreſſione.*



DIC $AC = a$, $BC = x$; & erunt $DC = 2x - a$, & $AB = 2a - x$. Erunt etiam AD $(= \sqrt{ACq - DCq}) = \sqrt{4ax - 4xx}$, & BD $(= \sqrt{BCq - DCq}) = \sqrt{4ax - 3xx - aa}$. Atque adeo rurſus AB $= \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$. Quare $2a - x = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$,

ſive $2a-x-\sqrt{4ax-4xx}=\sqrt{4ax-3xx-aa}$. Et partibus quadratis $4aa-3xx-\overline{4a+2x}\times\sqrt{4ax-4xx}=4ax-3xx-aa$; ſive $5aa-4ax=\overline{4a-2x}\sqrt{4ax-4xx}$. Et, partibus iterum quadratis, ac terminis rite diſpoſitis, $16x^4-80ax^3+144aaxx-104a^3x+25a^4=0$. Hanc æquationem divide per $2x-a$, & orietur $8x^3-36axx+54aax-25a^3=0$, æquatio cujus reſolutione dabitur x ex aſſumpto utcunque a. Habitis a & x, conſtitue triangulum, cujus latera erunt $2a-x$, a, & x; & perpendiculum, in latus $2a-x$ demiſſum, erit $2x-a$. PROBLEMATA GEOMETRICA.

Si poſuiſſem differentiam laterum trianguli eſſe d, & perpendiculum eſſe x; opus evaſiſſet aliquanto concinnius, prodeunte tandem æquatione $x^3=24ddx+48d^3$.

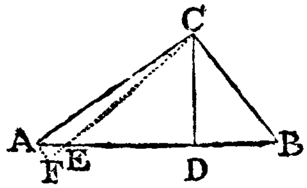
PROB. XV.

Invenire Triangulum, ABC, *cujus tria latera,* AB, AC, BC, *& perpendiculum,* CD, *ſunt in Geometricâ progreſſione.*

DIC AC$=x$, & BC$=a$; & erit AB $=\frac{xx}{a}$; et CD $=\frac{aa}{x}$. Eſt & AD $(=\sqrt{ACq-CDq})=\sqrt{xx-\frac{a^4}{xx}}$; & BD $(=\sqrt{BCq-DCq})=\sqrt{aa-\frac{a^4}{xx}}$; adeoque $\frac{xx}{a}$ (= AB) $=\sqrt{xx-\frac{a^4}{xx}}+\sqrt{aa-\frac{a^4}{xx}}$; ſive $\frac{xx}{a}-\sqrt{aa-\frac{a^4}{xx}}=\sqrt{xx-\frac{a^4}{xx}}$. Et, partibus æquationis quadratis, $\frac{x^4}{aa}-\frac{2xx}{a}\times\sqrt{aa-\frac{a^4}{xx}}+aa-\frac{a^4}{xx}=xx-\frac{a^4}{xx}$; hoc eſt, $x^4-aaxx+a^4=2aax\sqrt{xx-aa}$. Et, partibus iterum quadratis, $x^8-2aax^6+3a^4x^4-2a^6xx+a^8=4a^4x^4-4a^6xx$. Hoc eſt $x^8-2aax^6-a^4x^4+2a^6xx+a^8=0$. Divide hanc æquationem per $x^4-aaxx-a^4$, & orietur $x^4-aaxx-a^4$. Quare eſt $x^4=aaxx+a^4$. Et, extractâ radice, $xx=\frac{1}{2}aa+\sqrt{\frac{5}{4}a^4}$, ſive $x=a\sqrt{\frac{1}{2}+\sqrt{\frac{5}{4}}}$. Cape ergo a, ſive BC, cujuſvis longitudinis; & fac BC. AC :: AC. AB :: 1. $\sqrt{\frac{1}{2}+\sqrt{\frac{5}{4}}}$; & trianguli ABC, ex his lateribus conſtituti, perpendiculum DC erit ad latus BC in eâdem ratione.

CAPUT XIV.

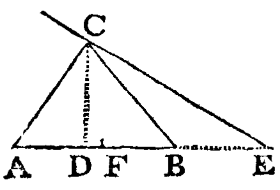
Idem aliter.



Cùm ſit AB . AC : : BC . DC dico angulum ACB rectum eſſe. Nam ſi negas, age CE conſtituentem angulum ECB rectum. Sunt ergo triangula BCE, DBC ſimilia, per 8. VI. Elem. adeoque EB . EC : : BC . DC. hoc eſt EB . EC :: AB . AC. Age AF perpendicularem CE, & propter parallelas AF, BC, erit EB . EC : : AE . FE : : AB . BC. Ergo, per 9. V. Elem. eſt AC=FC, hoc eſt Hypotenuſa trianguli rectanguli æqualis lateri, contra 19. I. Elem. Non eſt ergo angulus ECB rectus, & proinde ipſum ACB rectum eſſe oportet. Eſt itaque $ACq+BCq=ABq$. Sed eſt $ACq=AB\times BC$, ergo $AB\times BC+BCq=ABq$; &, extractâ radice, $AB=\frac{1}{2}BC+\sqrt{\frac{5}{4}BCq}$. Quamobrem cape BC . AB : : 1 . $\frac{1+\sqrt{5}}{2}$, & AC mediam proportionalem inter BC & AB, & triangulo ex his lateribus conſtituto, erunt AB, AC, BC, DC continuè proportionales.

PROB. XVI.

Super datâ baſi, AB, *triangulum*, ABC, *conſtituere, cujus vertex* C *erit ad rectam*, EC, *poſitione datam, baſis autem medium exiſtet Arithmeticum inter latera.*



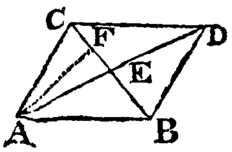
BAſis AB biſecetur in F, & producatur donec rectæ EC poſitione datæ occurrat in E, & ad ipſam demittatur perpendicularis CD; dictiſque $AB=a$, $FE=b$, & $BC-AB=x$, erit $BC=a+x$, $AC=a-x$. Et per 13. II. Elem. BD $\left(=\frac{BCq-ACq+ABq}{2AB}\right)=2x+\frac{1}{2}a$. Adeoque $FD=2x$, $DE=b+2x$, & CD $(=\sqrt{CBq-BDq})=\sqrt{\frac{3}{4}aa-3xx}$. Sed, propter datas poſitiones rectarum CE & AB, datur angulus CED; adeoque & ratio DE ad CD; quæ ſi ponatur d ad e dabit analogiam $d . e : : b+2x . \sqrt{\frac{3}{4}aa-3xx}$. Unde, multiplicatis extremis & mediis in ſe, oritur æquatio $eb+2ex=d\sqrt{\frac{3}{4}aa-3xx}$; cujus partibus quadratis & ritè diſpoſitis, fit $xx=\frac{\frac{3}{4}ddaa-eebb-4eebx}{4ee+3dd}$. Et, radice extractâ, $x=$

$-2eeb$

$\frac{-2ecb + d\sqrt{3eeaa - 3eebb + \frac{4}{3}ddaa}}{4ee + 3dd}$. Dato autem x, datur $BC = a + x$, & $AC = a - x$.

PROB. XVII.

Datis Parallelogrammi cujuſcunque lateribus, AB, BD, DC, & AC, & *unâ lineâ diagonali*, BC, *invenire alteram diagonalem*, AD.

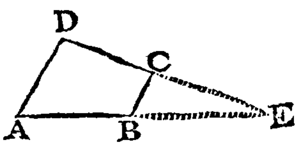


SIT E concurſus diagonalium, & ad diagonalem BC demitte normalem AF, &, per 13. II. Elementorum, erit $\frac{ACq - ABq + BCq}{2BC} = CF$; atque etiam $\frac{ACq - AEq + ECq}{2EC} = CF$. Quare cùm ſit $EC = \frac{1}{2}BC$, & $AE = \frac{1}{2}AD$, erit $\frac{ACq - ABq + BCq}{2BC} = \frac{ACq - \frac{1}{4}ADq + \frac{1}{4}BCq}{BC}$; &, factâ reductione, $AD = \sqrt{2ACq + 2ABq - BCq}$.

Unde obiter, in quolibet parallelogrammo ſumma quadratorum laterum æquatur ſummæ quadratorum diagonalium.

PROB. XVIII.

Datis Trapezii, ABCD, *angulis, perimetro, & areâ, determinare latera*.



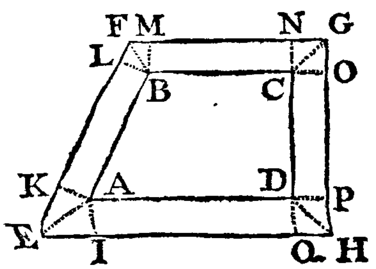
LAtera duo quælibet AB ac DC produc donec concurrant in E, ſitque $AB = x$, & $BC = y$; &, propter angulos omnes datos, dantur rationes BC ad CE & BE; quas pone d ad e & f; & erit $CE = \frac{ey}{d}$; & $BE = \frac{fy}{d}$; adeoque $AE = x + \frac{fy}{d}$. Dantur etiam rationes AE ad AD ac DE; quas pone g & h ad d; & erit $AD = \frac{dx + fy}{g}$; & $ED = \frac{dx + fy}{h}$; adeoque $CD = \frac{dx + fy}{h} - \frac{ey}{d}$; & ſumma omnium laterum $x + y + \frac{dx + fy}{g} + \frac{dx + fy}{h} - \frac{ey}{d}$; quæ, cùm detur, eſto a; & abbrevientur etiam termini ſcribendo $\frac{p}{r}$ pro dato $1 + \frac{d}{g} + \frac{d}{h}$, & $\frac{q}{r}$ pro dato $1 + \frac{f}{g} + \frac{f}{h} - \frac{e}{d}$, & habebitur æquatio $\frac{px + qy}{r} = a$.

Adhæc, propter datos omnes angulos, datur ratio BCq ad triangulum BCE; quam pone m ad n, & erit triang. $BCE = \frac{n}{m}yy$. Datur etiam ratio AEq ad triangulum ADE; quam pone m ad d,

CAPUT XIV. & erit triang. ADE $= \frac{ddxx + 2dfxy + ffyy}{dm}$. Quare cùm area AC, quæ eſt horum triangulorum differentia, detur, eſto bb, & erit $\frac{ddxx + 2dfxy + ffyy - dnyy}{dm} = bb$. Atque ita habentur duæ æquationes, ex quarum reductione omnia determinantur. Nempe ſuperior æquatio dat $\frac{ra - qy}{p} = x$; ſcribendo $\frac{ra - qy}{p}$ pro x in inferiori, provenit $\frac{drraa - 2dqray + dqqyy}{ppm} + \frac{2afry - 2fqyy}{pm} + \frac{ffyy - dnyy}{dm} = bb$. Et, abbreviatis terminis ſcribendo s pro dato $\frac{dqq}{pp} - \frac{2fq}{p} - \frac{ff}{d} - n$, & st pro dato $+ \frac{adqr}{pp} - \frac{afr}{q}$, ac stv pro dato $bbm - \frac{drraa}{pp}$, oritur $yy = 2ty + tv$, ſeu $y = t + \sqrt{tt + tv}$.

PROB. XIX.

Piſcinam, ABCD, *perambulatorio,* ABCD EFGH, *datæ areæ, & ejuſdem ubique latitudinis circundare.*



ESTO perambulatorii latitudo x, & ejus area aa. Et à punctis A, B, C, D, ad lineas EF, FG, GH & HE demiſſis perpendicularibus AK, BL, BM, CN, CO, DP, DQ, AI, perambulatorium dividetur in quatuor trapezia, IK, LM, NO, PQ, & in quatuor parallelogramma, AL, BN, CP, DI, latitudinis x, & ejuſdem longitudinis cum lateribus dati trapezii. Sit ergo ſumma laterum (AB+BC+CD+DA)$=b$, & erit ſumma parallelogrammorum $= bx$.

Porro ductis AE, BF, CG, DH, cùm ſit AI = AK, erit ang. AEI = ang. AEK$=\frac{1}{2}$IEK, ſive $\frac{1}{2}$DAB. Datur ergo ang. AEI, & proinde ratio ipſius AI ad IE, quam pone d ad e; & erit IE $= \frac{ex}{d}$. Hanc duc in $\frac{1}{2}$AI, ſive $\frac{1}{2}x$, & fiet area trianguli AEI $= \frac{exx}{2d}$. Sed, propter æquales angulos & latera, triangula AEI & AEK ſunt æqualia; adeoque trapezium IK (= 2 triang. AEI) $= \frac{exx}{d}$. Simili modo ponendo BL. LF :: $d.f$, & CN. NG :: $d.g$, & DP. PH :: $d.h$, (nam illæ etiam rationes dantur ex datis angulis B, C, ac D) habebitur trapezium LM $= \frac{fxx}{d}$; NO $= \frac{gxx}{d}$; & PQ$= \frac{hxx}{d}$. Quamobrem $\frac{exx}{d}$ +

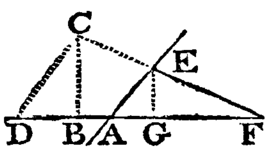
fxx

$\frac{fxx}{d} + \frac{gxx}{d} + \frac{hxx}{d}$, ſive $\frac{pxx}{d}$, ſcribendo p pro $e+f+g+h$, erit æquale trapeziis quatuor IK + LM + NO + PQ; & proinde $\frac{pxx}{d} + bx$, æquabitur toti perambulatorio aa. Quæ æquatio, dividendo omnes terminos per $\frac{p}{d}$ & extrahendo radicem ejus, evadet $x = \frac{-db + \sqrt{bbdd + 4aapd}}{2p}$. Latitudine Perambulatorii ſic inventâ, facile eſt ipſum deſcribere.

PROBLEMATA GEOMETRICA.

PROB. XX.

A dato puncto, C, *rectam lineam*, CF, *ducere, quæ cum aliis duabus poſitione datis rectis*, AE & AF, *triangulum datæ magnitudinis*, AEF, *comprehendet.*

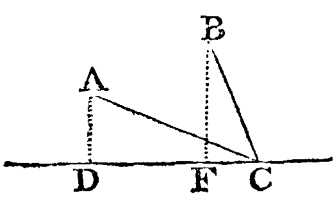


Age CD parallelam AE, & CB ac EG perpendiculares in AF, ſitque AD $= a$, CB $= b$, AF $= x$, & trianguli AEF area cc; &, propter proportionales DF. AF (: : DC. AE) : : CB. EG, hoc eſt $a + x . x : : b . \frac{bx}{a+x}$, erit $\frac{bx}{a+x} =$ EG. Hanc duc in $\frac{1}{2}$AF, & emerget $\frac{bxx}{2a+2x}$ quantitas areæ AEF, quæ proinde æquatur cc. Atque adeo, æquatione ordinatâ, eſt $xx = \frac{2ccx + 2cca}{b}$, ſeu $x = \frac{cc + \sqrt{c^4 + 2ccab}}{b}$.

Nihil ſecus recta per datum punctum ducitur, quæ triangulum, vel trapezium quodvis, in datâ ratione ſecabit.

PROB. XXI.

Punctum C *in datâ rectâ lineâ*, DF, *determinare, à quo ad alia duo poſitione data puncta*, A & B, *ductæ rectæ*, AC & BC, Vide Prop. 45. *datam habeant differentiam.*



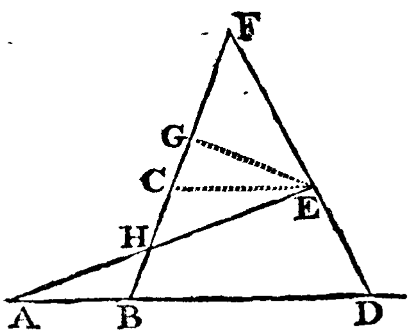
A Datis punctis ad datam rectam demitte perpendiculares AD & BF, & dic AD$=a$, BF$=b$, DF$=c$, DC$=x$; & erit AC$=\sqrt{aa+xx}$; FC$=x-c$; & BC$=\sqrt{bb+xx-2cx+cc}$. Sit jam data harum differentia d, exiſtente AC majori quàm BC; erit $\sqrt{aa+xx} - d = \sqrt{bb+xx-2cx+cc}$. Et, quadratis partibus, $aa + xx + dd - 2d\sqrt{aa+xx} = bb + xx - 2cx + cc$. Factâque reductione, & abbreviandi cauſâ pro datis $aa+dd-bb-cc$ ſcripto

CAPUT XIV. ſcripto $2ce$, emerget $ee + cx = d\sqrt{aa + xx}$. Iterumque quadratis partibus, $e^4 + 2ceex + cexx = ddaa + ddxx$. Et, æquatione reductâ, $xx = \frac{2eecx + e^4 - aadd}{dd - cc}$, ſeu $x = \frac{eec + \sqrt{e^4dd - aad^4 + aaddcc}}{dd - cc}$.

Haud ſecùs problema reſolvitur, ſi linearum, AC & BC, ſumma vel quadratorum ſumma aut differentia, vel proportio, vel rectangulum, vel angulus ab ipſis comprehenſus detur; vel etiam ſi vice rectæ DC, circumferentia circuli, aut alia quævis curva linea adhibeatur, modò calculus, in hoc ultimo præſertim caſu, referatur ad lineam conjungentem puncta A & B.

PROB. XXII.

Datis poſitione tribus rectis, AD, AE, BF, *quartam* DF *ducere, cujus partes,* DE, EF, *prioribus interceptæ, datarum erunt longitudinum.*



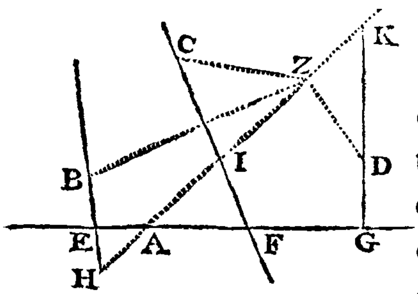
Ad BF demitte perpendicularem EG, ut & obliquam EC parallelam AD, & rectis tribus poſitione datis concurrentibus in A, B, & H, dic AB $= a$, BH $= b$, AH $= c$, ED $= d$, EF $= e$, & HE $= x$. Jam propter ſimilia triangula ABH, ECH, eſt AH. AB :: HE. EC $= \frac{ax}{c}$; & AH. HB :: HE. CH $= \frac{bx}{c}$. Adde HB, & fit CB $= \frac{bx + bc}{c}$. Inſuper, propter ſimilia triangula FEC, FDB, eſt ED.CB::EF.FC $= \frac{ebx + ebc}{dc}$. Denique (per 12 & 13 .II. Elem.) eſt $\frac{ECq - EFq}{2FC} + \frac{1}{2}FC\,(=CG) = \frac{HEq - ECq}{2CH} - \frac{1}{2}CH$; hoc eſt, $\frac{\frac{aaxx}{cc} - ee}{\frac{2ebx + 2ebc}{dc}} + \frac{ebx + ebc}{2dc} = \frac{xx - \frac{aaxx}{cc}}{\frac{2bx}{c}} - \frac{bx}{2c}$. Sive $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebx}{d} + \frac{ebc}{d} = \frac{ccx - aax - bbx}{b}$. Hic, abbreviandi cauſâ, pro $\frac{cc - aa - bb}{b} - \frac{eb}{d}$, ſcribe m; & erit $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebc}{d} = mx$; ac terminis omnibus multiplicatis per $x + c$, fiet $\frac{aadxx - eedcc}{eb} + \frac{ebcx}{d} + \frac{ebcc}{d} = mxx + mcx$. Iterum pro

aad

$\frac{aad}{cb} - m$ ſcribe p; pro $mc - \frac{ebc}{d}$ ſcribe $2pq$; & pro $-\frac{ebcc}{d} + \frac{eedcc}{eb}$ ſcribe prr; & evadet $xx = 2qx + rr$, & $x = q \pm \sqrt{qq+rr}$. Invento x, ſive HE, age EC parallelam AB, & cape FC.BC::$e.d$, & acta FED conditionibus quæſtionis ſatisfaciet.

PROB. XXIII.

Punctum Z *determinare, à quo ad quatuor poſitione datas rectas lineas,* FA, EB, FC, GD, *ſi aliæ quatuor lineæ,* ZA, ZB, ZC, *&* ZD, *in datis angulis ducantur, duarum è ductis,* ZA *&* ZB, *rectangulum, & aliarum duarum,* ZC *&* ZD, *ſumma detur.*



E lineis elige aliquam poſitione datam, FA; ut & poſitione non datam, ZA, quæ ad illam ducitur, ex quarum longitudinibus punctum Z determinetur; & cæteras poſitione datas lineas produc donec his, ſi opus eſt etiam productis, occurrant, ut vides. Dictiſque EA $= x$, & AZ $= y$, propter angulos trianguli AEH datos, dabitur ratio AE ad AH, quam pone p ad q, & erit AH $= \frac{qx}{p}$. Adde AZ, fitque ZH $= y + \frac{qx}{p}$. Et inde cùm propter datos angulos trianguli HZB detur ratio HZ ad BZ, ſi ea ponatur n ad p, habebitur ZB $= \frac{py + qx}{n}$.

Præterea ſi data EF dicatur a, erit AF $= a - x$; indeque, ſi propter datos angulos trianguli AFI, ſtatuatur AF ad AI in ratione p ad r, evadet AI $= \frac{ra - rx}{p}$. Hanc aufer ab AZ, & reſtabit IZ $= y - \frac{ra - rx}{p}$. Et, propter datos angulos trianguli ICZ, ſi ponatur IZ ad ZC in ratione m ad p, evadet ZC $= \frac{py - ra + rx}{m}$.

Ad eundem modum, ſi ponatur EG$=b$. AG.AK::$l.s$, & ZK.ZD::$p.l$. obtinebitur ZD $= \frac{sb - sx - ly}{p}$.

Jam ex ſtatu quæſtionis ſi duarum ZC & ZD ſumma, $\frac{py - ra + rx}{m} + \frac{sb - sx - ly}{p}$, ponatur æqualis dato alicui f; & aliarum duarum rectangulum, $\frac{pyy + qxy}{n}$, æquale gg; habebuntur duæ æqua-

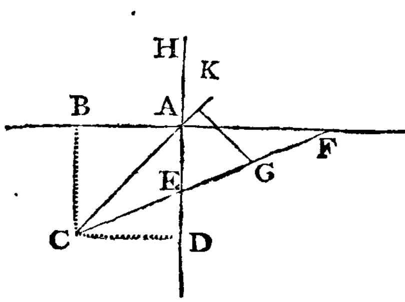
tiones

Caput XIV. tiones pro determinandis x & y. Per poſteriorem fit $x = \frac{ngg - pyy}{qy}$; & hunc ipſius x valorem ſcribendo pro eo in priori æquatione, evadet $\frac{py - ra}{m} + \frac{rngg - rpyy}{mqy} + \frac{sb - ly}{p} - \frac{sngg - spyy}{pqy} = f$. Et reducendo, $yy = \frac{apqry - bmqsy + fmpqy + ggmns - ggnpr}{ppq - ppr - mlq + mps}$. Et abbrevi. cauſâ ſcripto $2b$ pro $\frac{apqr - bmqs + fmpq}{ppq - ppr - mlq + mps}$, & kk pro $\frac{ggmns - ggpnr}{ppq - ppr - mlq + mps}$, fiet $yy = 2by + kk$, ſive $y = b \pm \sqrt{bb + kk}$. Cujus æquationis ope cùm y innoteſcit, æquatio $\frac{ngg - pyy}{qy} = x$ dabit x: quod ſufficit ad determinandum punctum z.

Ad eundem ferè modum punctum determinatur, à quo ad plures vel pauciores poſitione datas rectas totidem aliæ rectæ ducantur, eâ lege ut aliquarum ſumma, vel differentia, vel contentum detur; aut æquetur cæterarum ſummæ, vel differentiæ, vel contento, vel ut alias quaſlibet habeant aſſignatas conditiones.

PROB. XXIV.

Angulum rectum, EAF, *datâ rectâ,* EF, *ſubtendere, quæ tranſibit per datum punctum,* C, *a lineis rectum angulum comprehendentibus æquidiſtans.*



Quadratum ABCD compleatur, & linea EF biſcetur in G. Tum dic CB vel CD eſſe a, EG vel FG eſſe b, & CG eſſe x; eritque CE $= x - b$, & CF $= x + b$. Dein cùm $CFq - BCq = BFq$, erit BF $= \sqrt{xx + 2bx + bb - aa}$. Denique propter ſimilia triangula CDE, FBC, eſt CE.CD :: CF.BF, ſive $x - b . a :: x + b . \sqrt{xx + 2bx + bb - aa}$. Unde $ax + ab = \overline{x - b} \times \sqrt{xx + 2bx + bb - aa}$. Cujus æquationis utrâque parte quadratâ, & prodeuntibus terminis in ordinem redactis, prodit $x^4 = \begin{matrix} 2aa \\ +2bb \end{matrix} xx \pm \frac{2aabb}{b^4}$. Et, extractâ radice, ſicut fit in æquationibus quadraticis, prodit $xx = aa + bb + \sqrt{a^4 + 4aabb}$. Adeoque $x = \sqrt{aa + bb + \sqrt{a^4 + 4aabb}}$. CG ſic inventa dat CE vel CF, quæ determinando punctum E vel F problemati ſatisfacit.

 Idem

Idem aliter.

Sit CE $= x$, CD $= a$, & EF $= b$, eritque CF $= x + b$, & BF $= \sqrt{xx + 2bx + bb - aa}$. Et proinde, cùm ſit CE.CD::CF.BF, ſive $x . a :: x+b . \sqrt{xx+2bx+bb-aa}$, erit $ax + ab = x\sqrt{xx+2bx+bb-aa}$. Hujus æquationis partibus quadratis, & terminis in ordinem redactis, prodibit $x^4 + 2bx^3 {+bb \atop -2aa} xx - 2aabx - aabb = 0$, æquatio biquadratica, cujus radicis inveſtigatio difficilior eſt quàm in priori caſu. Sic autem inveſtigari poteſt. Pone $x^4 + 2bx^3 {+bb \atop -2aa} xx - 2aabx + a^4 = aabb + a^4$; & extractâ utrobique radice, $xx + bx - aa = \pm a\sqrt{aa+bb}$.

Ex his occaſionem nactus ſum tradendi *Regulam de electione terminorum* ad ineundum calculum.

Scilicet cùm duorum terminorum talis obvenit affinitas, ſive ſimilitudo relationis, ad cæteros terminos quæſtionis, ut oporteret æquationes per omnia ſimiles ex utrovis adhibito produci; aut ambos, ſi ſimul adhiberentur, eaſdem in æquatione finali dimenſiones, & eandem omninò formam, ſignis fortè + & − exceptis, habituros eſſe; (id quod facile proſpicitur) tunc neutrum adhibere convenit; ſed eorum vice tertium quemvis eligere, qui ſimilem utrique relationem gerit; puta ſemiſummam vel ſemidifferentiam, vel medium proportionale forſan, aut quamvis aliam quantitatem utrique indifferenter & ſine compare relatam.

Sic in præcedente problemate, cùm viderim lineam EF pariter ad utramque AB & AD referri (quod patebit ſi ducas itidem EF in angulo BAH) atque adeo nullâ ratione ſuaderi poſſem, cur ED potiùs quàm BF, vel AE potiùs quàm AF, vel CE potius quàm CF, pro quærendâ quantitate adhiberentur; vice punctuorum E & F undè hæc ambiguitas proficiſcitur, ſumpſi, in ſolutione priori, intermedium G, quod parem relationem ad utramque linearum AB & AD obſervat. Deinde ab hoc G non demiſi perpendiculum ad AF, pro quærendâ quantitate, quia potui eâdem ratione demiſiſſe ad AD. Et eapropter in neutrum, CB vel CD, demiſi, ſed inſtitui CG quærendum eſſe, quod nullum admittit compar; & ſic æquationem biquadraticam obtinui ſine terminis imparibus.

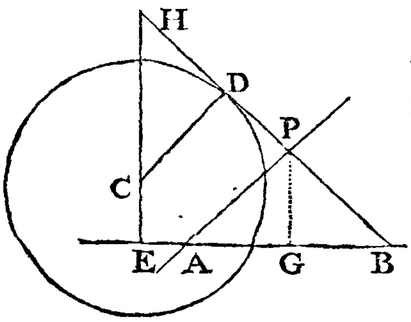
 Potui

CAPUT XIV. Potui etiam, animadverſo quod punctum G jaceat in peripheriâ circuli centro A radio EG deſcripti, demiſiſſe GK perpendiculum in diagonalem AC, & quæſiviſſe AK vel CK, quippe quæ ſimilem etiam utrique, AB & AD, relationem gerunt; atque ita in æquationem quadraticam $yy = -\frac{1}{2}ey + \frac{1}{2}bb$ incidiſſem, poſito $AK = y$, $AC = e$, & $EG = b$. Et AK ſic invento, erigendum fuiſſet perpendiculum KG præfato circulo occurrens in G, per quod CF tranſiret.

Ad hanc regulam animum advertens, in Prob. 9 & 10. ubi trianguli latera germana, BC & AC, determinanda erant, quæſivi potiùs ſemidifferentiam quàm alterutrum eorum. Sed regulæ hujus utilitas ex vigeſimo octavo Problemate magis eluceſcet.

PROB. XXV.

Ad Circulum, centro C *radio* CD *deſcriptum, ducere Tangentem* DB*, cujus pars,* PB*, inter rectas poſitione datas,* AP*,* AB*, ſita, ſit datæ longitudinis.*



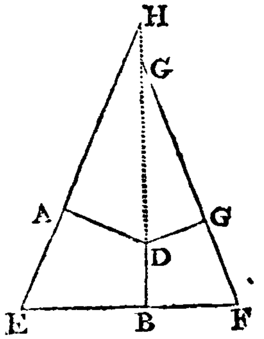
A Centro C ad alterutram rectarum poſitione datarum, puta AB, demitte normalem CE, eamque produc donec Tangenti DB occurrat in H. Ad eandem AB demitte etiam normalem PG, & dictis $EA = a$, $EC = b$, $CD = c$, $BP = d$, & $PG = x$, propter ſimilia triangula PGB, CDH, erit $GB\ (\sqrt{dd - xx}) .\ PB :: CD .\ CH = \frac{cd}{\sqrt{dd - xx}}$. Adde EC; & fiet $EH = b + \frac{cd}{\sqrt{dd - xx}}$. Porro eſt $PG . GB :: EH . EB = \frac{b}{x}\sqrt{dd - xx} + \frac{cd}{x}$. Adhæc, propter angulum PAG datum, datur ratio PG ad AG; quâ poſitâ e ad f, erit $AG = \frac{fx}{e}$. Adde EA & BG, & habebitur denuò $EB = a + \frac{fx}{e} + \sqrt{dd - xx}$. Eſt itaque $\frac{cd}{x} + \frac{b}{x}\sqrt{dd - xx} = a + \frac{fx}{e} + \sqrt{dd - xx}$: & per tranſpoſitionem terminorum $a + \frac{fx}{e} - \frac{cd}{x} = \frac{b - x}{x}\sqrt{dd - xx}$. Et partibus æquationis quadratis, $aa + \frac{2afx}{e} - \frac{2acd}{x} + \frac{ffxx}{ee} - \frac{2cdf}{e} + \frac{ccdd}{xx} = \frac{bbdd}{xx} - bb - \frac{2bdd}{x} + 2bx + dd - xx$. Et, per debitam reductionem,

$$\frac{x^4 \begin{matrix} +2aef \\ -2bee \end{matrix} x^3 \begin{matrix} +aaee \\ +bbee \\ -ddee \\ -2cdef \end{matrix} xx \begin{matrix} +2bddee \\ -2acdee \end{matrix} x \begin{matrix} +ccddee \\ -bbddee \end{matrix}}{ee + ff} = 0.$$

PROB.

PROB. XXVI.

Invenire punctum D, *à quo tres rectæ,* DA, DB, DC, *ad totidem alias poſitione datas rectas,* AE, BF, CF, *perpendiculariter demiſſæ, datam inter ſe rationem obtineant.*



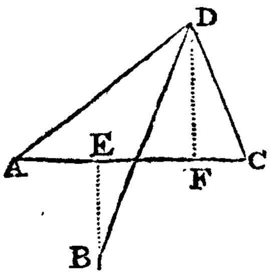
E Rectis poſitione datis producatur una, puta BF, ut & ejus perpendicularis, BD, donec reliquis, AE & CF, occurrant, BF quidem in E & F, BD autem in H & G. Jam ſit $EB = x$ & $EF = a$; eritque $BF = a - x$. Cùm autem propter datam poſitionem rectarum EF, EA, & FC, anguli E & F, adeoque rationes laterum triangulorum EBH & FBG dentur; ſit EB ad BH ut d ad e; & erit $BH = \frac{ex}{d}$, & EH $(= \sqrt{EBq + BHq}) = \sqrt{xx + \frac{eexx}{dd}}$, hoc eſt $= \frac{x}{d}\sqrt{dd+ee}$. Sit etiam BF ad EG ut d ad f; & erit $BG = \frac{fa - fx}{d}$, & FG $(= \sqrt{BFq + BGq}) = \sqrt{aa - 2ax + xx + \frac{ffaa - 2ffax + ffxx}{dd}}$, hoc eſt $= \frac{a-x}{d}\sqrt{dd+ff}$. Præterea dicatur $BD = y$, & erit $HD = \frac{ex}{d} - y$, & $GD = \frac{fa - fx}{d} - y$; adeoque, cùm ſit AD. HD (: : EB. EH) : : $d. \sqrt{dd+ee}$, & DC. GD (: : BF. FG) : : $d. \sqrt{dd+ff}$, erit $AD = \frac{ex - dy}{\sqrt{dd+ee}}$, & $DC = \frac{fa - fx - dy}{\sqrt{dd+ff}}$. Denique, ob datas rationes linearum BD, AD, DC, ſit BD. AD : : $\sqrt{dd+ee}. h - d$; & erit $\frac{hy - dy}{\sqrt{dd+ee}}$ (= AD) $= \frac{ex - dy}{\sqrt{dd+ee}}$, ſive $hy = ex$. Sit etiam BD. DC : : $\sqrt{dd+ff}. k - d$; & erit $\frac{ky - dy}{\sqrt{dd+ff}}$ (= DC =) $\frac{fa - fx - dy}{\sqrt{dd+ff}}$, ſive $ky = fa - fx$. Eſt itaque $\frac{ex}{h}$ $(= y) = \frac{fa - fx}{k}$; &, per reductionem, $\frac{fha}{ek + fh} = x$. Quare cape EB. EF : : $h. \frac{ek}{f} + h$; dein BD. EB : : $e. h$; & habebitur punctum quæſitum D.

PROB. XXVII.

Invenire punctum D, *à quo tres rectæ,* DA, DB, DC, *ad data tria puncta* A, B, C, *ductæ, datam inter ſe rationem obtineant.*

E Datis tribus punctis junge duo quævis, puta A & C; & à tertio, B, ad lineam conjungentem, AC, demitte perpendiculum

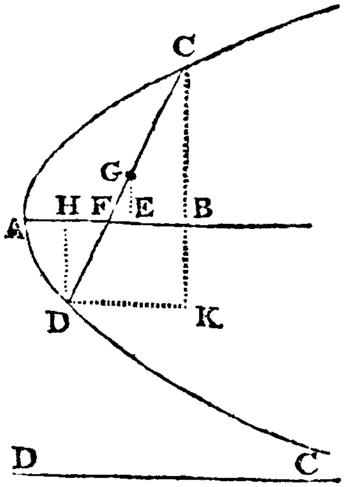
CAPUT XIV. BE, ut & perpendiculum DF à puncto quæſito D; dictiſque $AE=a$, $AC=b$, $EB=c$, $AF=x$, & $FD=y$, erit $ADq=xx+yy$; $FC=b-x$; $CDq\ (=FCq+FDq)=bb-2bx+xx+yy$; $EF=x-a$, ac $BDq\ (=EFq+\overline{EB+FD}\ \text{quad.})=xx-2ax+aa+cc+2cy+yy$. Jam cùm ſit AD ad CD in datâ ratione, ſit iſta ratio d ad e; & erit $CD=\frac{e}{d}\sqrt{xx+yy}$. Cùm etiam ſit AD ad BD in datâ ratione, ſit iſta ratio d ad f, & erit $BD=\frac{f}{d}\sqrt{xx+yy}$. Adeoque eſt $\frac{eexx+eeyy}{dd}\ (=CDq)=bb-2bx+xx+yy$; & $\frac{ffxx+ffyy}{dd}\ (=BDq)=xx-2ax+aa+cc+2cy+yy$. In quibus ſi, abbreviandi cauſâ, pro $\frac{dd-ee}{d}$ ſcribatur p, & q pro $\frac{dd-ff}{d}$, emerget $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy=0$, & $aa+cc-2ax+2cy+\frac{q}{d}xx+\frac{q}{d}yy=0$. Per priorem eſt $\frac{2bqx-bbq}{p}=\frac{q}{d}xx+\frac{q}{d}yy$. Quare in poſteriori pro $\frac{q}{d}xx+\frac{q}{d}yy$ ſcribe $\frac{2bqx-bbq}{p}$, & orietur $\frac{2bqx-bbq}{p}+aa+cc-2ax+2cy=0$. Iterum, abbreviandi cauſâ, ſcribe m pro $a-\frac{bq}{p}$, & $2cn$ pro $\frac{bbq}{p}-aa-cc$; & erit $2mx+2cn=2cy$; terminiſque per $2c$ diviſis, $\frac{mx}{c}+n=y$. Quamobrem in æquatione $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy=0$, pro yy ſcribe quadratum de $\frac{mx}{c}+n$, & habebitur $bb-2bx+\frac{p}{d}xx+\frac{pmm}{dcc}xx+\frac{2pmn}{dc}x+\frac{pnn}{d}=0$. Ubi denuò ſi, abbreviandi cauſâ, $\frac{b}{r}$ ſcribatur pro $\frac{p}{d}+\frac{pmm}{dcc}$, $\frac{sb}{r}$ pro $b-\frac{pmn}{dc}$, & $\frac{tbb}{r}$ pro $bb+\frac{pnn}{d}$, habebitur $xx=2sx-tb$. Et, extractâ radice, $x=s\pm\sqrt{ss-tb}$. Invento x, æquatio $\frac{mx}{c}+n=y$ dabit y; & ex datis x & y, hoc eſt AF & FD, determinatur punctum quæſitum D.



PROB. XXVIII.

Rectam, DC, *datæ longitudinis in datam Conicam ſectionem,* DAC, *ſic inſcribere, ut ea per punctum,* G, *poſitione datum tranſeat.*

SIT AF axis Curvæ, & à punctis, D, G & C, ad hunc demitte normales DH, GE, & CB. Jam ad determinandam poſitionem



tionem rectæ DC, puncti D, aut C, inventio proponi poteſt; ſed cùm hæc ſint germana & adeo paria, ut ad alterutrum determinandum operatio ſimilis evaſura eſſet, ſive quærerem CG, CB, aut AB; ſive comparia DG, DH, aut AH; ea propter de tertio aliquo puncto proſpicio, quod utrumque D & C ſimiliter reſpectet, & unà determinet: et hujuſmodi video eſſe punctum F. PROBLEMATICA GEOMETRICA.

Jam ſit AE $= a$, EG $= b$, DC $= c$, EF $= z$; & præterea cùm relatio inter AB & BC habeatur in æquatione, quam ſuppono pro Conicâ ſectione determinandâ datam eſſe, ſit AB $= x$, & BC $= y$; & erit FB $= x - a + z$. Et propter GE. EF :: CB. FB erit iterùm FB $= \frac{yz}{b}$. Ergo $x - a + z = \frac{yz}{b}$.

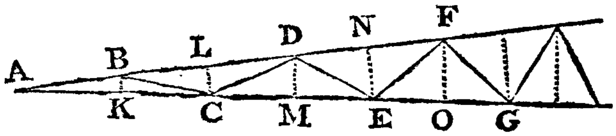
His ita præparatis, tolle x per æquationem quæ curvam deſignat. Quemadmodum ſi Curva ſit Parabola, per æquationem $rx = yy$ deſignata, ſcribe $\frac{yy}{r}$ pro x; & orietur $\frac{yy}{r} - a + x = \frac{yz}{b}$. Et, extractâ radice, $y = \frac{rz}{2b} \pm \sqrt{\frac{rrzz}{4bb} + ar - rz}$. Unde patet $\sqrt{\frac{rrzz}{bb} + 4ar - 4rz}$ eſſe differentiam gemini valoris y, id eſt linearum $+$ BC & $-$ DH, adeoque (demiſſo DK in CB normali) valere CK. Eſt autem FG. GE :: DC. CK, hoc eſt $\sqrt{bb + zz}.\ b :: c.\ \sqrt{\frac{rrzz}{bb} + 4ar - 4rz}$. Ducendoque quadrata extremorum & mediorum in invicem, & facta ordinando, orietur $z^4 = \frac{4bbrz^3 \begin{smallmatrix}-4abbr \\ -bbrr\end{smallmatrix} zz + 4b^4rz \begin{smallmatrix}-4ab^4r \\ +b^4cc\end{smallmatrix}}{rr}$, æquatio quatuor tantùm dimenſionum, quæ ad octo dimenſiones aſcendiſſet, ſi quæſiviſſem CG vel CB aut AB.

PROB. XXIX.

Datum angulum per datum numerum multiplicare vel dividere.

IN angulo quovis FAG inſcribe lineas AB, BC, CD, DE, *&c.* ejuſdem cujuſvis longitudinis; & erunt triangula, ABC, BCD, CDE, DEF, *&c.* Iſoſcelia; adeoque per 32. I. Elem. erit ang. CBD = ang.

CAPUT XIV. ang. A+ACB=2 ang. A. Et ang. DCE=ang. A+ADC=3 ang. A. Et ang. EDF=A+AED=4 ang. A. Et ang. FEG=5 ang. A, & ſic deinceps. Poſitis jam AB, BC, CD, &c. radiis æqualium circulorum, perpendicula BK, CL, DM, &c. demiſſa



in AC, BD, CE, &c. erunt ſinus iſtorum angulorum, & AK, BL, CM, DN, &c. ſinus complementorum ad rectum. Vel poſitâ AB diametro, illæ AK, BL, CM, &c. erunt chordæ. Sit ergo AB = $2r$, & AK $= x$: dein ſic operare.

$$AB \,.\, AK :: AC \,.\, AL.$$
$$2r \,.\, x :: 2x \,.\, \frac{xx}{r}.$$
$$\left.\begin{array}{r} AL - AB \\ \text{Et } \frac{xx}{r} - 2r \end{array}\right\} = BL, \text{ Duplicatio.}$$
$$AB \,.\, AK :: AD\,(2AL - AB) \,.\, AM.$$
$$2r \,.\, x :: \frac{2xx}{r} - 2r \,.\, \frac{x^3}{rr} - x.$$
$$\left.\begin{array}{r} AM - AC \\ \text{Et } \frac{x^3}{rr} - 3x \end{array}\right\} = CM, \text{ Triplicatio.}$$
$$AB \,.\, AK :: AE\,(2AM - AC) \,.\, AN.$$
$$2r \,.\, x :: \frac{2x^3}{rr} - 4x \,.\, \frac{x^4}{r^3} - \frac{2xx}{r}.$$
$$\left.\begin{array}{r} AN - AD \\ \text{Et } \frac{x^4}{r^3} - \frac{4xx}{r} + 2r \end{array}\right\} = DN, \text{ Quadruplicatio.}$$
$$AB \,.\, AK :: AF\,(2AN - AD) \,.\, AO.$$
$$2r \,.\, x :: \frac{2x^4}{r^3} - \frac{6xx}{r} + 2r \,.\, \frac{x^5}{r^4} - \frac{3x^3}{rr} + x.$$
$$\left.\begin{array}{r} AO - AE \\ \text{Et } \frac{x^5}{r^4} - \frac{5x^3}{rr} + 5x \end{array}\right\} = EO, \text{ Quintuplicatio.}$$

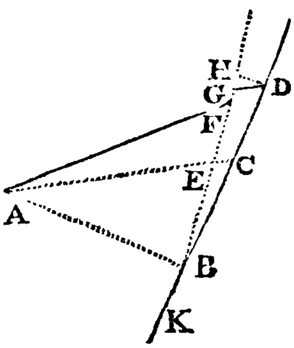
Et ſic deinceps. Quòd ſi velis angulum in aliquot partes dividere, pone q pro BL, CM, DN, &c. Et habebis $xx - 2rr = qr$ ad biſectionem: $x^3 - 3rrx = qrr$ ad triſectionem: $x^4 - 4rrxx + 2r^4 = qr^3$ ad quadriſectionem: $x^5 - 5rrx^3 + 5r^4x = qr^4$ ad quinquiſectionem, &c.

PROB.

PROB. XXX.

PROBLEMATA GEOMETRICA.

Cometæ in lineâ rectâ, BD, *uniformiter progredientis poſitionem curſûs ex tribus obſervationibus determinare.*



SIT A oculus ſpectatoris, B locus Cometæ in primâ obſervatione, C in ſecundâ, ac D in tertiâ; quærenda erit inclinatio lineæ BD ad lineam AB. Ex obſervationibus itaque dantur anguli BAC, BAD; adeoque, ſi BH ducatur ad AB normalis & occurrens AC & AD in E & F, ex aſſumpto utcunque AB dabuntur BE & BF, tangentes nempe præfatorum angulorum reſpectu radii AB. Sit ergo $AB = r$, $BE = b$, & $BF = c$. Porrò ex datis obſervationum intervallis dabitur ratio BC ad BD, quæ ſi ponatur b ad e, & agatur DG parallela AC, cùm ſit BE ad BG in eâdem ratione, & BE dicta fuerit b, erit $BG = e$; adeoque $GF = e - c$. Adhæc, ſi demittatur DH normalis ad BG, propter triangula ABF & DHF ſimilia & ſimiliter ſecta lineis AE ac DG, erit FE.AB::FG.HD, hoc eſt $c - b . a :: e - c . \frac{ae - ac}{c - b} = HD$. Erit etiam FE.FB::FG.FH, hoc eſt $c - b . c :: e - c . \frac{ce - cc}{c - b} = FH$; cui adde BF, ſive c, & fit $BH = \frac{ce - cb}{c - b}$. Quare eſt $\frac{ce - cb}{c - b}$ ad $\frac{ae - ac}{c - b}$, (ſive $ce - cb$ ad $ae - ac$, vel $\frac{ce - cb}{e - c}$ ad a) ut BH ad HD; hoc eſt, ut tangens anguli HDB, ſive ABK, ad radium. Quare, cùm a ſupponatur eſſe radius, erit $\frac{ce - cb}{e - c}$ tangens anguli ABK: adeoque factâ reſolutione, erit ut $e - c$ ad $e - b$ (ſive GF ad GE) ita c (ſive tangens anguli BAF) ad tangentem anguli ABK.

Dic itaque, ut tempus inter primam & ſecundam obſervationem, ad tempus inter primam ac tertiam, ita tangens anguli BAE, ad quartam proportionalem. Dein, ut differentia inter illam quartam proportionalem & tangentem anguli BAF, ad differentiam inter eandem quartam proportionalem & tangentem anguli BAE, ita tangens anguli BAF, ad tangentem anguli ABK.

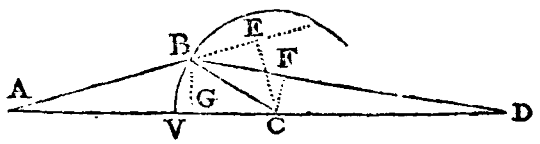
PROB.

CAPUT XIV.

P R O B. XXXI.

Radiis à puncto lucido ad ſphæricam ſuperficiem refringentem divergentibus, invenire concurſus ſingulorum refractorum cum axe ſphæræ per punctum illud lucidum tranſeunte.

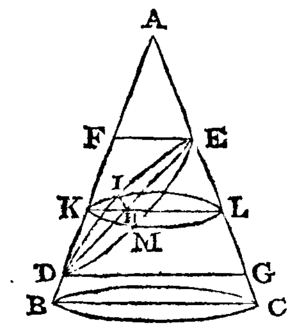
SIT A punctum illud lucidum, & BV ſphæra, cujus axis AD, centrum C, & vertex V; ſitque AB radius incidens, & BD refractus ejus, ac demiſſis ad radios iſtos perpendicularibus CE & CF, ut & BG perpendiculari ad AD, actâque BC, dic $AC = a$, VC, vel BC, $= r$, $CG = x$, & $CD = z$: eritque $AG = a - x$, $BG = \sqrt{rr - xx}$, $AB = \sqrt{aa - 2ax + rr}$; &, propter ſim. tri. ABG & ACE, $CE = \frac{a\sqrt{rr - xx}}{\sqrt{aa - 2ax + rr}}$. Item $GD = z + x$, $BD = \sqrt{zz + 2zx + rr}$: & propter ſim. tri. DBG ac DCF, $CF = \frac{z\sqrt{rr - xx}}{\sqrt{zz + 2zx + rr}}$. Præterea cùm ratio ſinuum incidentiæ & refractionis, adeoque CE ad CF detur, pone illam rationem eſſe a ad f; & erit $\frac{fa\sqrt{rr - xx}}{\sqrt{aa - 2ax + rr}} = \frac{az\sqrt{rr - xx}}{\sqrt{zz + 2zx + rr}}$; ac multiplicando in crucem, dividendoque per $a\sqrt{rr - xx}$, erit $f\sqrt{zz + 2zx + rr} = z\sqrt{aa - 2ax + rr}$: & quadrando, ac redigendo terminos in ordinem, $zz = \frac{2ffxz + ffrr}{aa - 2ax + rr - ff}$. Denique pro dato $\frac{ff}{a}$ ſcribe p, & q pro dato $a + \frac{rr}{a} - p$, & erit $zz = \frac{2pxz + prr}{q - 2x}$, ac $z = \frac{px + \sqrt{ppxx - 2prrx + pqrr}}{q - 2x}$. Inventum eſt itaque z; hoc eſt longitudo CD, adeoque punctum quæſitum D, quo refractus BD concurrit cum axe. Q. E. F.



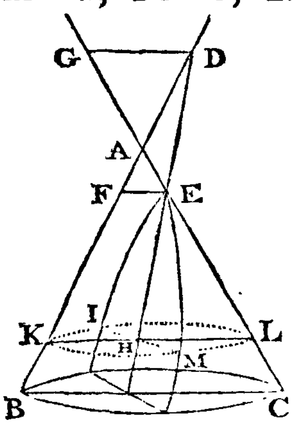
Poſui hic incidentes radios divergentes eſſe, & in Medium denſius incidere; ſed mutatis mutandis Problema perinde reſolvitur ubi convergunt, vel incidunt è denſiori Medio in rarius.

PROB. XXXII.

Si Conus plano quolibet ſecetur, invenire figuram ſectionis.



SIT ABC conus circulari baſi BC inſiſtens; IEM ejus ſectio quæſita; KILM alia quælibet ſectio parallela baſi, & occurrens priori ſectioni in HI; & ABC tertia ſectio, perpendiculariter biſecans priores duas, in EH & KL, & conum in triangulo ABC. Et producto EH donec occurrat ipſi AK in D, actiſque EF ac DG parallelis KL, & occurrentibus AB & AC in F ac G, dic EF $=a$, DG $=b$, ED $=c$, EH $=x$, & HI $=y$; & propter ſim. tri.



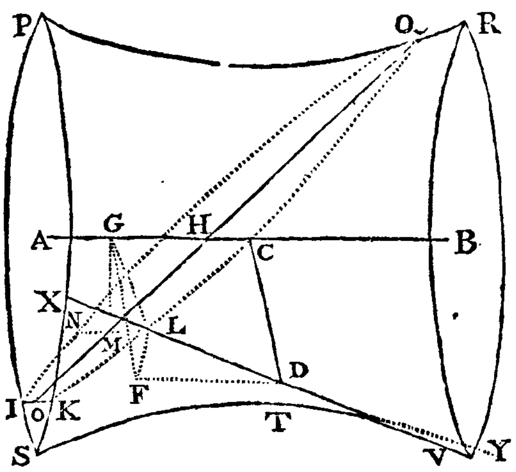
EHL, EDG, erit ED. DG : : EH. HL $= \frac{bx}{c}$. Dein propter ſim. tri. DEF, DHK, erit DE. EF : : DH. ($c-x$ in Fig. 1, & $c+x$ in Fig. 2.) HK $= \frac{ac \mp ax}{c}$. Deinque cùm ſectio KIL ſit parallela baſi, adeoque circularis; erit HK × HL = HIq; hoc eſt $\frac{ab}{c}\,x \mp \frac{ab}{cc}\,xx = yy$, æquatio quæ exprimit relationem inter EH (x) & HI (y) hoc eſt inter axem & ordinatim applicatam ſectionis EIM; quæ æquatio cùm ſit ad Ellipſin in Fig. 1, & ad Hyperbolam in Fig. 2, patet ſectionem illam perinde Ellipticam vel Hyperbolicam eſſe.

Quòd ſi ED nullibi occurrat AK, ipſi parallela exiſtens, tunc erit HK = EF (a), & inde $\frac{ab}{c}\,x$ (HK × HL) $= yy$, æquatio ad Parabolam.

PROB. XXXIII.

Si recta, XY, *circa axem* AB, *ad diſtantiam* CD, *in datâ inclinatione ad planum* DCB, *convolvatur, & ſolidum,* PQRVTS, *iſtâ convolutione generatum, ſecetur plano quolibet* INQLK; *invenire figuram Sectionis.*

ESTO BHQ, vel GHO, inclinatio axis, AB, ad planum ſectionis; & L quilibet concurſus rectæ XY cum plano illo. Age



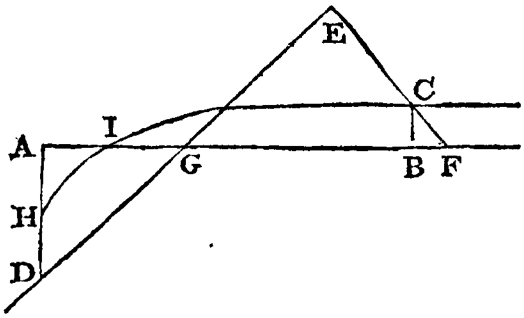
DF parallelam AB, & ad AB, DF & HO demitte perpendiculares LG, LF, LM, ac junge FG & MG. Dictiſque CD $= a$, CH $= b$, HM $= x$, & ML $= y$; &, propter datum angulum GHO, poſito MH. HG $:: d.e$: erit $\frac{ex}{d}$ = GH, & $b + \frac{ex}{d}$ = GC vel FD. Adhæc, propter angulum datum LDF (nempe inclinationem rectæ XY ad planum GCDF) poſito FD.FL$::g.h$, erit $\frac{hb}{g} + \frac{hex}{dg}$ = FL; cujus quadrato adde FGq, (DCq ſeu aa) & emerget GL$q = aa + \frac{hhbb}{gg} + \frac{2hhbex}{dgg} + \frac{hheexx}{ddgg}$. Hinc aufer MG$q$ (HMq − HGq ſeu $xx - \frac{ee}{dd}xx$) & reſtabit $\frac{aagg + hhbb}{gg} + \frac{2hhbe}{dgg}x + \frac{hhee - ddgg + eegg}{ddgg}xx$ (= MLq) $= yy$: æquatio quæ exprimit relationem inter x & y, hoc eſt inter HM axem ſectionis, & ML ordinatim applicatam. Et proinde, cùm in hâc æquatione x & y ad duas tantùm dimenſiones aſcendant, patet figuram INQLK eſſe conicam ſectionem. Utpote ſi angulus MHG major ſit angulo LDF, Ellipſis erit hæc figura; ſi minor, Hyperbola; ſi æqualis vel Parabola, vel, coincidentibus inſuper punctis C & H, parallelogrammum.

PROB. XXXIV.

Si ad AF *erigatur perpendiculum,* AD, *datæ longitudinis, & normæ,* DEF, *crus unum,* ED, *continuò tranſeat per punctum* D, *dum alterum crus* EF, *æquale* AD, *dilabatur ſuper* AF; *invenire curvam* HIC, *quam crus,* EF, *medio ejus puncto,* C, *deſcribit.*

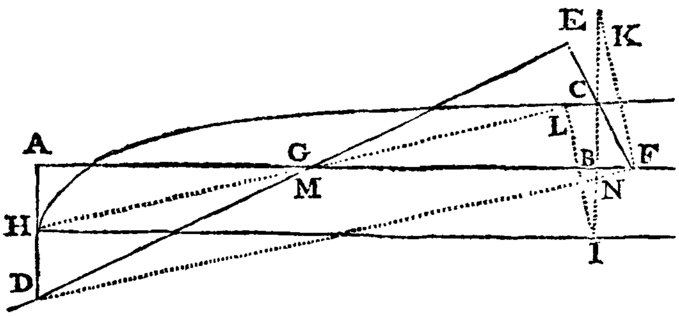
SIT EC vel CF $= a$, perpendiculum CB $= y$, AB $= x$: & propter ſimilia triangula FBC, FEG, erit BF $(\sqrt{aa - yy}.)$. BC + CF $(y + a)$ ::

PROBLEMATA GEOMETRICA.



EF ($2a$.). EG+GF (AG+GF) ſeu AF. Quare $\frac{2ay+2aa}{\sqrt{aa-yy}}$ (= AF = AB + BF) $= x+\sqrt{aa-yy}$. Jam, multiplicando per $\sqrt{aa-yy}$, fit $2ay+2aa = aa-yy+x\sqrt{aa-yy}$, ſeu $2ay+aa+yy=x\sqrt{aa-yy}$: & quadrando partes diviſas per $\sqrt{a+y}$, acordinando, prodit $y^3+3ayy \begin{smallmatrix}+3aa\\+xx\end{smallmatrix} y \begin{smallmatrix}+a^3\\-axx\end{smallmatrix}=0$.

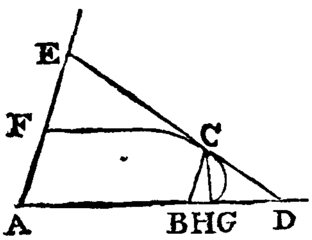
Idem aliter.



In BC cape hinc inde BI, & CK æquales CF; & age KF, HI, HC, ac DF; quarum HC ac DF occurrant ipſis AF & IK in M & N: & in HC demitte normalem IL. Eritque angulus K $=\frac{1}{2}$BCF $=\frac{1}{2}$EGF = GFD = AMH = MHI = CIL; adeoque triangula rectangula KBF, FBN, HLI & ILC ſimilia. Dic ergo FC$=a$, HI$=x$, & IC$=y$; & erit BN ($2a-y$.). BK (y) : : LC. LH : : CIq(yy.) HIq (xx.) adeoque $2axx-yxx=y^3$. Ex quâ æquatione facilè colligitur hanc curvam eſſe Ciſſoidem Veterum, ad circulum cujus centrum ſit A ac radius AH, pertinentem.

PROB. XXXV.

Si datæ longitudinis recta ED, *angulum datum*, EAD, *ſubtendens, ita moveatur, ut termini ejus,* D *&* E, *anguli iſtius latera,* AD *&* AE, *perpetim contingant; proponatur Curvam* FCG *determinare, quam punctum quodvis* C, *in rectâ iſtâ,* ED, *datum, deſcribit.*



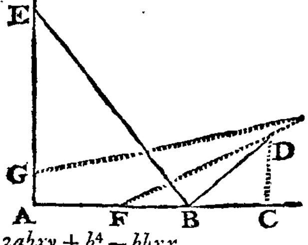
A Dato puncto C age CB parallelam EA; & dic AB$=x$, BC$=y$, CE$=a$, & CD$=b$: & propter ſimilia triangula DCB, DEA, erit EC. AB : : CD. BD. hoc eſt $a.x : : b.$ BD $=\frac{bx}{a}$. Præterea demiſſo perpendiculo CH, propter datum angulum DAE vel DBC, adeoque datam rationem laterum trianguli

CAPUT XIV. trianguli rectanguli BCH ſit $a.\ e :: BH.\ BH$, & erit $BH = \frac{ey}{a}$. Aufer hanc de BD, & reſtabit $HD = \frac{bx-ey}{a}$. Jam in triangulo BCH, propter angulum rectum BHC, eſt $BCq - BHq = CHq$; hoc eſt, $yy - \frac{eeyy}{aa} = CHq$. Similiter in triangulo CDH, propter angulum CHD rectum, eſt $CDq - CHq = DHq$, hoc eſt $bb - yy + \frac{eeyy}{aa}$ ($= HDq = \frac{bx-ey}{a}$ quadrato) $= \frac{bbxx - 2bexy + eeyy}{aa}$. Et, per reductionem, $yy = \frac{2be}{aa}xy + \frac{aabb - bbxx}{aa}$: ubi cùm incognitæ quantitates ſint duarum tantùm dimenſionum, patet curvam eſſe Conicam ſectionem. Præterea, extractâ radice, fit $y = \frac{bex \pm b\sqrt{eexx - aaxx + a^4}}{aa}$. Ubi in termino radicali coefficiens ipſius xx eſt $ee - aa$. Atqui erat $a.\ e :: BC.\ BH$; & BC neceſſariò major eſt linea quàm BH, nempe Hypotenuſa trianguli rectanguli major latere; ergo a major quàm e, & $ee - aa$ negativa eſt quantitas; atque adeo curva erit Ellipſis.

PROB. XXXVI.

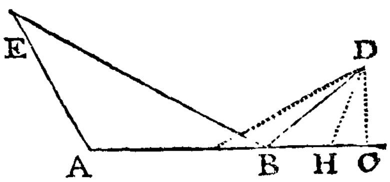
Si norma, EBD, *ita moveatur, ut ejus crus unum,* EB, *continuò ſubtendat angulum rectum* EAB, *dum terminus alterius cruris,* BD, *deſcribat curvam aliquam lineam* FDG; *invenire lineam iſtam,* FDG, *quam punctum* D *deſcribit.*

A Puncto D ad latus AC demitte perpendiculum DC; & dictis $AC = x$, & $DC = y$, atque $EB = a$, & $BD = b$; in triangulo BDC, propter angulum rectum ad C, eſt $BCq = BDq - DCq = bb - yy$.



Ergo $BC = \sqrt{bb-yy}$, & $AB = x - \sqrt{bb-yy}$. Præterea propter ſimilia triangula BEA. DBC, eſt $BD.\ DC :: EB.\ AB$, hoc eſt $b.\ y :: a.\ x - \sqrt{bb-yy}$. Quare $bx - b\sqrt{bb-yy} = ay$, ſive $bx - ay = b\sqrt{bb-yy}$. Et partibus quadratis ac debitè reductis, $yy = \frac{2abxy + b^4 - bbxx}{aa+bb}$. Et extractâ radice, $y = \frac{abx \pm bb\sqrt{aa+bb-xx}}{aa+bb}$. Unde patet iterum Curvam eſſe Ellipſin.

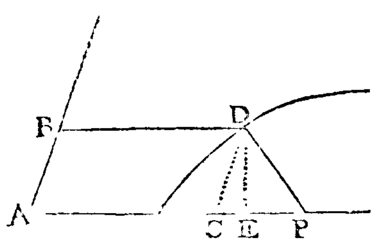
Hæc ita ſe habent, ubi anguli EBD & EAB recti ſunt: ſed ſi anguli iſti ſunt alteriùs cujuſvis magnitudinis, dummodo ſint æquales,



quales, ſic procedendum erit. Demittatur DC perpendicularis ad AC ut antè, & agatur DH conſtituens angulum DHA æqualem angulo HAE, puta obtuſum: dictiſque EB $= a$, BD $= b$, AH $= x$, & HD $= y$, propter ſimilia triangula EAB, BHD, erit BD. DH : : EB. AB; hoc eſt $b . y : : a$. AB $= \frac{ay}{b}$. Aufer hanc de AH, & reſtabit BH $= x - \frac{ay}{b}$. Præterea in triangulo DHC, propter omnes angulos datos, adeoque datam rationem laterum, aſſume DH ad HC in ratione quâvis datâ, puta b ad e; & cùm DH ſit y, erit HC $= \frac{ey}{b}$, & HB $\times$ HC $= \frac{exy}{b} - \frac{aeyy}{bb}$. Denique per 12. II. Elem. in triangulo BHD, eſt BDq = BHq + DHq + 2BH$\times$HC, hoc eſt $bb = xx - \frac{2axy}{b} + \frac{aayy}{bb} + yy + \frac{2exy}{b} - \frac{2aeyy}{bb}$. Et, extractâ radice, $x = \frac{ay - ey \pm \sqrt{eeyy - bbyy + bbbb}}{b}$. Ubi cùm b ſit major e, hoc eſt $ee - bb$ negativa quantitas, patet iterum curvam eſſe Ellipſin. PROBLEMATA GEOMETRICA.

PROB. XXXVII.

In dato angulo, PAB, *actis utcunque rectis*, BD, PD, *in datâ ratione, hâc ſemper lege, ut* BD *ſit parallela* AP, *&* PD *terminetur ad punctum*, P, *in rectâ* AP *datum; invenire locum puncti* D.



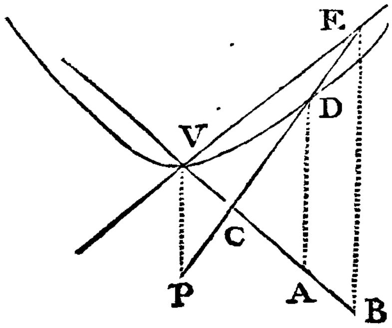
Age CD parallelam AB, & DE perpendicularum AP; ac dic AP $= a$, CP $= x$, & CD $= y$; ſitque BD ad PD in ratione d ad e; & erit AC, vel BD, $= a - x$, & PD $= \frac{ea - ex}{d}$. Sit inſuper, propter datum angulum DCE, ratio CD ad CE, d ad f; & erit CE $= \frac{fy}{d}$, & EP $= x - \frac{fy}{d}$. Atqui propter angulos ad E rectos, eſt CDq − CEq (= EDq) = PDq − EPq, hoc eſt, $yy - \frac{ffyy}{dd} = \frac{eeaa - 2eeax + eexx}{dd} - xx + \frac{2fxy}{d} - \frac{ffyy}{dd}$. Ac deletis utrobique $-\frac{ffyy}{dd}$, terminiſque ritè diſpoſitis, $yy = \frac{2fxy}{d} + \frac{eeaa - 2eeax + eexx - ddxx}{dd}$. Et, extractâ radice, $y = \frac{fx}{d} \pm \frac{\sqrt{eeaa - 2eeax \begin{smallmatrix} + ee \\ - dd \\ + ff \end{smallmatrix} xx}}{d}$. Ubi cùm x & y in æquatione penultimâ non niſi

Caput XIV. niſi ad duas dimenſiones aſcendant, locus puncti D erit Conica ſectio; eaque Hyperbola, Parabola, vel Ellipſis, prout $ee - dd + ff$, (coefficiens ipſius xx in æquatione poſteriori), ſit majus, æquale, vel minus nihilo.

PROB. XXXVIII.

Rectis duabus, VE *&* VC, *poſitione datis, & ab aliâ rectâ,* PE, *circa polum poſitione datum,* P, *vertente, ſectis utcunque in* C *&* E; *ſi recta intercepta,* CE, *dividatur in partes,* CD, DE, *rationem datam habentes, proponatur invenire locum puncti* D.

AGE VP, eique parallelas DA, EB occurrentes VC in A & B. Dic $VP = a$, $VA = x$, & $AD = y$: & cùm detur ratio CD ad DE, vel converſè ratio CD ad CE, hoc eſt ratio DA ad EB, ſit iſta ratio d ad e, & erit $EB = \frac{ey}{d}$. Præterea, cùm detur angulus EVB, adeoque ratio EB ad VB, ſit iſta ratio e ad f; & erit $VB = \frac{fy}{d}$. Denique, propter ſimilia triangula



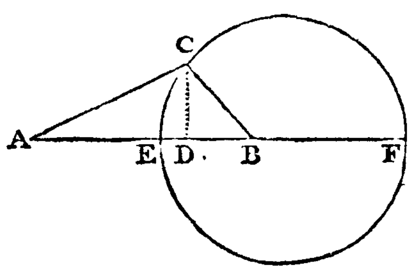
CEB, CDA, CPV, eſt EB.CB::DA.CA::VP.VC; & componendo, EB + VP. CB + VC:: DA + VP.CA + VC. Hoc eſt, $\frac{ey}{d} + a. \frac{fy}{d} :: y + a. x$. Ductiſque extremis & mediis in ſe, $eyx + dax = fyy + fay$. Ubi cùm indefinitæ quantitates, x & y, non niſi ad duas dimenſiones aſcendant, ſequitur curvam VD, in quâ punctum D perpetim reperitur, eſſe conicam ſectionem, eamque Hyperbolam; quia una ex indefinitis quantitatibus, nempe x eſt unius tantùm dimenſionis, & in termino exy multiplicatur per alteram indefinitam quantitatem y.

PROB. XXXIX.

Si rectæ duæ, AC, BC, *à duobus poſitione datis punctis,* A *&* B, *in datâ quâvis ratione ad tertium quodvis punctum,* C, *ducantur; invenire locum puncti concurſûs* C.

JUnge AB; & ad hanc demitte normalem CD: dictiſque $AB = a$, $AD = x$, $DC = y$: erit $AC = \sqrt{xx + yy}$; $BD = a - x$; & $BC (= \sqrt{BDq + DCq})$ $= \surd$

$=\sqrt{xx-2ax+aa+yy}$. Jam cùm detur ratio AC ad BC, ſit iſta d ad e; &, extremis & mediis in ſe ductis, erit $e\sqrt{xx+yy}=d\sqrt{xx-2ax+aa+yy}$. Et, per reductionem, $\sqrt{\frac{ddaa-2ddax}{ee-dd}-xx}=y$. Ubi cùm xx ſit negativum, & ſolâ unitate affectum, atque etiam angulus ADC rectus; patet curvam, in quâ punctum C locatur, eſſe circulum. Nempe in rectâ AB cape puncta E & F, ita ut ſint $d.\ e$: : AE. BE : : AF. BF, & erit EF circuli hujus diameter.

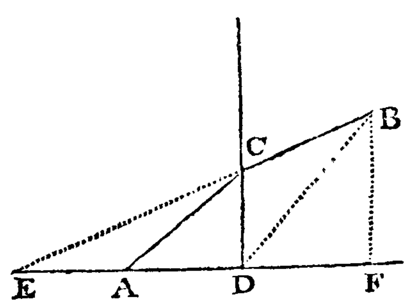


Et hinc è converſo patet hoc Theorema: Quòd in circuli cujuſvis diametro EF, infinitè productâ, datis utcunque duobus punctis, A & B, hâc lege, ut ſit AE. AF : : BE. BF, & à punctis hiſce actis duabus rectis, AC, BC, concurrentibus ad circulum in puncto quovis C; erit AC ad BC in datâ ratione AE ad BE.

PROB. XL.

Si punctum lucidum, A, *radios verſus refringentem ſuperficiem planam,* CD, *ejiciat; invenire radium,* AC, *cujus refractus,* CB, *impinget in datum punctum* B.

A Puncto iſto lucido ad refringens planum demitte perpendiculum AD, & cum eo, utrinque producto, concurrat refractus radius BC in E, & perpendiculum à puncto B demiſſum, in F, & agatur BD; dictiſque AD $=a$, DB $=b$, BF $=c$, DC $=x$, ſtatue rationem ſinuum incidentiæ & refractionis, hoc eſt ſinuum angulorum CAD, CED, eſſe d ad e: & cùm EC & AC (ut notum eſt) ſint in eâdem ratione, & AC ſit $\sqrt{aa+xx}$, erit EC $=\frac{d}{e}\sqrt{aa+xx}$.

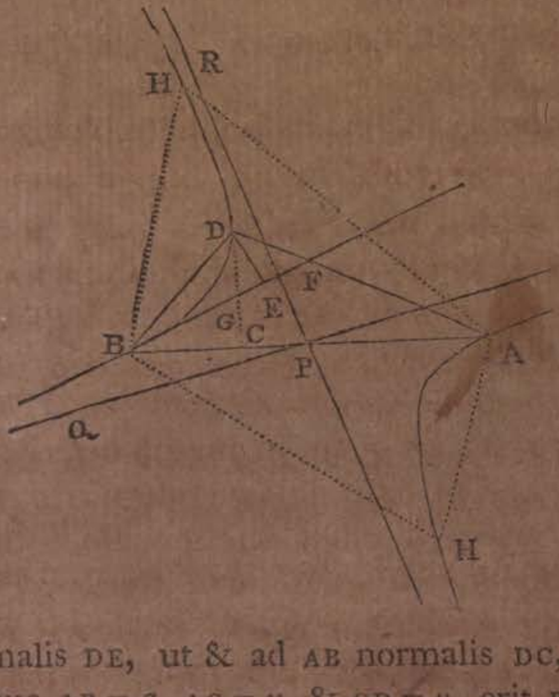


Præterea eſt ED $(=\sqrt{ECq-CDq})=\sqrt{\frac{ddaa+ddxx}{ee}-xx}$; & DF $=\sqrt{bb-cc}$; atque EF $=\sqrt{bb-cc}+\sqrt{\frac{ddaa+ddxx}{ee}-xx}$. Denique, propter ſimilia triangula ECD, EBF, eſt ED. DC : : EF. FB; &, ductis extremorum

CAPUT XIV. extremorum & mediorum valoribus in ſe, $c\sqrt{\frac{ddaa+ddxx}{ee}-xx} = x\sqrt{bb-cc}+x\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, ſive $\overline{c-x}\sqrt{\frac{ddaa+ddxx}{ee}-xx} = x\sqrt{bb-cc}$. Et, partibus æquationis quadratis & ritè diſpoſitis, $x^4 - 2cx^3 + \frac{\left.\begin{matrix}ddcc\\+ddaa\\-eebb\end{matrix}\right\} xx - 2ddaacx + ddaacc}{dd-ee} = 0$.

PROB. XLI.

Invenire locum verticis trianguli, D, *cujus baſis,* AB, *datur, & anguli ad baſem,* DAB, DBA, *datam habent differentiam.*



UBI angulus ad Verticem, ſive, quod perinde eſt, ubi ſumma angulorum ad baſem datur, docuit Euclides locum verticis eſſe circumferentiam circuli; III. 29. Euclid. propoſuimus igitur inventionem loci ubi differentia angulorum ad baſem datur. Sit angulus DBA major angulo DAB, ſitque ABF eorum data differentia, rectâ BF occurrente AD in F. Inſuper ad BF demittatur normalis DE, ut & ad AB normalis DC, occurrens BF in G. Dictiſque AB $= a$, AC $= x$, & CD $= y$, erit BC $= a - x$. Jam in triangulo BCG, cùm dentur omnes anguli, dabitur ratio laterum, BC & GC; ſit iſta d ad a, & erit CG $= \frac{aa-ax}{d}$. Aufer hanc de DC, ſive y, & reſtabit DG $= \frac{dy-aa+ax}{d}$. Præterea, propter ſimilia triangula BGC, DGE, eſt BG. BC : : DG. DE. Eſt autem in triangulo BGC, $a.d$: : CG. BC. Adeoque $aa.dd$: : CGq. BCq; & componendo, $aa+dd.dd$: : BGq. BCq. Et, extractis radicibus, $\sqrt{aa+dd}.d$ (: : BG. BC) : : DG. DE. Ergo DE $= \frac{dy-aa+ax}{\sqrt{aa+dd}}$. Adhæc, cùm angulus ABF ſit differentia angulorum BAD & ABD, adeoque anguli BAD & FBD æquentur, ſimilia erunt

triangula rectangula CAD & EBD, & proinde latera proportionalia DA. DC : : DB. DE. Sed eſt DC $= y$; DA $(=\sqrt{ACq + DCq}) = \sqrt{xx+yy}$; DB $(=\sqrt{BCq+DCq}) = \sqrt{aa-2ax+xx+yy}$; & ſuprà erat DE $= \frac{dy-aa+ax}{\sqrt{aa+dd}}$. Quare eſt $\sqrt{xx+yy}.\,y :: \sqrt{aa-2ax+xx+yy}.\, \frac{dy-aa+ax}{\sqrt{aa+dd}}$. Et extremorum & mediorum quadratis in ſe ductis, $aayy-2axyy+xxyy+y^4 =$

$$\frac{ddxxyy+ddy^4-2aadxxy-2aady^3+2adyx^3+2adxy^3+a^4xx+a^4yy-2a^3x^3-2a^3xyy+aax^4+aaxxyy}{aa+dd}.$$

Duc omnes terminos in $aa+dd$, & prodeuntes redige in debitum ordinem, & orietur $x^4 \begin{array}{c} -2a \\ +\frac{2d}{a}y \end{array} x^3 \begin{array}{c} -2dy \\ +aa \end{array} xx \begin{array}{c} +\frac{2dy^3}{a} \\ +\frac{2dd}{a}yy \end{array} x \begin{array}{c} -ddyy \\ -2dy^3 \\ -y^4 \end{array} = 0$. Divide hanc æquationem per $xx-ax \begin{array}{c} +dy \\ +yy \end{array}$, & orietur $xx \begin{array}{c} -a \\ +\frac{2d}{a}y \end{array} x \begin{array}{c} -yy \\ -dy \end{array} = 0$.

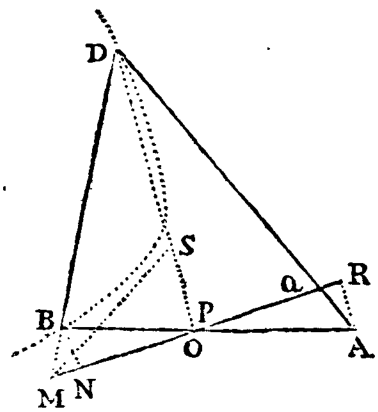
Duæ itaque prodierunt æquationes in ſolutione hujus Problematis. Prior, $xx-ax \begin{array}{c} +dy \\ +yy \end{array} = 0$, eſt ad circulum, locum nempe puncti D, ubi angulus FBD ſumitur ad alias partes rectæ BF quàm in figurâ deſcribitur, exiſtente angulo ABF ſummâ angulorum DAB, DBA ad baſem, adeoque angulo ADB ad verticem dato. Poſterior, $xx \begin{array}{c} -a \\ +\frac{2d}{a}y \end{array} x \begin{array}{c} -dy \\ -yy \end{array} = 0$ eſt, ad Hyperbolam, locum puncti D, ubi angulus FBD ſitum obtinet à rectâ BF quem in Figurâ deſcripſimus: hoc eſt, ita ut angulus ABF ſit differentia angulorum DAB, DBA ad baſem. Hyperbolæ autem hæc eſt determinatio. Biſeca AB in P. Age PQ conſtituentem angulum BPQ æqualem dimidio anguli ABF. Huic erige normalem PR, & erunt PQ, PR Aſſymptoti hujus Hyperbolæ, & B punctum per quod Hyperbola tranſibit.

Et hinc prodit tale Theorema. *Hyperbolæ rectangulæ diametro quàvis* AB *ductâ, & à terminis ejus ad Hyperbolæ puncta duo quævis,* D *&* H, *ductis rectis* AD, BD, AH, BH; *hæ rectæ angulos,* DAH, DBH, *ad terminos diametri conſtituent æquales.*

Idem breviùs.

Ad PROB. XXIV. *Regulam* de commodâ terminorum, ad ineundum calculum, electione tradidi; ubi obvenit ambiguitas in electione. Hîc differentia angulorum ad baſem eodem modo ſe

CAPUT XIV. habet ad utrumque angulum; &, in conſtructione Schematis, æque potuit addi ad angulum minorem DAB, ducendo ab A rectam ipſi BF parallelam, ac ſubtrahi ab angulo majori DBA ducendo rectam BF. Quamobrem nec addo nec ſubtraho; ſed dimidium ejus uni angulorum addo, alteri ſubtraho. Deinde, cùm etiam ambiguum ſit utrum AC vel BC pro termino indefinito, cui ordinatim applicata DC inſiſtit, adhibeatur, neutrum adhibeo; ſed biſeco AB in P, & adhibeo PC: vel potiùs actâ MPQ conſtituente hinc inde angulos, APQ, BPM, æquales dimidio differentiæ angulorum ad baſem, ita ut ea cum rectis AD, BD conſtituat angulos, DQP, DMP, æquales; ad MQ demitto normales, AR, BN, DO, & adhibeo DO pro ordinatim applicatâ, ac PO pro indefinitâ lineâ cui inſiſtit. Voco itaque PO$=x$, DO$=y$, AR, vel BN, $=b$, & PR, vel PN, $=c$. Et propter ſimilia triangula BNM, DOM, erit BN. DO :: MN.MO. Et dividendo, DO$-$BN$(y-b)$.DO(y):: MO$-$MN (ON ſive $c-x$). MO. Quare MO$=$



$\frac{cy-xy}{y-b}$. Similiter, ex alterâ parte, propter ſimilia triangula ARQ, DOQ, erit AR. DO :: RQ. QO: & componendo, DO + AR $(y+b)$. DO (y) :: QO + RQ (OR ſive $c+x$). QO. Quare QO$=\frac{cy+xy}{y+b}$. Denique, propter æquales angulos DMQ, DQM, æquantur MO & QO; hoc eſt, $\frac{cy-xy}{y-b}=\frac{cy+xy}{y+b}$. Divide omnia per y, & multiplica per denominatores, & orietur $cy+cb-xy-xb=cy-cb+xy-xb$, ſive $cb=xy$, notiſſima æquatio ad Hyperbolam.

Quin etiam locus puncti D ſine calculo Algebraico prodire potuit. Eſt enim ex ſuperioribus DO$-$BN.ON::DO.MO(QO)::DO$+$AR.OR. Hoc eſt DO $-$ BN. DO $+$ BN :: ON. OR; & mixtim, DO. BN :: $\frac{ON+OR}{2}$ (NP). $\frac{OR-ON}{2}$ (OP). Adeoque DO $\times$ OP $=$ BN $\times$ NP.

PROB. XLII.

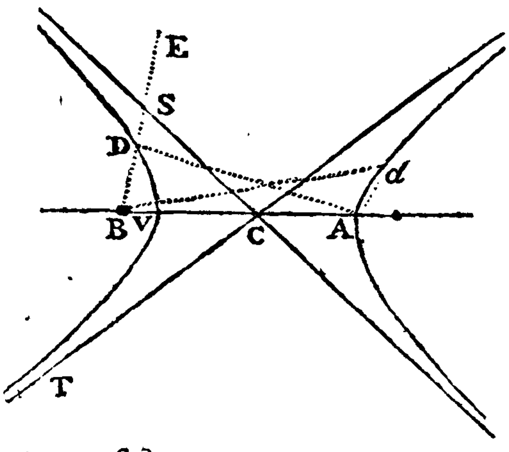
Locum verticis trianguli invenire cujus baſis datur, & angulorum ad baſem unus dato angulo differt à duplo alterius.

IN Schemate noviſſimo ſuperioris Problematis ſit ABD triangulum iſtud, AB baſis biſecta in P, APQ vel BPM triens anguli dati,

dati, quo angulus DBA excedit duplum anguli DAB: & angulus DMQ erit duplus anguli DQM. Ad MQ demitte perpendicula, AR, BN, DO; & angulum DMQ biſeca rectâ MS occurrente DO in S; & erunt triangula DOQ, SOM ſimilia; adeoque OQ. OM : : OD. OS; & dividendo, OQ − OM. OM : : SD. OS : : (per 3. VI. Elem.) DM. OM. Quare (per 9. V. Elem.) OQ − OM = DM. Dictis jam PO $= x$, OD $= y$, AR, vel BN, $= b$, & PR, vel PN, $= c$, erit, ut in ſuperiori Problemate, OM $= \frac{cy - xy}{y - b}$, et OQ $= \frac{cy + xy}{y + b}$; adeoque OQ − OM $= \frac{-2bcy + 2xyy}{yy - bb}$. Pone jam DO$q$ + OMq = DMq; hoc eſt, $yy + \frac{cc - 2cx + xx}{yy - 2by + bb} yy = \frac{4bbcc - 8bcxy + 4xxyy}{y^4 - 2bbyy + b^4} yy$. Et, per debitam reductionem, orietur tandem $y^4 * \begin{matrix} + cc \\ - 2bb \\ - 2cx \\ - 3xx \end{matrix} yy \begin{matrix} + 2bxx \\ + 4bcx \\ + 2bcc \end{matrix} y \begin{matrix} + b^4 \\ - 3bbcc \\ - 2bbcx \\ + bbxx \end{matrix} = 0$.

Divide omnia per $y - b$, & evadet $y^3 + byy \begin{matrix} - bb \\ + cc \\ - 2cx \\ - 3xx \end{matrix} y \begin{matrix} - b^3 \\ + 2bcc \\ + 2bcx \\ - bxx \end{matrix} = 0$. Quare punctum D eſt ad curvam trium dimenſionum; quæ tamen evadit Hyperbola, ubi angulus BPM ſtatuitur nullus, ſive angulorum ad baſem unus, DBA, duplus alterius, DAB. Tunc enim, BN ſive b evaneſcente, æquatio fiet $yy = 3xx + 2cx - cc$.

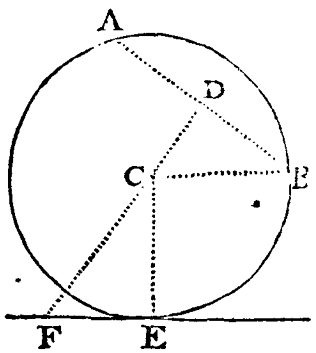
Ex hujus autem æquationis conſtructione tale elicitur Theorema. *Si centro* C, *Aſymptotis* CS, CT, *angulum* SCT 120 *graduum continentibus, deſcribatur Hyperbola quævis* DV, *cujus ſemiaxis ſint* CV, CA: *produc* CV *ad* B, *ut ſit* VB = VC; *& ab* A *&* B *actis utcunque rectis,* AD, BD, *concurrentibus ad Hyperbolam, erit angulus* BAD *dimidium anguli* ABD *triens vero anguli* ADE, *quem recta* AD *comprehendit cum* BD *productâ. Hoc intelligendum eſt de Hyperbolâ quæ tranſit per punctum* V. *Quòd ſi ab iiſdem punctis,* A *&* B, *actæ rectæ,* Ad, Bd, *conveniant ad conjugatam Hyperbolam, quæ tranſit per* A: *tunc externorum angulorum trianguli ad baſem ille ad* B *erit duplus alterius ad* A.



PROB. XLIII.

Circulum per data duo puncta deſcribere, qui rectam poſitione datam continget.

SUnto A & B puncta data, & EF recta poſitione data, & requiratur circulum ABE, per iſta puncta deſcribere, qui contingat rectam iſtam FE. Junge AB, & eam biſeca in D. Ad D erige normalem, DF, occurrentem rectæ FE in F, & circuli centrum incidet in hanc noviſſimè ductam DF, puta in C. Junge ergo CB, & ad FE demitte CE normalem; eritque E punctum contactûs, ac CB, CE æquales inter ſe, utpote radii circuli quæſiti. Jam cùm puncta, A, B, D, & F, dentur, eſto DB $= a$, ac DF $= b$; & ad determinandum centrum circuli quæratur DC, quam ideo dic x. Jam in triangulo CDB, propter angulum ad D rectum, eſt $\sqrt{DBq + DCq}$, hoc eſt $\sqrt{aa+xx}$, = CB. Eſt & DF − DC, ſive $b - x$, = CF. Et in triangulo rectangulo CFE cùm dentur anguli, dabitur ratio laterum CF & CE; ſit iſta d ad e; & erit CE $= \frac{e}{d} \times$ CF hoc eſt $= \frac{eb - ex}{d}$. Pone jam CB & CE, (radios nempe circuli quæſiti), æquales inter ſe; & habebitur æquatio $\sqrt{aa+xx} = \frac{eb - ex}{d}$. Cujus partibus quadratis & multiplicatis per dd, oritur $aadd + ddxx = eebb - 2cebx + eexx$. Sive $xx = \dfrac{+cebb - 2eebx - aadd}{dd - ee}$. Et, extractâ radice, $x = \dfrac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$. Inventa eſt ergo longitudo DC, adeoque centrum C, quo circulus per puncta A & B deſcribendus eſt, ut contingat rectam FE.

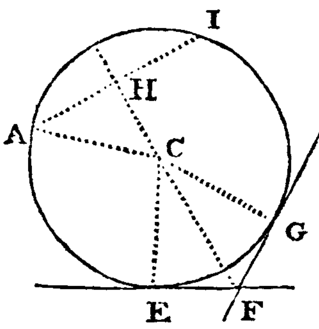


PROB. XLIV.

Circulum per datum punctum deſcribere, qui rectas duas poſitione datas continget.

Reſolvitur ut Prop. 43. Nam dato puncto A, datur & aliud punctum E.

ESTO datum punctum A, & ſint EF, FG rectæ duæ poſitione datæ, & AEG circulus quæſitus eaſdem contingens, ac tranſiens per punctum iſtud A. Rectâ CF biſecetur angulus EFG, & centrum



centrum circuli in ipſâ reperietur. Sit iſtud C; & ad EF & FG demiſſis perpendiculis, CE, CG, erunt E ac G puncta contactûs. Jam in triangulis CEF, CGF, cùm anguli ad E & G, ſint recti, & anguli ad F ſemiſſes ſint anguli EFG, dantur omnes anguli adeoque ratio laterum CF & CE, vel CG. Sit iſta d ad e, & ſi ad determinandum centrum circuli quæſiti C, aſſumatur $CF = x$, erit CE vel $CG = \frac{ex}{d}$. Præterea ad FC demitte normalem AH, & cùm punctum A detur, dabuntur etiam rectæ AH & FH. Dicantur iſtæ a & b, & ab FH, ſive b, ablato FC, ſive x, reſtabit $CH = b - x$. Cujus quadrato, $bb - 2bx + xx$, adde quadratum ipſius AH, ſive aa, & ſumma $aa + bb - 2bx + xx$, erit ACq per 47. I. Elem. ſiquidem angulus AHC ex Hypotheſi ſit rectus. Pone jam radios circuli, AC & CG, inter ſe æquales; hoc eſt pone æqualitatem inter eorum valores, vel inter quadrata eorum; & habebitur æquatio $aa + bb - 2bx + xx = \frac{eexx}{dd}$. Aufer utrobique xx, & mutatis omnibus ſignis, erit $-aa - bb + 2bx = xx - \frac{eexx}{dd}$. Duc omnia in dd, ac divide per $dd - ee$, & evadet $\frac{-aadd - bbdd + 2bddx}{dd - ee} = xx$, Cujus æquationis extracta radix eſt $x = \frac{bdd - d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$. Inventa eſt itaque longitudo FC, adeoque punctum C, quod centrum eſt circuli quæſiti.

Problemata Geometrica.

Si inventus valor x, ſive FC, auferatur de b, ſive HF, reſtabit $HC = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$, eadem æquatio quæ in priori problemate prodiit, ad determinandum longitudinem DC.

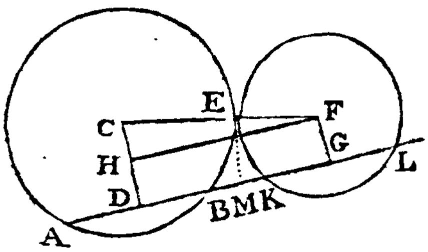
PROB. XLV.

Circulum per data duo puncta deſcribere, qui alium circulum poſitione datum continget. Vide Prop. 21.

SINT A, B puncta data, EK circulus poſitione & magnitudine datus, F centrum ejus, ABE circulus quæſitus per puncta A & B tranſiens, ac tangens alterum circulum in E, & C centrum ejus. Ad AB productam demitte perpendicula CD, & FG; & age

CF,

CAPUT XIV. CF, ſecantem circulos in puncto contactûs E, ac age etiam FH parallelam DG, & occurrentem CD in H. His conſtructis dic AD vel DB $=a$, DG vel HF $=b$, GF $=c$, & EF (radium nempe circuli dati) $=d$, atque DC $=x$: & erit CH (=CD − FG) $=x-c$; & CFq (=CHq + HFq) $=xx-2cx+cc+bb$; atque CBq(=CDq+DBq) $=xx+aa$; adeoque CB vel CE $=\sqrt{xx+aa}$. Huic adde EF, & habebitur CF $=d+\sqrt{xx+aa}$; cujus quadratum $dd+aa+xx+2d\sqrt{xx+aa}$, æquatur valori ejuſdem CFq prius invento, nempe $xx-2cx+cc+bb$. Aufer utrobique xx, & reſtabit $dd+aa+2d\sqrt{xx+aa}=cc+bb-2cx$. Aufer inſuper $dd+aa$, & habebitur $2d\sqrt{xx+aa}=cc+bb-dd-aa-2cx$. Jam, abbreviandi cauſâ, pro $cc+bb-dd-aa$, ſcribe $2gg$, & habebitur $2d\sqrt{xx+aa}=2gg-2cx$, ſive $d\sqrt{xx+aa}=gg-cx$. Et partibus æquationis quadratis, erit $ddxx+ddaa=g^4-2ggcx+ccxx$. Utrinque aufer $ddaa$ & $ccxx$, & reſtabit $ddxx-ccxx=g^4-ddaa-2ggcx$. Et partibus æquationis diviſis per $dd-cc$, habebitur $xx=\frac{g^4-ddaa-2ggcx}{dd-cc}$. Atque, per extractionem radicis affectæ, $x=\frac{-ggc+\sqrt{g^4dd-d^4aa+ddaacc}}{dd-cc}$.

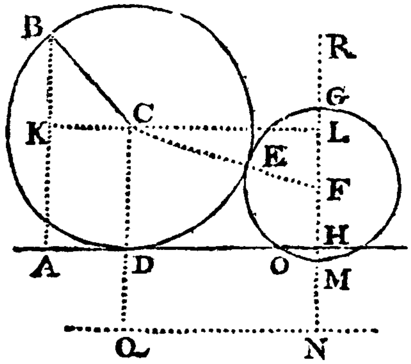


Inventâ igitur x, ſive longitudine DC, biſeca AB in D, & ad D erige perpendiculum DC $=\frac{-ggc+d\sqrt{g^4-aadd+aacc}}{dd-cc}$. Dein centro C, per punctum A vel B, deſcribe circulum ABE; nam hic continget alterum circulum EK, & tranſibit per utrumque punctum A, B. Q. E. F.

PROB. XLVI.

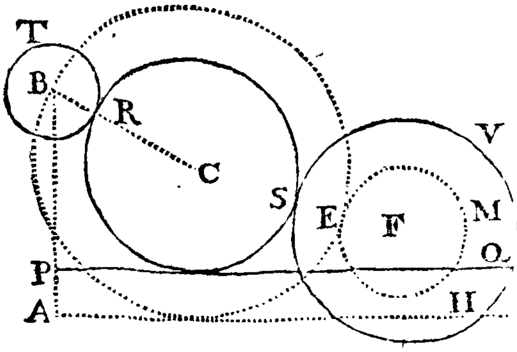
Circulum per datum punctum deſcribere qui datum circulum, & rectam lineam poſitione datam continget.

SIT circulus iſte deſcribendus BD, ejus centrum C, punctum per quod deſcribi debet B, recta quam continget AD, punctum contactûs D, circulus quem continget GEM, ejus centrum F, & punctum contactûs E. Junge CB, CD, CF; & CD erit perpendicularis ad AD, atque CF ſecabit circulos in puncto contactûs E.



Produc CD ad Q, ut ſit DQ=EF; & per Q age QN parallelam AD. Denique à B & F ad AD & QN demitte perpendicula BA, FN; & à C ad AB & FN perpendicula CK, CL. Et cùm ſit BC=CD vel AK, erit BK (= AB − AK) = AB − BC, adeoque BKq = ABq − 2AB × BC + BCq. Aufer hoc de BCq, & reſtabit 2AB × BC − ABq, pro quadrato de CK. Eſt itaque AB × $\overline{2BC-AB}$ = CKq; & eodem argumento erit FN × $\overline{2FC-FN}$ = CLq, atque adeo $\frac{CKq}{AB}$ + AB = 2BC, & $\frac{CLq}{FN}$ + FN = 2FC. Quamobrem, ſi pro AB, CK, FN, KL, & CL, ſcribas a, y, b, c, & $c-y$, erit $\frac{yy}{2a}+\frac{1}{2}a$ = BC, & $\frac{cc-2cy+yy}{2b}+\frac{1}{2}b$ = FC. De FC aufer BC, & reſtabit EF = $\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{2a}-\frac{1}{2}a$. Jam ſi puncta, ubi FN producta ſecat rectam AD & circulum GEM, notentur literis H, G, & M; & in HG productâ capiatur HR=AB; cùm ſit HN (=DQ= EF) = GF, addendo FH utrinque, erit FN = GH; adeoque AB − FN (=HR−GH) = GR; & AB−FN+2EF, hoc eſt $a-b+2$EF = RM; & $\frac{1}{2}a-\frac{1}{2}b$ + EF = $\frac{1}{2}$RM. Quare, cùm ſuprà fuerit EF = $\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{2a}-\frac{1}{2}a$, ſi hoc ſcribatur pro EF, habebitur $\frac{1}{2}$RM = $\frac{cc-2cy+yy}{2b}-\frac{yy}{2a}$. Dic ergo RM d, & erit $d=\frac{cc-2cy+yy}{b}-\frac{yy}{a}$. Duc omnes terminos in a & b, & orietur $abd=acc-2acy+ayy-byy$. Aufer utrinque $acc-2acy$, & reſtabit $abd-acc+2acy=ayy-byy$. Divide per $a-b$, & orietur $\frac{abd-acc+2acy}{a-b}=yy$. Et, extractâ radice $y=\frac{ac}{a-b}\pm\sqrt{\frac{aabd-abbd+abcc}{aa-2ab+bb}}$. Quæ concluſiones ſic abbreviari poſſunt. Pone $c.b::d.e$; dein $a-b.a::c.f$; & erit $fe-fc-2fy=yy$, ſive $y=f\pm\sqrt{ff+fe-fc}$. Invento y, ſive KC vel AD, cape AD $=f\pm\sqrt{ff+fe-fc}$; ad D erige perpendiculum DC (= BC) = $\frac{KCq}{2AB}+\frac{1}{2}$AB; & centro C, intervallo CB vel CD, deſcribe circulum BDE; nam hic tranſiens per datum punctum B, tanget rectam AD in D, & circulum GEM in E. Q. E. F.

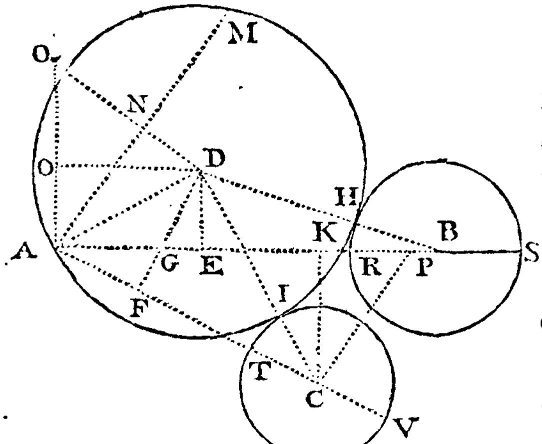
Hinc circulus etiam deſcribi poteſt, qui duos datos circulos & rectam poſitione datam continget. Sint enim circuli dati RT, SV; eorum centra B, F; & recta poſitione data PQ. Centro F, radio



FS − BR deſcribe circulum EM. A puncto B, ad rectam PQ demitte perpendiculum BP; & producto eo ad A, ut ſit PA = BR, per A age AH parallelam PQ; & circulus deſcribatur qui tranſeat per punctum B, tangatque rectam AH, & circulum EM. Sit ejus centrum C; junge BC ſecantem circulum RT in R; & eodem centro C, radio vero CR, deſcriptus circulus, RS, tanget circulos RT, SV, & rectam PQ, ut ex conſtructione manifeſtum eſt.

PROB. XLVII.

Circulum deſcribere qui per datum punctum tranſibit, & alios duos poſitione, & magnitudine datos circulos continget.



ESTO datum punctum A, ſintque circuli poſitione, & magnitudine dati TIV, RHS, centra eorum C & B, circulus deſcribendus AIH, centrum ejus D, & puncta contactûs I & H. Junge AB, AC, AD, DB, ſecetque AB producta circulum RHS in punctis R & S, & AC producta circulum TIV in T & V. Et à punctis D & C demiſſis perpendiculis, DE ad AB, & DF ad AC (occurrente AB in G) atque CK ad AB; in triangulo ADB erit $ADq - DBq + ABq = 2AE \times AB$, per 13. II. Elem. Sed $DB = AD + BR$, adeoque $DBq = ADq + 2AD \times BR + BRq$. Aufer hoc de $ADq + ABq$, & reſtabit $ABq - 2AD \times BR - BRq$, pro $2AE \times AB$. Eſt & $ABq - BRq = \overline{AB - BR} \times \overline{AB + BR} = AR \times AS$. Quare $AR \times AS - 2AD \times BR = 2AE \times AB$. Et $\frac{AR \times AS - 2AB \times AE}{BR} = 2AD$.

Et ſimili ratiocinio in triang. ADC emerget iterum $2AD = \frac{TAV - 2CAF}{CT}$. Quare $\frac{RAS - 2BAE}{BR} = \frac{TAV - 2CAF}{CT}$. Et $\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR} = \frac{2CAF}{CT}$.

$\frac{2CAF}{CT}$. Et $\overline{\frac{TAV}{CT}-\frac{RAS}{BR}+\frac{2BAE}{BR}}\times\frac{CT}{2AC}=AF$. Unde, cùm ſit AK.AC::AF.AG, erit $AG=\overline{\frac{TAV}{CT}-\frac{RAS}{BR}+\frac{2BAE}{BR}}\times\frac{CT}{2AK}$. Aufer hoc de AE, ſive $\frac{2KAE}{CT}\times\frac{CT}{2AK}$, & reſtabit $GE=\overline{\frac{RAS}{BR}-\frac{TAV}{CT}-\frac{2BAE}{BR}+\frac{2KAE}{CT}}\times\frac{CT}{2AK}$. Unde, cùm ſit KC.AK::GE.DE, erit $DE=\overline{\frac{RAS}{BR}-\frac{TAV}{CT}-\frac{2BAE}{BR}+\frac{2KAE}{CT}}\times\frac{CT}{2KC}$. In AB cape AP, quæ ſit ad AB ut CT ad BR; & erit $\frac{2PAE}{CT}=\frac{2BAE}{BR}$; adeoque $\frac{2PK\times AE}{CT}=\frac{2BAE}{BR}-\frac{2KAE}{CT}$: adeoque $DE=\overline{\frac{RAS}{BR}-\frac{TAV}{CT}-\frac{2PK\times AE}{CT}}\times\frac{CT}{2KC}$. Ad AB erige ergo perpendiculum $AQ=\overline{\frac{RAS}{BR}-\frac{TAV}{CT}}\times\frac{CT}{2KC}$, & in eo cape $QO=\frac{PK\times AE}{KC}$; & erit AO = DE.

PROBLEMATA GEOMETRICA.

Junge DO, DQ, CP; & triangula DOQ, CKP erunt ſimilia, quippe quorum anguli ad O & K ſunt recti, & latera (KC.PK::AE, vel DO.QO) proportionalia. Anguli ergo OQD, KPC æquales ſunt, & proinde QD perpendicularis eſt ad CP. Quamobrem, ſi agatur AN parallela CP, & occurrens QD in N, angulus ANQ erit rectus, & triangula AQN, PCK ſimilia; adeoque PC.KC::AQ.AN. Unde, cùm AQ ſit $\overline{\frac{RAS}{BR}-\frac{TAV}{CT}}\times\frac{CT}{2KC}$, AN erit $\overline{\frac{RAS}{BR}-\frac{TAV}{CT}}\times\frac{CT}{2PC}$. Produc AN ad M, ut ſit NM=AN; & erit AD = DM, adeoque circulus quæſitus tranſibit per punctum M. Cum ergo punctum M datum ſit, ex his, ſine ulteriori Analyſi, talis emergit Problematis reſolutio.

In AB cape AP, quæ ſit ad AB ut CT ad BR; junge CP, eique parallelam age AM, quæ ſit ad $\frac{RAS}{BR}-\frac{TAV}{CT}$, ut CT ad PC: & ope *Prob.* 45. per puncta A & M deſcribe circulum, AIHM, qui tangat alterutrum circulorum, TIV, RHS; & idem circulus tanget utrumque. Q.E.F.

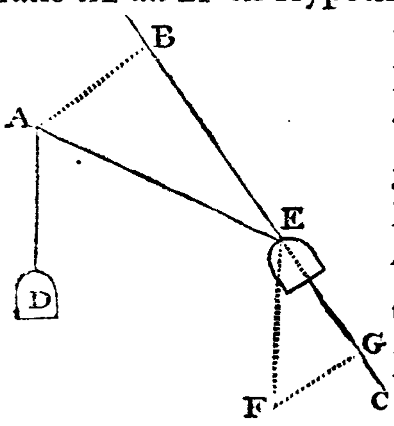
Et hinc circulus etiam deſcribi poteſt, qui tres circulos, poſitione & magnitudine datos, continget. Sunto trium datorum circulorum radii A, B, C, & centra D, E, F. Centris E & F, radiis $B\pm A$, $C\pm A$ deſcribantur duo circuli, & tertius circulus qui hoſce tangat, tranſeatque per punctum D. Sit hujus radius G, & centrum H; & eodem centro H, radio $G\pm A$, deſcriptus circulus continget tres primos circulos, ut fieri oportuit.

Caput XIV.

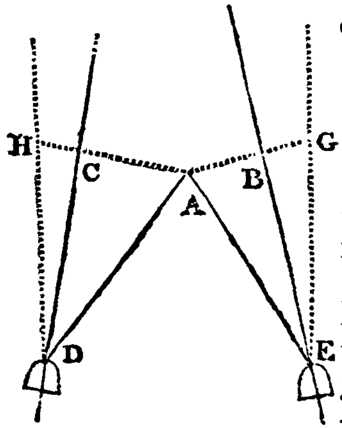
PROB. XLVIII.

Si ad extremitates fili, DAE, *circa paxillum,* A, *labentis, appendantur pondera duo,* D & E, *quorum pondus* E *labitur per lineam obliquam,* BG: *Invenire locum ponderis* E, *ubi pondera hæc in æquilibrio conſiſtunt.*

PUTA factum; & ipſi AD age parallelam EF quæ ſit ad AE, ut pondus E ad pondus D. Et à punctis A & F ad lineam BG demitte perpendicula AB, FG. Jam cùm pondera ex Hypotheſi ſint ut lineæ AE, EF, exponantur pondera per lineas iſtas; pondus D per lineam AE, & pondus E per lineam EF. Ergo Corpus E proprii ponderis vi directâ EF tendit verſus F; & vi obliquâ EG tendit verſus G. Et idem Corpus E ponderis D vi directâ AE trahitur verſus A; & vi obliquâ BE trahitur verſus B. Cùm itaque pondera ſe mutuò ſuſtineant in æquilibrio, vis, quâ pondus E trahitur verſus B, æqualis eſſe debet vi contrariæ, quâ tendit verſus G; hoc eſt BE æqualis eſſe debet ipſi EG. Jam verò datur ratio AE ad EF ex Hypotheſi; &, propter datum angulum FEG, datur etiam ratio FE ad EG, cui BE æqualis eſt. Ergo datur ratio AE ad BE. Datur etiam AB longitudine. Et inde triangulum ABE, & punctum E, facilè dabitur. Nempe dic $AB=a$, $BE=x$, & erit $AE=\sqrt{aa+xx}$. Sit inſuper AE ad BE in datâ ratione d ad e; & erit $e\sqrt{aa+xx}=dx$. Et, partibus æquationis quadratis & reductis, $eeaa=ddxx-eexx$, ſive $\frac{ea}{\sqrt{dd-ee}}=x$. Inventa eſt igitur longitudo BE, quæ determinat locum ponderis E. Q. E. F.



Quòd ſi pondus utrumque per lineam obliquam deſcendat, Computum ſic inſtitui poteſt. Sint CD, BE obliquæ lineæ, poſitione datæ, per quas pondera iſta, D & E, deſcendunt. A paxillo A ad has lineas demitte perpendicula, AC, AB; iiſque productis occurrant, in punctis G & H, lineæ EG, DH, à ponderibus perpendiculariter ad Horizontem erectæ; & vis quâ pondus E conatur deſcendere



deſcendere juxta lineam perpendicularem, hoc eſt tota gravitas ipſius E, erit ad vim quâ pondus idem conatur deſcendere juxta lineam obliquam BE, ut GE ad BE; atque vis, quâ conatur juxta lineam iſtam obliquam BE deſcendere, erit ad vim quâ conatur juxta lineam AE deſcendere, hoc eſt ad vim quâ filum AE diſtenditur, ut BE ad AE. Adeoque gravitas ipſius E erit ad tenſionem fili AE ut GE ad AE. Et eâdem ratione gravitas ipſius D erit ad tenſionem fili AD ut HD ad AD. Sit itaque fili totius, DA+AE, longitudo c; ſitque pars ejus AE $= x$; & erit altera pars AD $= c-x$. Et quoniam eſt AE$q-$AB$q=$BEq; & AD$q-$AC$q=$CDq; ſit inſuper AB $= a$, & AC $= b$; & erit BE $=\sqrt{xx-aa}$ & CD $=\sqrt{xx-2cx+cc-bb}$. Adhæc, cùm triangula, BEG, CDH, dentur ſpecie, ſit BE.EG::f.E; & CD.DH::$f.g$; & erit EG $=\frac{E}{f}\sqrt{xx-aa}$; & DH $=\frac{g}{f}\sqrt{xx-2cx+cc-bb}$. Quamobrem cùm ſit GE.AE:: pondus E. tenſionem AE; et HD.AD:: pondus D. tenſionem AD, & tenſiones iſtæ æquentur inter ſe, erit $\frac{Ex}{\frac{E}{f}\sqrt{xx-aa}}$ = tenſioni AE = tenſioni AD $=\frac{Dc-Dx}{\frac{g}{f}\sqrt{xx-2cx+cc-bb}}$. Cujus æquationis reductione provenit

$$gx\sqrt{xx-2cx+cc-bb}=\overline{Dc-Dx}\sqrt{xx-aa};$$ ſive $$\begin{matrix}gg\\-DD\end{matrix}x^4\begin{matrix}-2ggc\\+2DDc\end{matrix}x^3\begin{matrix}+ggcc\\-ggbb\\-DDcc\\+DDaa\end{matrix}xx-2DDcaax+DDccaa=0.$$

Si caſum deſideras quo hoc Problema per Regulam & circinum conſtrui queat, pone pondus D ad pondus E ut ratio $\frac{BE}{EG}$ ad rationem $\frac{CD}{DH}$; & evadet $g=$ D, adeoque vice præcedentis æquationis habebitur hæc, $\begin{matrix}aa\\-bb\end{matrix}xx-2aacx+aacc=0$; ſive $x=\frac{ac}{a+b}$.

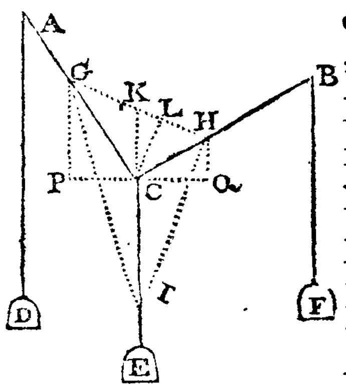
PROBLEMATA GEOMETRICA.

CAPUT XIV.

PROB. XLIX.

Si ad filum, DACBF, *circa paxillos duos,* A, B, *labile, appendantur tria pondera,* D, E, F; D *&* F *ad extremitates fili, &* E *ad medium ejus punctum,* C, *inter paxillos poſitum: Ex datis ponderibus & ſitu paxillorum invenire ſitum puncti* C, *ad quod medium pondus appenditur, ubi pondera conſiſtunt in æquilibrio.*

CUM tenſio fili AC æquetur tenſioni fili AD, & tenſio fili BC tenſioni fili BF; tenſiones filorum, AC, BC, EC, erunt ut pondera, D, F, E. In eâdem ponderum ratione cape partes filorum CG, CH, CI. Compleatur triangulum GHI. Produc IC, donec ea occurrat GH in K, & erit GK=KH, & CK=$\frac{1}{2}$CI, adeoque C centrum gravitatis trianguli GHI. Nam per C agatur ipſi CE perpendiculare, PQ; & huic à punctis G & H perpendicularia, GP, HQ. Et ſi vis, quâ filum AC vi ponderis D trahit punctum C verſus A, exponatur per lineam GC, vis, quâ filum iſtud trahet idem punctum verſus P, exponetur per lineam CP; & vis, quâ trahit illud verſus K, exponetur per lineam GP. Et ſimiliter vires, quibus filum BC vi ponderis F trahit idem punctum C verſus B, Q & K, exponentur per lineas CH, CQ, HQ; & vis, quâ filum CE vi ponderis E trahit punctum illud C verſus E, exponetur per lineam CI. Jam cùm punctum C viribus æquipollentibus ſuſtineatur in æquilibrio, ſumma virium, quibus fila AC & BC ſimul trahunt punctum C verſus K, æqualis erit vi contrariæ, quâ filum EC trahit punctum illud verſus E; hoc eſt ſumma, GP+HQ, æqualis erit ipſi CI: & vis, quâ filum AC trahit punctum C verſus P, æqualis erit vi contrariæ, quâ filum BC trahit idem punctum C verſus Q; hoc eſt linea PC æqualis lineæ CQ. Quare cùm PG, CK & QH parallelæ ſint, erit etiam GK=KH, & CK $(=\frac{GP+HQ}{2})=\frac{1}{2}$CI. Quod erat oſtendendum. Reſtat itaque triangulum GCH determinandum; cujus latera, GC & LC, dantur, unà cum lineâ CK, quæ à vertice C ad medium baſis ducitur, Demittatur itaque à vertice, C, ad baſem,

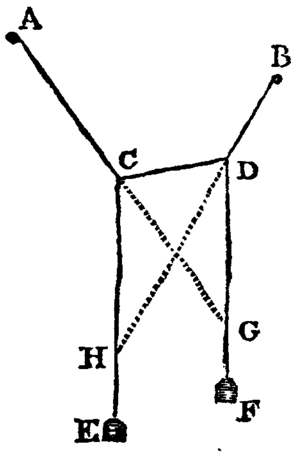


GH,

GH, perpendiculum CL; & erit $\frac{GCq - CHq}{2GH} = KL = \frac{GCq - KCq - GKq}{2GK}$. Pro 2GK ſcribe GH, &, rejecto communi diviſore GH, & ordinatis terminis, erit $GCq - 2KCq + CHq = 2GKq$; ſive $\sqrt{\frac{1}{2}GCq - KCq + \frac{1}{2}CHq} = GK$. Invento GK vel KH, dantur ſimul anguli GCK, KCH, ſive DAC, FBC. Quare à punctis A & B in datis iſtis angulis, DAC, FBC, duc lineas, AC, BC, concurrentes in puncto C: & iſtud C erit punctum quod quæritur. PROBLEMATA GEOMETRICA.

Cæterùm quæſtiones omnes, quæ ſunt ejuſdem generis, non ſemper opus eſt per Algebram ſigillatim ſolvere, ſed ex ſolutione unius plerumque conſectatur ſolutio alterius. Ut ſi jam proponeretur hæc quæſtio.

Filo, ACDB, *in datas partes*, AC, CD, DB, *diviſo, & extremitatibus ejus ad paxillos duos*, A, B, *poſitione datos ligatis, ſi ad puncta diviſionum*, C *ac* D, *appendantur pondera duo*, E *&* F; *ex dato pondere* F, *& ſitu punctorum* C *ac* D, *cognoſcere pondus* E.



EX præcedentis Problematis ſolutione ſatis facilè colligetur hæcce ſolutio hujus. Produc lineas AC, BD, donec occurrant lineis DF, CE, in G & H; & erit pondus E ad pondus F ut DG ad CH.

Et hinc obiter patet ratio componendi ſtateram ex ſolis filis, quà pondus corporis cujuſvis E, ex unico dato pondere F cognoſci poteſt.

PROB. L.

Lapide in puteum decidente, ex ſono lapidis fundum percutientis, altitudinem putei cognoſcere.

SIT altitudo putei x; & ſi lapis motu uniformiter accelerato deſcendat per ſpatium quodlibet datum a in tempore dato b, & ſonus motu uniformi tranſeat per idem ſpatium datum a in tempore dato d, lapis deſcendet per ſpatium x, in tempore $b\sqrt{\frac{x}{a}}$; ſonus autem, qui fit à lapide in fundum putei impingente, aſcen-

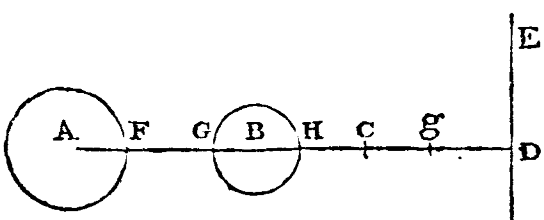
 det

CAPUT XIV. det per idem ſpatium x, in tempore $\frac{dx}{a}$. Ut enim ſunt ſpatia gravibus decidentibus deſcripta, ita ſunt quadrata temporum deſcenſûs. Vel ut radices ſpatiorum, hoc eſt ut $\sqrt{x}$ & $\sqrt{a}$, ita ſunt ipſa tempora. Et ut ſpatia x & a, per quæ ſonus tranſit, ita ſunt tempora tranſitûs. Ex horum temporum, $b\sqrt{\frac{x}{a}}$ & $\frac{dx}{a}$, ſummâ, conflatur tempus à lapide demiſſo ad ſoni reditum. Hoc tempus ex obſervatione cognoſci poteſt. Sit ipſum t; & erit $b\sqrt{\frac{x}{a}} + \frac{dx}{a} = t$. Ac $b\sqrt{\frac{x}{a}} = t - \frac{dx}{a}$. Et partibus quadratis $\frac{bbx}{a} = tt - \frac{2tdx}{a} + \frac{ddxx}{aa}$. Et, per reductionem, $xx = \frac{2adt + abb}{dd}x - \frac{aatt}{dd}$. Et, extractâ radice, $x = \frac{adt + \frac{1}{2}abb}{dd} - \frac{ab}{2dd}\sqrt{bb + 4dt}$.

PROB. LI.

Dato globo, A, *poſitione parietis,* DE, *& centri globi,* B, *à pariete diſtantiâ,* BD; *invenire molem globi* B, *eâ lege, ut in ſpatiis liberis, & vi gravitatis deſtitutis, ſi globus* A, *cujus centrum in lineâ* BD, *quæ ad parietem perpendicularis eſt, ultra* B *productâ conſiſtit, uniformi cum motu verſus* D *feratur, donec is impingat in alterum quieſcentem globum* B; *globus iſte* B, *poſtquam reflectitur à pariete, denuò occurrat globo* A *in dato puncto* C.

SIT globi A celeritas ante reflectionem a, & erit per PROB. XII. celeritas globi A poſt reflectionem $= \frac{aA - aB}{A + B}$, & celeritas globi B poſt reflexionem $= \frac{2aA}{A + B}$. Ergo celeritas globi A ad celeritatem globi B eſt ut A − B ad 2A. In GD cape gD = GH diametro nempe globi B; & celeritates iſtæ erunt ut GC ad Gg + gC. Nam ubi Globus A impegit in globum B; punctum G, quod in ſuperficie globi B exiſtens movetur in lineâ AD, perget per ſpatium Gg, antequam globus ille B impinget in parietem, & per ſpatium gC poſtquam à pariete reflectitur (hoc eſt per totum ſpatium Gg + gC) in eodem tempore quo globi A punctum F perget per ſpatium



GC:

GC: eò ut globus uterque rurſus conveniant, & in ſe mutuò impingant, in puncto dato C. Quamobrem cùm dentur intervalla BC & CD, dic $BC=m$, $BD+CD=n$, & $BG=x$; & erit $GC=m+x$, & $Gg+gC=GD+DC-2gD=GB+BD+DC-2GH=x+n-4x$, ſeu $=n-3x$. Suprà erat $A-B$ ad $2A$ ut celeritas globi A ad celeritatem globi B; & celeritas globi A ad celeritatem globi B ut GC ad $Gg+gC$; adeoque $A-B$ ad $2A$ ut GC ad $Gg+gC$. Ergo cùm ſit $GC=m+x$, & $Gg+gC=n-3x$, erit $A-B$ ad $2A$ ſicut $m+x$ ad $n-3x$. Porrò globus A eſt ad globum B ut cubus radii ejus AF ad cubum radii alterius GB; hoc eſt, ſi ponas radium AF eſſe s, ut s^3 ad x^3. Ergo $s^3-x^3.2s^3$ (: : $A-B.\ 2A$) : : $m+x.\ n-3x$. Et, ductis extremis & mediis in ſe, habebitur æquatio $s^3n-3s^3x-nx^3+3x^4=2ms^3+2s^3x$. Et, per reductionem, $3x^4-nx^3-5s^3x\ {+s^3n \atop -2s^3m}=0$. Cujus æquationis conſtructione dabitur globi B ſemidiameter x; quo dato, datur etiam Globus ille. Q. E. F.

PROBLEMATA GEOMETRICA.

Nota verò, quòd ubi punctum C jacet ad contrarias partes globi B, debet ſignum quantitatis $2m$ mutari, & ſcribi $3x^4-nx^3-5s^3x\ {+s^3n \atop +2s^3m}=0$.

Si datus eſſet Globus B, & quæreretur Globus A, eâ lege, ut globi duo poſt reflexionem convenirent in C, quæſtio foret facilior. Nempe in inventâ æquatione noviſſimâ ſupponendum eſſet x dari & s quæri. Quâ ratione per debitam reductionem illius æquationis, tranſlatis terminis $-5s^3x+s^3n-2s^3m$ ad æquationis partem contrariam, ac diviſâ utrâque parte per $5x-n+2m$, emergeret $\frac{3x^4-nx^3}{5x-n+2m}=s^3$. Ubi per ſolam extractionem radicis cubicæ obtinebitur s.

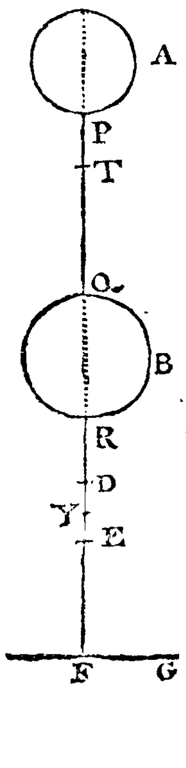
Quòd ſi dato Globo utroque quæreretur punctum C, in quo poſt reflexionem ambo in ſe mutuò impingerent: cùm ſuprà fuerit $A-b$ ad $2A$ ut GC ad $Gg+gC$; ergo, invertendo & componendo, $3A-B$ erit ad $A-B$ ut $2Gg$ ad diſtantiam quæſitam GC.

PROB.

PROB. LII.

Si globi duo, A *&* B, *tenui jungantur filo,* PQ, *& pendente globo* B *à globo* A, *ſi demittatur globus* A, *ita ut globus uterque ſimul ſolâ gravitatis vi, in eâdem lineâ perpendiculari* PQ *cadere incipiat; dein globus inferior* B, *poſtquam à fundo, ſeu plano horizontali* FG, *ſurſùm reflectitur, ſuperiori decidenti globo* A *occurrat in puncto quodam,* D: *ex datâ fili longitudine* PQ, *& puncti illius,* D, *à fundo diſtantiâ,* DF, *invenire altitudinem,* PF, *à quâ globus ſuperior,* A, *ad hunc effectum demitti debet.*

SIT fili PQ longitudo a. In perpendiculo PQRF ab F ſurſùm cape FE æqualem globi inferioris diametro QR; ita ut cùm globi illius punctum infimum, R, incidit in fundum ad F, punctum ejus ſupremum, Q, occupet locum E; ſitque ED diſtantia, per quam globus ille, poſtquam à fundo reflectitur, aſcendendo tranſit, antequam globo ſuperiori decidenti occurrat in puncto D. Igitur ob datam puncti D à fundo diſtantiam, DF, globique inferioris diametrum, EF, dabitur eorum differentia DE. Sit ea $=b$. Sitque altitudo, quam globus ille inferior, antequam impingit in fundum, cadendo deſcribit (RF vel QE) $=x$, ſiquidem ea ignoretur. Et invento x, ſi eidem addantur EF & PQ, habebitur altitudo PF, à quâ globus ſuperior ad effectum deſideratum demitti debet.



Cùm igitur ſit PQ $=a$, & QE $=x$, erit PE $=a+x$. Aufer DE, ſeu b, & reſtabit PD $=a+x-b$. Eſt autem tempus deſcenſûs globi A ut radix ſpatii cadendo deſcripti, ſeu $\sqrt{a+x-b}$; & tempus deſcenſûs globi alterius, B, ut radix ſpatii cadendo deſcripti, ſeu $\sqrt{x}$; & tempus aſcenſûs ejuſdem ut differentia radicis illius & radicis ſpatii, quod cadendo tantùm à Q ad D deſcriberetur. Nam hæc differentia eſt ut tempus deſcenſûs à D ad E; quod æquale eſt tempori aſcenſûs ab E ad D. Eſt autem differentia illa $\sqrt{x}-\sqrt{x-b}$. Unde tempus deſcenſûs & aſcenſûs conjunctim erit ut $2\sqrt{x}-\sqrt{x-b}$. Quamobrem, cùm hoc tempus æquetur tempori deſcenſûs globi ſuperioris, erit $\sqrt{a+x-b}=2\sqrt{x}-\sqrt{x-b}$. Cujus æquationis partibus quadratis habebitur $a+x-b=5x-b-$

$4\sqrt{xx-bx}$, ſeu $a=4x-4\sqrt{xx-bx}$; &, ordinatâ æquatione, $4x-a=4\sqrt{xx-bx}$. Cujus partes iterum quadrando oritur $16xx-8ax+aa=16xx-16bx$, ſeu $aa=8ax-16bx$. Et, diviſis omnibus per $8a-16b$, fiet $\frac{aa}{8a-16b}=x$. Fac igitur ut $8a-16b$ ad a ita a ad x, & habebitur x, ſeu QE. Q. E. I. PROBLEMATA GEOMETRICA.

Quòd ſi ex dato QE quæreretur fili longitudo PQ, ſeu a; eadem æquatio, $aa=8ax-16bx$, extrahendo affectam radicem quadraticam, daret $a=4x-\sqrt{16xx-16bx}$. Id eſt, ſi ſumas QY mediam proportionalem inter QD & QE, erit PQ$=4$EY. Nam media illa proportionalis erit $\sqrt{x\times\overline{x-b}}$, ſeu $\sqrt{xx-bx}$; quod ſubductum de x, ſeu QE, relinquit EY; cujus quadruplum eſt $4x-4\sqrt{xx-bx}$.

Sin verò ex datis tum QE, ſeu x, tum fili longitudine PQ, ſeu a, quæreretur punctum D, in quo globus ſuperior in inferiorem incidit: puncti illius à dato puncto E diſtantia DE, ſeu b, è præcedente æquatione $aa=8ax-16bx$, eruetur; transferendo aa & $16bx$ ad æquationis partes contrarias cum ſignis mutatis, & omnia dividendo per $16x$. Orietur enim $\frac{8ax-aa}{16x}=b$. Fac igitur ut $16x$, ad $8x-a$ ita a ad b, & habebitur b, ſeu DE.

Hactenus ſuppoſui globos tenui filo connexos ſimul dimitti. Quòd ſi nullo connexi filo diverſis temporibus dimittantur, ita ut globus ſuperior A, verbi gratiâ, priùs dimiſſus, deſcenderit per ſpatium PT, antequam globus alter incipiat cadere; & ex datis diſtantiis, PT, PQ, ac DE quæratur altitudo, PF, à quâ globus ſuperior dimitti debet, eâ lege ut in inferiorem incidat ad punctum D: ſit PQ$=a$, DE$=b$, PT$=c$, & QE$=x$; & erit PD$=a+x-b$ ut ſuprà. Et tempora, quibus globus ſuperior cadendo deſcribat ſpatia PT ac TD, & globus inferior priùs cadendo dein reaſcendendo deſcribat ſummam ſpatiorum QE+ED, erunt ut $\sqrt{\text{PT}}$, $\sqrt{\text{PD}}-\sqrt{\text{PT}}$, & $2\sqrt{\text{QE}}-\sqrt{\text{QD}}$; hoc eſt, ut $\sqrt{c}$, $\sqrt{a+x-b}-\sqrt{c}$, & $2\sqrt{x}-\sqrt{x-b}$. At ultima duo tempora, propterea quòd ſpatia, TD, & QE+ED, ſimul deſcribuntur, æqualia ſunt. Ergo $\sqrt{a+x-b}-\sqrt{c}=2\sqrt{x}-\sqrt{x-b}$. Et partibus quadratis, $a+c-2\sqrt{ca+cx-cb}=4x-4\sqrt{xx-bx}$. Pone $a+c=e$, & $a-b=f$; & erit, per debitam reductionem, $4x-e+2\sqrt{cf+cx}=4\sqrt{xx-bx}$; &, partibus quadratis, $ee-8ex+16xx+4cf+4cx$

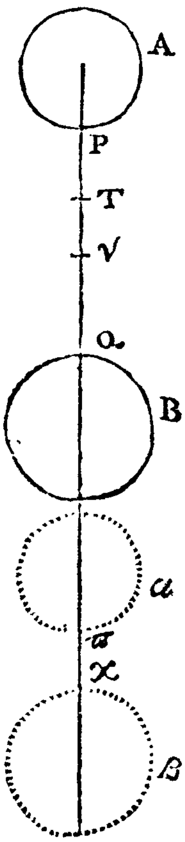
CAPUT XIV. $+4cx+\overline{16x-4e}\sqrt{cf+cx}=16xx-16bx$. Ac deletis utrobique $16xx$, & pro $ee+4cf$ ſcripto m, nec non pro $8e-16b-4c$ ſcripto n, habebitur, per debitam reductionem, $\overline{16x-4e}\sqrt{cf+cx}=nx-m$. Et partibus quadratis, $256cfxx+256cx^3-128cefx-128cexx+16ceef+16ceex=nnxx-2mnx+mm$. Et, ordinatâ æquatione,

$$256cx^3\ \begin{array}{l}+256cf\\-128ce\\-nn\end{array}xx\ \begin{array}{l}-128cef\\+16cee\\+2mn\end{array}x\ \begin{array}{l}+16ceef\\-mm\end{array}=0.$$

Cujus æquationis conſtructione, dabitur x ſeu QE; cui ſi addas datas diſtantias, PQ, & EF, habebitur altitudo PF, quam oportuit invenire.

PROB. LIII.

Si globi duo quieſcentes, ſuperior A, *&* *inferior* B, *diverſis temporibus dimittantur; & globus inferior eo temporis momento cadere incipiat, ubi ſuperior cadendo jam deſcripſit ſpatium* PT; *invenire loca* α, β, *quæ globi illi cadentes occupabunt, ubi eorum intervallum,* $\varpi\chi$, *dato æquale eſt.*



CUM dentur diſtantiæ, PT, PQ, & $\varpi\chi$, dic primam a, ſecundam b, tertiam c; & pro P ϖ, ſeu ſpatio quod globus ſuperior, antequam pervenit ad locum quæſitum α, cadendo deſcribit, ponatur x. Jam tempora, quibus globus ſuperior deſcribit ſpatia, PT, Pϖ, Tϖ, & inferior ſpatium Qχ, ſunt ut $\sqrt{PT}$, $\sqrt{P\varpi}$, $\sqrt{P\varpi}-\sqrt{PT}$, & $\sqrt{Q\chi}$. Quorum temporum poſteriora duo, eo quòd globi cadendo ſimul deſcribant ſpatia Tϖ & Qχ, ſunt æqualia. Unde & $\sqrt{P\varpi}-\sqrt{PT}$ æquale erit $\sqrt{Q\chi}$. Erat $P\varpi=x$, & $PT=a$; & ad Pϖ addendo $\varpi\chi$, ſeu c, & à ſummâ auferendo PQ, ſeu b, habebitur $Q\chi=x+c-b$. Quamobrem, his ſubſtitutis, fiet $\sqrt{x}-\sqrt{a}=\sqrt{x+c-b}$. Et æquationis partibus quadratis, orietur $x+a-2\sqrt{ax}=x+c-b$. Ac, deleto utrobique x, & ordinatâ æquatione, habebitur $a+b-c=2\sqrt{ax}$. Et partibus quadratis erit quadratum de $a+b-c$ æquale

quale $4ax$, & quadratum illud diviſum per $4a$ æquale x; ſeu $4a$ ad $a+b-c$ ſicut $a+b-c$ ad x. Ex invento autem x, ſeu Pϖ, datur globi ſuperioris decidentis locus quæſitus α. Et per locorum diſtantiam ſimul datur etiam locus inferioris β. PROBLEMATA GEOMETRICA.

Et hinc ſi punctum quæratur, ubi globus ſuperior cadendo tandem impinget in inferiorem; ponendo diſtantiam ϖχ nullam eſſe, ſeu delendo c, dic $4a$ ad $a+b$ ut $a+b$ ad x, ſeu Pϖ; & punctum ϖ erit quod quæris.

Et viciſſim ſi detur punctum illud ϖ, vel χ, in quo globus ſuperior incidit in inferiorem, & quæratur locus, T, quem ſuperioris globi decidentis punctum imum, P, tunc occupabat, cùm globus inferior incipiebat cadere: quoniam eſt $4a$ ad $a+b$ ut $a+b$ ad x, ſeu, ductis extremis & mediis in ſe, $4ax=aa+2ab+bb$; &, per æquationis debitam ordinationem, $aa=4ax-2ab-bb$: extrahe radicem quadraticam, & proveniet $a=2x-b-2\sqrt{xx-bx}$. Cape ergo Vϖ mediam proportionalem inter Pϖ & Qϖ, & verſus V cape VT=VQ, & erit T punctum quod quæris. Nam Vϖ erit $=\sqrt{P\varpi\times Q\varpi}$; hoc eſt $=\sqrt{x\times\overline{x-b}}$, ſeu $=\sqrt{xx-bx}$; cujus duplum ſubductum de $2x-b$, ſeu de 2Pϖ−PQ, hoc eſt de PQ+2Qϖ, relinquit PQ−2VQ, ſeu PV−VQ, hoc eſt PT.

Si denique globorum, poſtquam ſuperior incidit in inferiorem, & impetu in ſe invicem facto inferior acceleratur, ſuperior retardatur, deſiderantur loci, ubi inter cadendum diſtantiam datæ rectæ æqualem acquirent: quærendus erit primò locus, ubi ſuperior impingit in inferiorem; dein, ex cognitis tum magnitudinibus globorum tum eorum, ubi in ſe impingunt, celeritatibus, inveniendæ ſunt celeritates, quas proxime poſt reflexionem habebunt; idque per modum PROB. XII. Poſtea quærenda ſunt loca ſumma, ad quæ globi celeritatibus hiſce, ſi ſurſum ferantur, aſcenderent; & inde cognoſcentur ſpatia, quæ globi datis temporibus poſt reflexionem cadendo deſcribent, ut & differentia ſpatiorum: & viciſſim, ex aſſumptâ illâ differentiâ, per Analyſin regredietur ad ipſa ſpatia cadendo deſcripta.

Ut ſi globus ſuperior incidit in inferiorem ad punctum π, & poſt reflexionem celeritas ſuperioris deorſum tanta ſit, ut ſi ſur-

CAPUT XIV.



ſum eſſet, aſcendere faceret globum illum per ſpatium πN, & inferioris celeritas deorſum tanta eſſet, ut, ſi ſurſum eſſet, aſcendere faceret globum illum inferiorem per ſpatium πM; tum tempora, quibus globus ſuperior viciſſim deſcenderet per ſpatia $N\pi$, NG, & inferior per ſpatia $M\pi$, MH, forent ut $\surd N\pi$, $\surd NG$, $\surd M\pi$, $\surd MH$; adeoque tempora, quibus globus ſuperior conficeret ſpatium πG, & inferior ſpatium πH, forent ut $\surd NG - \surd N\pi$, ad $\surd MH - \surd M\pi$. Pone hæc tempora æqualia eſſe, & erit $\surd NG - \surd N\pi = \surd MH - \surd M\pi$. Et inſuper, cùm detur diſtantia GH, pone $\pi G + GH = \pi H$. Et harum duarum æquationum reductione ſolvetur problema. Ut ſi ſit $M\pi = a$, $N\pi = b$, $GH = c$, $\pi G = x$; erit, juxta poſteriorem æquationem, $x + c = \pi H$. Adde $M\pi$; fiet $MH = a + c + x$. Ad πG adde $N\pi$, & fiet $NG = b + x$. Quibus inventis, juxta priorem æquationem, erit $\sqrt{b+x} - \surd b = \sqrt{a+c+x} - \surd a$. Scribatur e pro $a + c$, & $\surd f$ pro $\surd a - \surd b$; & æquatio fiet $\sqrt{b+x} = \sqrt{e+x} - \surd f$. Et, partibus quadratis, $b + x = e + x + f - 2\sqrt{ef + fx}$, ſeu $e + f - b = 2\sqrt{ef + fx}$. Pro $e + f - b$ ſcribe g, & fiet $g = 2\sqrt{ef + fx}$; &, partibus quadratis, $gg = 4ef + 4fx$; &, per reductionem, $\frac{gg}{4f} - e = x$.

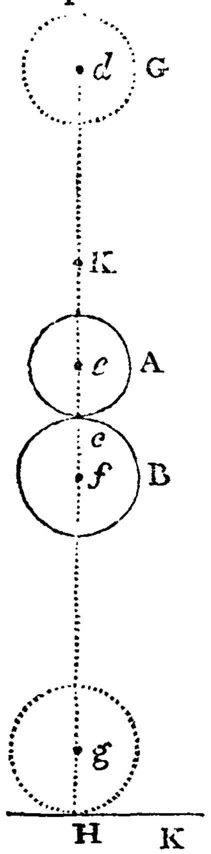
PROB. LIV.

Si duo ſint globi, A, B, *quorum ſuperior*, A, *ab altitudine*, G, *decidens, in alterum inferiorem*, B, *à fundo*, H, *verſus ſuperiora reſilientem, incidat; & hi globi ita per reflexionem ab invicem denuò recedant, ut globus* A, *vi reflexionis illius, ad altitudinem priorem*, G, *redeat, idque eodem tempore quo globus inferior*, B, *ad fundum*, H, *revertitur; dein globus* A *rurſus decidat, & in globum* B, *à fundo reſilientem, denuò incidat, idque in eodem loco*, AB, *ubi priùs in ipſum incidebat; & ſic perpetuò globi ab invicem reſiliant, rurſuſque ad eundem locum redeant: Ex datis globorum magnitudinibus, poſitione fundi & loco* G, *à quo globus ſuperior decidit, invenire locum ubi globi in ſe mutuò impingent.*

SIT c centrum globi A, & f centrum globi B; d centrum loci G, in quo globus ſuperior in maximâ eſt altitudine; g centrum loci globi inferioris ubi in fundum impingit; a ſemidiameter

meter globi A, b ſemidiameter globi B, c punctum contactûs globorum in ſe mutuò impingentium, & H punctum contactûs globi inferioris & fundi. Et celeritas globi A, ubi in globum B impingit, ea erit, quæ generatur caſu globi ab altitudine de; adeoque eſt ut $\sqrt{dc}$. Hâc eâdem celeritate reflecti debet globus A verſus ſuperiora, ut ad locum priorem, G redeat. Et globus B eâdem celeritate deorſùm reflecti debet, quâ aſcenderat, ut eodem tempore redeat ad fundum quo inde receſſerat. Ut autem hæc

PROBLEMATA GEOMETRICA.



duo eveniant, globorum motus inter reflectendum æquales eſſe debent. Motus autem ex globorum celeritatibus & magnitudinibus componuntur; adeoque quod fit ex globi unius mole & celeritate æquale erit ei, quod fit ex globi alterius mole & celeritate. Unde ſi factum ex unius globi mole & celeritate dividatur per molem alterius globi, habebitur celeritas alterius globi proximè ante & poſt reflexionem; ſeu ſub fine aſcenſûs & initio deſcenſûs. Erit igitur hæc celeritas ut $\frac{A\sqrt{de}}{B}$; ſeu, cùm globi ſint ut cubi radiorum, ut $\frac{a^3\sqrt{de}}{b}$. Ut autem hujus celeritatis quadratum ad quadratum celeritatis globi A proximè ante reflexionem, ita altitudo ad quam globus B hâc celeritate, ſi occurſu globi A in eum decidentis non impediretur, aſcenderet, ad altitudinem ed à quâ globus A deſcendit. Hoc eſt ut $\frac{Aq}{Bq}de$ ad de, ſeu ut Aq ad Bq, vel a^6 ad b^6, ita altitudo illa prior ad x; ſi modò pro altitudine poſteriore cd ponatur x. Ergo hæc altitudo, ad quam nimirum, B ſi non impediretur, aſcenderet, eſt $\frac{a^6}{b^6}x$. Sit ea fK. Ad fK adde fg, ſeu dH$-dc-cf-g$H; hoc eſt $p-x$, ſi modò pro dato dH$-cf-g$H ſcribas p, & x pro incognito de; & habebitur K$g = \frac{a^6}{b^6}x+p-x$. Unde celeritas globi B, ubi decidit à K ad fundum, hoc eſt ubi decidit per ſpatium Kg quod centrum ejus inter decidendum deſcriberet, erit ut $\sqrt{\frac{a^6}{b^6}x+p-x}$. At globus ille decidit à loco Bcf ad fundum, eodem tempore quo globus ſuperior, A, aſcendit à loco Ace ad ſummam altitudinem d,

aut

CAPUT XIV. aut viciſſim deſcendit à d ad locum ACE; & proinde, cùm gravium cadentium celeritates æqualibus temporibus æqualiter augeantur, celeritas globi B deſcendendo ad fundum tantùm augebitur, quanta eſt celeritas tota, quam globus A eodem tempore cadendo à d ad e acquirat, vel aſcendendo ab e ad d amittat. Ad celeritatem itaque quam globus B habet in loco Bcf adde celeritatem, quam globus A habet in loco Ace; & ſumma, quæ eſt ut $\sqrt{de} + \frac{a^3\sqrt{de}}{b^3}$, ſeu $\sqrt{x} + \frac{a^3}{b^3}\sqrt{x}$, erit celeritas globi B, ubi is in fundum incidit. Proinde $\sqrt{x} + \frac{a^3}{b^3}\sqrt{x}$ æquabitur $\sqrt{\frac{a^6}{b^6}x + p - x}$. Pro $\frac{a^3+b^3}{b^3}$ ſcribe $\frac{r}{s}$; & pro $\frac{a^6-b^6}{b^6}$, $\frac{rt}{ss}$; & æquatio illa fiet $\frac{r}{s}\sqrt{x} = \sqrt{\frac{rt}{ss}x + p}$; &, partibus quadratis, $\frac{rr}{ss}x = \frac{rt}{ss}x + p$. Aufer utrobique $\frac{rt}{ss}x$, duc omnia in ss, ac divide per $rr-rt$; & orietur $x = \frac{ssp}{rr-rt}$. Quæ quidem æquatio prodiiſſet ſimplicior, ſi modò aſſumpſiſſem $\frac{p}{s}$ pro $\frac{a^3+b^3}{b^3}$: prodiiſſet enim $\frac{ss}{p-t} = x$. Unde faciendo ut ſit $p-t$ ad s ut s ad x, habebitur x, ſeu ed; cui ſi addas ec, habebitur dc, & punctum c in quo globi in ſe mutuò impingent. Q. E. F.

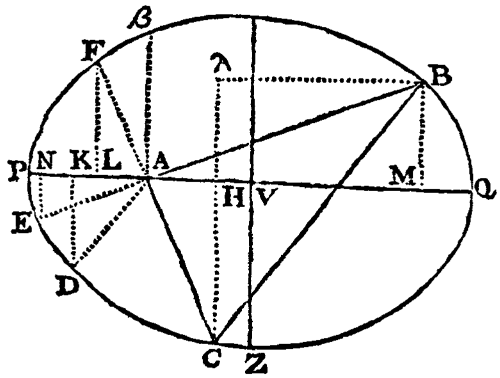
PROB. LV.

Erectis alicubi terrarum tribus baculis, ad Horizontale planum in punctis, A, B, *&* C, *perpendicularibus, quorum is qui in* A *ſit ſex pedum, qui in* B *octodecim pedum, & qui in* C *octo pedum, exiſtente lineâ* AB *triginta trium pedum: contingit quodam die extremitatem umbræ baculi* A, *tranſire per puncta* B *&* C; *baculi autem* B *per* A *&* C; *ac baculi* C *per punctum* A. *Quæritur declinatio ſolis & elevatio Poli, ſive dies locuſque, ubi hæc evenerint?*

QUoniam umbra baculi cujuſque deſcripſit Conicam ſectionem, ſectionem nempe Coni radioſi cujus vertex eſt baculi ſummitas; fingam BCDEF eſſe hujuſmodi curvam (ſive ea ſit Hyperbola, Parabola vel Ellipſis) quam umbra baculi A eo die deſcripſit; ponendo AD, AE, AF ejus umbras fuiſſe cùm BC, BA, CA reſpectivè fuerunt umbræ baculorum B & C. Et præterea fingam PAQ eſſe lineam Meridionalem, ſive axem hujus curvæ, ad quem demiſſæ perpendiculares, BM, CH, DK, EN, & FL, ſunt ordinatim

dinatim applicatæ. Has verò ordinatim applicatas indefinitè deſignabo literâ y, & axis partes interceptas, AM, AH, AK, AN, & AL, literâ x. Fingam denique æquationem $aa \dotplus bx \dotplus cxx = yy$, ipſarum x & y relationem (*i. e.* naturam Curvæ) deſignare, aſſumendo aa, b & c tanquam cognitas, ut ex Analyſi tandem inveniantur. Ubi incognitas quantitates x & y, duarum tantùm dimenſionum poſui, quia æquatio eſt ad Conicam ſectionem; & ipſius y dimenſiones impares omiſi, quia ipſa eſt ordinatim applicata ad axem. Signa autem ipſorum b & c, quia indeterminata ſunt, deſignavi notulâ $\dotplus$, quam indifferenter pro + aut − uſurpo, & ejus oppoſitum $\dotminus$ pro ſigno contrario. At ſignum quadrati aa affirmativum poſui, quia baculum A, umbras in adverſas plagas (C & F, B & E) projicientem, concava pars curvæ neceſſariò complectitur; & proinde ſi ad punctum A erigatur perpendiculum Aβ, hoc alicubi occurret curvæ, puta in β; hoc eſt, ordinatim applicata y, ubi x nullum eſt, erit reale. Nam inde ſequitur, quadratum ejus, quod in eo caſu eſt aa, affirmativum eſſe.

PROBLEMATA GEOMETRICA.

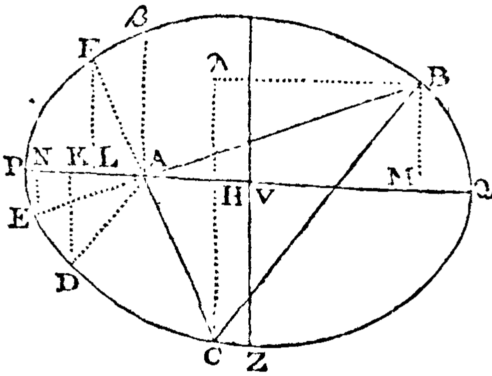


Conſtat itaque quòd æquatio hæc fictitia $aa \dotplus bx \dotplus cxx = yy$, ſicut terminis ſuperfluis non referta, ſic neque reſtrictior eſt, quàm ut ad omnes hujus problematis conditiones ſe extendat, Hyperbolam, Ellipſin vel Parabolam quamlibet deſignatura, prout ipſorum aa, b, c, valores determinabuntur, aut nulli fortè reperientur. Quid autem valent, quibuſque ſignis b & c debent affici, & inde quænam ſit hæc curva, ex ſequenti Analyſi conſtabit.

Analyſeos pars prior.

Cum umbræ ſint ut altitudines baculorum erit BC. AD : : AB. AE (: : 18. 6.) : : 3. 1. Item CA. AF (: : 8. 6.) : : 4. 3. Quare nominatis AM$=r$, MB$=s$, AH$=t$, & HC$= \dotplus v$; ex ſimilitudine triangulorum, AMB, ANE, & AHC, ALF, erunt AN $= -\frac{r}{3}$. NE $= -\frac{s}{3}$. AL $= -\frac{3t}{4}$. Et LF $= \dotminus \frac{3v}{4}$: quarum ſigna ſignis ipſarum, AM, MB, AH, HC, contraria poſui, quia tendunt ad contrarias plagas reſpectu puncti

CAPUT XIV.



puncti A, à quo ducuntur, axiſve PQ cui inſiſtunt. His autem pro x & y in æquatione fictitiâ, $aa \pm bx \pm cxx = yy$, reſpectivè ſcriptis,

r & s dabunt $aa \pm br \pm crr = ss$.

$-\frac{r}{3}$ & $-\frac{s}{3}$ dabunt $aa \mp \frac{br}{3} \pm \frac{1}{9}crr = \frac{1}{9}ss$.

t & $\pm v$ dabunt $aa \pm bt \pm ctt = vv$.

$-\frac{3}{4}t$ & $\pm\frac{3}{4}v$ dabunt $aa \pm \frac{3}{4}bt \pm \frac{9}{16}ctt = \frac{9}{16}vv$.

Jam è primâ & ſecundâ harum, exterminando ss ut obtineatur r, prodit $\frac{2aa}{\pm b} = r$: unde patet $\pm b$ eſſe affirmativum. Item è tertiâ & quartâ, exterminando vv ut obtineatur t, prodit $\frac{aa}{3b} = t$. Et ſcriptis inſuper $\frac{2aa}{b}$ pro r in primâ, & $\frac{aa}{3b}$ pro t in tertiâ, oriuntur $3aa \pm \frac{4a^4c}{bb} = ss$, & $\frac{4}{3}aa \pm \frac{a^4c}{9bb} = vv$.

Porrò demiſſâ Bλ perpendiculari in CH, erit BC. AD (: : 3. 1.) : : Bλ. AK : : Cλ. DK. Quare, cùm ſit Bλ ($=$ AM $-$ AH $= r - t$) $= \frac{5aa}{3b}$, erit AK $= \frac{5aa}{9b}$; vel potiùs $= -\frac{5aa}{9b}$. Item, cùm ſit Cλ ($=$ CH $\pm$ BM $=$ $v \pm s$) $= \sqrt{\frac{4aa}{3} \pm \frac{a^4c}{9bb}} \pm \sqrt{3aa \pm \frac{4a^4c}{bb}}$, erit DK ($= \frac{1}{3}$Cλ) $= \sqrt{\frac{4aa}{27} + \frac{a^4c}{81bb}} \pm \sqrt{\frac{1}{3}aa \pm \frac{4a^4c}{9bb}}$. Quibus in æquatione, $aa + bx \pm cxx = yy$, pro AK ac DK, ſive x, & y, reſpectivè ſcriptis, prodit $\frac{4aa}{9} \pm \frac{25a^4c}{81bb} = \frac{13}{27}aa \pm \frac{37a^4c}{81bb} \pm 2\sqrt{\frac{4aa}{27} \pm \frac{a^4c}{81bb}} \times \sqrt{\frac{aa}{3} \pm \frac{4a^4c}{9bb}}$. Et, per reductionem, $-bb \mp 4aac = \pm 2\sqrt{36b^4 \pm 51aabbc + 4a^4cc}$; &, partibus quadratis, iterumque reductis, exit $0 = 143b^4 \pm 196aabbc$, ſive $\frac{-143bb}{196aa} = \pm c$. Unde conſtat $\pm c$ negativam eſſe, adeoque æquationem fictitiam, $aa \pm bx \pm cxx = yy$, hujus eſſe formæ, $aa + bx - cxx = yy$; & ideo curvam quam deſignat Ellipſin eſſe. Ejus vero centrum & axes duo ſic eruuntur.

Ponendo $y = 0$ (ſicut in Figuræ verticibus P & Q contingit) habebitur $aa + bx = cxx$; &, extractâ radice, $x = \frac{b}{2c} \pm \sqrt{\frac{bb}{4cc} + \frac{aa}{c}} = \frac{\text{AQ}}{\text{AP}}$. Adeoque ſumpto AV $= \frac{b}{2c}$, erit V centrum Ellipſis, & VQ vel VP ($\sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$) ſemiaxis maximus. Si porrò ipſius AV valor $\frac{b}{2c}$ pro x

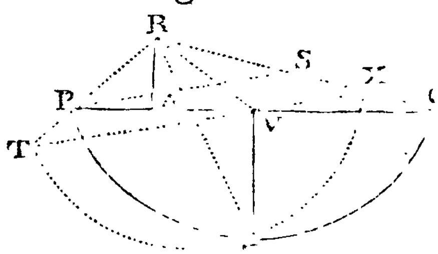
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in æquatione, $aa+bx-cxx=yy$, ſcribatur, fiet $aa+\frac{bb}{4c}=yy$. Quare est $aa+\frac{bb}{4c}=$ VZq, hoc eſt, quadrato ſemiaxis minimi. Denique in valoribus ipſarum, AV, VQ, VZ, jam inventis, ſcripto $\frac{143bb}{196aa}$ pro c, exeunt $\frac{98aa}{143b}=$ AV, $\frac{112aa\sqrt{3}}{143b}=$ VQ, & $\frac{8a\sqrt{3}}{\sqrt{143}}=$ VZ.

PROBLEMATA GEOMETRICA.

Analyſeos pars altera.

Supponatur jam baculum puncto A inſiſtens eſſe AR, & erit RPQ planum meridionale, ac RPZQ conus radioſus, cujus vertex eſt R. Sit inſuper TXZ planum ſecans Horizontem in VZ, ut & meridionale planum in TVX; quæ ſectio ſit ad axem mundi conive perpendicularis; & ipſum planum TXZ erit ad eundem axem perpendiculare, & conum ſecabit in peripheriâ circuli TZX, quæ ab ejus vertice pari ubique intervallo, RX, RZ, RT, diſtabit. Quamobrem, ſi PS ipſi TX parallela ducatur, fiet RS=RP, propter æquales, RX, RT; nec non SX=XQ, propter æquales, PV, VQ. Unde eſt RX, vel RZ $(=\frac{RS+RQ}{2})=\frac{RP+RQ}{2}$. Denique ducatur RV; & cùm VZ perpendiculariter inſiſtat plano RPQ, (ſectio utique exiſtens planorum eidem perpendiculariter inſiſtentium) fiet triangulum RVZ rectangulum ad V.



Dictis jam RA$=d$, AV$=e$, VP, vel VQ, $=f$, & VZ$=g$; erit AP$=f-e$, & RP$=\sqrt{ff-2ef+ee+dd}$. Item AQ$=f+e$, & RQ$=\sqrt{ff+2ef+ee+dd}$: adeoque RZ$(=\frac{RP+RQ}{2})=\frac{\sqrt{ff-2ef+ee+dd}+\sqrt{ff+2ef+ee+dd}}{2}$. Cujus quadratum $\frac{dd+ee+ff}{2}+\frac{1}{2}\sqrt{f^4-2eeff+e^4+2ddff+2ddee+d^4}$, eſt æquale (RV$q$+VZ$q$=RA$q$+AV$q$+VZ$q$=) $dd+ee+gg$. Jam, reductione factâ, eſt $\sqrt{f^4-2eeff+e^4+2ddff+2ddee+d^4}=dd+ee-ff+2gg$; &, partibus quadratis ac in ordinem redactis, $ddff=ddgg+eegg-ffgg+g^4$; ſive $\frac{ddff}{gg}=dd+ee-ff+gg$. Denique 6, $\frac{98aa}{143b}$, $\frac{112aa\sqrt{3}}{143b}$, & $\frac{8a\sqrt{3}}{\sqrt{143}}$ (valoribus ipſorum AR, AV, VQ, & VZ) pro d, e, f, ac g reſtitutis, oritur $36-\frac{196a^4}{143bb}+\frac{192aa}{143}=\frac{36\times14\times14aa}{143bb}$; & inde, per reductionem, $\frac{49a^4+36\times49aa}{48aa+1287}=bb$.

 In primo Schemate eſt AMq+MBq=ABq, hoc eſt $rr+ss=33\times33$. Erat autem $r=\frac{2aa}{b}$, & $ss=3aa-\frac{4a^4c}{bb}$; unde $rr=\frac{4a^4}{bb}$, & (ſubſtituto $\frac{143bb}{196aa}$ pro c) $ss=\frac{4aa}{49}$: quare $\frac{4a^4}{bb}+\frac{4aa}{49}=33\times33$: & inde, per reductionem, iterum reſultat $\frac{4\times49a^4}{53361-4aa}=bb$. Ponendo igitur æqualitatem inter duo bb, & dividendo utramque partem æquationis per 49, fit $\frac{a^4+36aa}{48aa+1287}=\frac{4k^4}{53361-4aa}$. Cujus partibus in crucem multiplicatis, ordinatis, ac diviſis per 49, exit $4a^4=981aa+39204$, cujus radix aa eſt $\frac{981+\sqrt{1589625}}{8}=280⌞2254144$.

Suprà inventum fuit $\frac{4\times49a^4}{53361-4aa}=bb$, ſive $\frac{14aa}{\sqrt{53361-4aa}}=b$. Unde AV $(\frac{98aa}{143b})$ eſt $\frac{7\sqrt{53361-4aa}}{143}$; & VP vel VQ $(\frac{112aa\sqrt{3}}{143b})$ eſt $\frac{8}{143}\sqrt{160083-12aa}$. Hoc eſt, ſubſtituendo 280⌞2254144 pro aa, ac terminos in decimales numeros reducendo, AV = 11⌞188297; & VP vel VQ = 22⌞147085. Adeoque AP (PV − AV) = 10⌞958788; & AQ (AV+VQ) 33⌞335382.

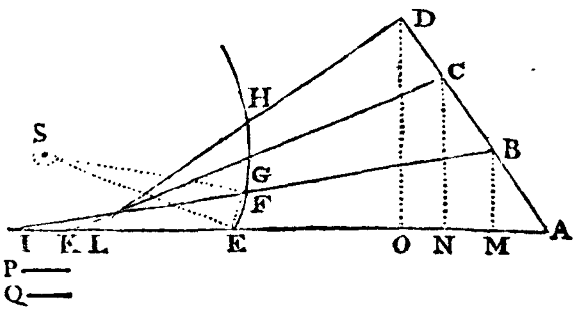
Denique ſi $\frac{1}{6}$ AR, ſive 1, ponatur radius, erit $\frac{1}{6}$ AQ, ſive 5⌞555897, tangens anguli ARQ, 79 gr. 47′. 48′; & $\frac{1}{6}$ AP, ſive 1⌞826465, tangens anguli ARP, 61 gr. 17′. 57″. Quorum angulorum ſemiſumma, 70 gr. 32′. 52″, eſt complementum declinationis ſolis; & ſemidifferentia, 9 gr. 14′. 56′, complementum latitudinis Loci. Proinde declinatio ſolis erat 19 gr. 27′. 8″; & Latitudo loci 80 gr. 45′. 4″. Quæ erant invenienda.

PROB. LVI.

E *Cometæ, motu uniformi rectilineo per Cœlum trajicientis, locis quatuor obſervatis, diſtantiam à terrâ, motûſque determinationem, in Hypotheſi Copernicæâ colligere.*

Si è centro Cometæ, in locis quatuor obſervatis, ad planum Eclipticæ demittantur totidem perpendicula; ſintque A, B, C, D, puncta in plano illo in quæ perpendicula incidunt; per puncta illa agatur recta AD, & hæc ſecabitur à perpendiculis in eâdem ratione cum lineâ quam Cometa motu ſuo deſcribit; hoc eſt, ita ut ſit AB ad AC ut tempus inter primam & ſecundam obſervationem ad tempus inter primam ac tertiam; & AB ad AD ut tempus

tempus illud inter primam & ſecundam obſervationem ad tempus inter primam & quartam. Ex obſervationibus itaque dantur rationes linearum, AB, AC, AD, ad invicem.



Inſuper in eodem Eclipticæ plano ſit S Sol; EH arcus lineæ Eclipticæ in quâ terra movetur; E, F, G, H, loca quatuor terræ temporibus obſervationum; E locus primus, F ſecundus, G tertius, H quartus. Jungantur AE, BF, CG, DH, & producantur, donec tres poſteriores priorem ſecent in I, K & L; BF in I, CG in K, DH in L: et erunt anguli, AIB, AKC, ALD, differentiæ longitudinum obſervatarum Cometæ; AIB differentia longitudinum loci primi Cometæ & ſecundi; AKC differentia longitudinum loci primi ac tertii; & ALD differentia longitudinum loci primi & quarti. Dantur itaque ex obſervationibus anguli AIB, AKC, ALD.

Junge SE, SF, EF; &, ob data puncta S, E, F, datumque angulum ESF, dabitur angulus SEF. Datur etiam angulus SEA, utpote differentia longitudinis Cometæ & Solis tempore obſervationis primæ. Quare ſi complementum ejus ad duos rectos, nempe angulum SEI, addas angulo SEF, dabitur angulus IEF. Trianguli igitur IEF dantur anguli unà cum latere EF, adeoque datur etiam latus IE. Et ſimili argumento dantur KE & LE. Dantur igitur poſitione lineæ quatuor, AI, BI, CK, DL; adeoque Problema huc redit, ut lineis quatuor poſitione datis, quintam inveniamus, quæ ab his in datâ ratione ſecabitur.

Demiſſis ad AI perpendiculis, BM, CN, DO, ob datum angulum AIB, datur ratio BM ad MI. Eſt & BM ad CN in datâ ratione BA ad CA; &, ob datum angulum CKN, datur ratio CN ad KN. Quare datur etiam ratio BM ad KN; & inde ratio quoque BM ad MI−KN, hoc eſt ad MN+IK. Cape P ad IK ut eſt AB ad BC; &, cùm ſit MA ad MN in eâdem ratione, erit etiam P+MA ad IK+MN in eâdem ratione; hoc eſt in ratione datâ. Quare datur ratio BM ad P+MA. Et ſimili argumento, ſi capiatur Q ad IL in ratione AB ad BD, dabitur ratio BM ad Q+MA. Et proinde ratio BM ad ip-

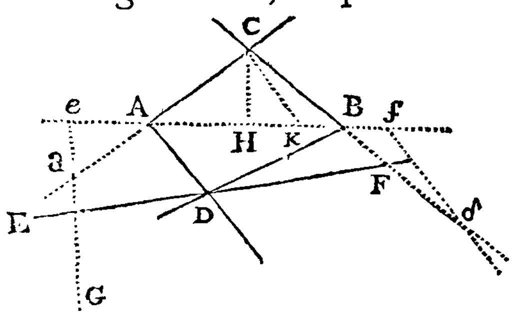
CAPUT XIV. ſorum P+MA & Q+MA differentiam quoque dabitur. At differentia illa, nempe P−Q vel Q−P, datur. Et proinde dabitur BM. Dato autem BM, ſimul dantur P+MA, & MI, & inde MA, ME, AE, & angulus EAB.

His inventis, erige ad A lineam plano Eclipticæ perpendicularem, quæ ſit ad lineam EA, ut tangens latitudinis Cometæ in obſervatione primâ ad radium; & iſtius perpendicularis terminus erit locus centri Cometæ in obſervatione primâ. Unde datur diſtantia Cometæ à Terrâ tempore illius obſervationis. Et eodem modo, ſi è puncto B erigatur perpendicularis, quæ ſit ad lineam BF, ut tangens latitudinis Cometæ in obſervatione ſecundâ ad radium, habebitur locus centri Cometæ in obſervatione illâ ſecundâ. Et acta linea à loco primo ad locum ſecundum ea eſt, in quâ Cometa per Cœlum trajicit.

PROB. LVII.

Si angulus datus, CAD, *circa punctum angulare,* A, *poſitione datum, & angulus datus,* CBD, *circa punctum angulare,* B, *poſitione datum, eâ lege circumvolvantur, ut crura,* AD, BD, *ad rectam poſitione datam,* EF, *ſeſe ſemper interſecent: invenire lineam illam curvam, quam reliquorum crurum,* AC, BC, *interſectio,* C, *deſcribit.*

PRoduc CA ad d, ut ſit Ad = AD, & CB ad δ, ut ſit Bδ = BD. Fac angulum Ade æqualem angulo ADE, & angulum Bδf æqualem angulo BDF; & produc AB utrinque, donec ea occurrat de & δf in e & f. Produc etiam cd ad G, ut ſit dG = δf; & à puncto C ad lineam AB, ipſi ed parallelam, age CH, & ipſi $f\delta$ parallelam CK. Et concipiendo lineas eG, $f\delta$ immobiles manere, dum anguli, CAD, CBD, lege præſcriptâ circa polos A & B volvantur, ſemper erit Gd æqualis ipſi δf, & triangulum CHK dabitur ſpecie. Dic itaque A$e = a$, eG $= b$, B$f = c$, AB $= m$, BK $= x$, & CK $= y$. Et erit BK . CK : : Bf . $f\delta$. Ergo $f\delta = \frac{cy}{x} =$ Gd. Aufer hoc de Ge, & reſtabit $ed = b - \frac{cy}{x}$. Cùm detur ſpecie tri-



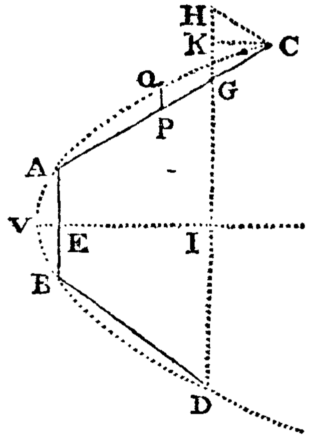
angulum

angulum CKH, pone CK. CH : : $d. e$; & CH. HK : : $e. f$; & erit CH $= \frac{ey}{d}$, & HK $= \frac{fy}{d}$. Adeoque AH $= m - x - \frac{fy}{d}$. Eſt autem AH. HC : : Ae. ed, hoc eſt $m - x - \frac{f}{d}y. \frac{ey}{d} : : a. b - \frac{cy}{x}$. Ergo ducendo media & extrema in ſe, fiet $mb - \frac{mcy}{x} - bx + cy - \frac{bf}{d}y + \frac{cfyy}{dx} = \frac{acy}{d}$. Duc omnes terminos in dx, coſque in ordinem redige; PROBLEMATA GEOMETRICA.

& fiet $fcyy \begin{array}{l} +dc \\ -ae \\ -fb \end{array} xy - dcmy - bdxx + bdmx = 0$. Ubi, cùm incognitæ quantitates, x & y, ad duas tantùm dimenſiones aſcendunt, patet curvam lineam, quam punctum C deſcribit, eſſe Conicam Sectionem. Pone $\frac{ae + fb - dc}{c} = 2p$, & fiet $yy = \frac{2p}{f}xy + \frac{dm}{f}y + \frac{bd}{fc}xx - \frac{bdm}{fc}x$. Et extractâ radice, $y = \frac{p}{f}x + \frac{dm}{2f} \pm \sqrt{\frac{pp}{ff}xx + \frac{bd}{fc}xx + \frac{pdm}{ff}x - \frac{bdm}{fc}x + \frac{ddmm}{4ff}}$. Unde colligitur Curvam Hyperbolam eſſe, ſi ſit $\frac{bd}{fc}$ affirmativum, vel negativum & minus quàm $\frac{pp}{ff}$; Parabolam, ſi ſit $\frac{bd}{fc}$ negativum & æquale $\frac{pp}{ff}$; Ellipſin vel Circulum, ſi ſit $\frac{bd}{fc}$ & negativum & majus quàm $\frac{pp}{ff}$. Q. E. I.

PROB. LVIII.

Parabolam deſcribere quæ per data quatuor puncta tranſibit.



Sint puncta illa data A, B, C, D. Junge AB, & eam biſeca in E. Et per E age rectam aliquam, VE, quam concipe diametrum eſſe Parabolæ, puncto V exiſtente vertice ejus. Junge AC, ipſique AB parallelam age DG, occurrentem AC in G. Dic AB $= a$, AC $= b$, AG $= c$, GD $= d$. In AC cape AP cujuſvis longitudinis, & à P age PQ parallelam AB: & concipiendo Q punctum eſſe Parabolæ, dic AP $= x$, PQ $= y$; & æquationem quamvis ad Parabolam aſſume, quæ relationem inter AP & PQ exprimat: ut quòd ſit $y = e + fx \pm \sqrt{gg + hx}$.

Jam

CAPUT XIV. Jam ſi ponatur AP ſive $x=0$ (puncto P incidente in ipſum A) fiet PQ, ſive $y,=0$, ut & $=-$ AB. Scribendo autem in æquatione aſſumptâ 0 pro x, fiet $y=e\pm\sqrt{gg}$, hoc eſt $=e\pm g$. Quorum valorum ipſius y major $e+g$, eſt$=0$; minor, $e-g,=-$AB, ſive $-a$. Ergo $e=-g$; & $e-g$, hoc eſt, $-2g,=-a$, ſive $g=\frac{1}{2}a$. Atque adeo vice æquationis aſſumptæ habebitur hæc, $y=-\frac{1}{2}a+fx\pm\sqrt{\frac{1}{4}aa+bx}$.

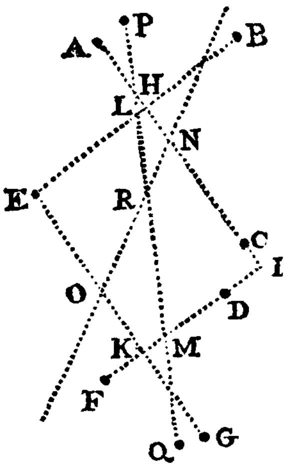
Adhæc ſi ponatur AP ſive $x=$ AC, ita ut punctum P incidat in C, fiet iterum PQ$=0$. Pro x igitur in æquatione noviſſimâ ſcribe AC ſive b, & pro y, 0; & fiet $0=-\frac{1}{2}a+fb+\sqrt{\frac{1}{4}aa+bb}$, ſive $\frac{1}{2}a-fb=\sqrt{\frac{1}{4}aa+bb}$; &, partibus quadratis, $-afb+ffbb=bb$; ſive $ffb-fa=b$. Atque ita vice aſſumptæ æquationis habebitur iſthæc, $y=-\frac{1}{2}a+fx\pm\sqrt{\frac{1}{4}aa+ffbx-fax}$.

Inſuper ſi ponatur AP ſive $x=$ AG, ſive c; fiet PQ, ſive $y,=-$GD, ſive $-d$. Quare pro x & y in æquatione noviſſimâ ſcribe c, & $-d$; & fiet $-d=-\frac{1}{2}a+fc-\sqrt{\frac{1}{4}aa+ffbc-fac}$; ſive $\frac{1}{2}a-d-fc=\sqrt{\frac{1}{4}aa+ffbc-fac}$. Et, partibus quadratis, $-ad-fac+dd+2dcf+ccff=ffbc-fac$. Et, æquatione ordinatâ & reductâ, $ff=\frac{2d}{b-c}f+\frac{dd-ad}{bc-cc}$. Pro $b-c$, hoc eſt pro GC, ſcribe k; & æquatio illa fiet $ff=\frac{2d}{k}f+\frac{dd-ad}{kc}$. Et extractâ radice, $f=\frac{d}{k}\pm\sqrt{\frac{ddc+ddk-adk}{kkc}}$. Invento autem f, æquatio ad Parabolam, *viz.* $y=-\frac{1}{2}a+fx\pm\sqrt{\frac{1}{4}aa+ffbx-fax}$, plenè determinatur: cujus itaque conſtructione Parabola etiam determinabitur. Conſtructio autem ejus hujuſmodi eſt. Ipſi BD parallelam age CH occurrentem DG in H. Inter DG ac DH cape mediam proportionalem DK, & ipſi CK parallelam age EI biſecantem AB in E, & occurrentem DG in I. Dein produc IE ad V; ut ſit EV. EI :: EBq. DI$q-$EBq; & erit V vertex, VE diameter, & $\frac{\text{BE}q}{\text{VE}}$ latus rectum Parabolæ quæſitæ.

PROB. LIX.

Conicam ſectionem per data quinque puncta deſcribere.

Sint puncta iſta A, B, C, D, E. Junge AC, BE, ſe mutuo ſecantes in H. Age DI parallelam BE, & occurrentem AC in I. Item EK parallelam AC, & occurrentem DI productæ in K. Produc



ID ad F, & EK ad G; ut ſit AHC. BHE : : AIC. FID : : EKG. FKD ; & erunt puncta F ac G in conicâ ſectione, ut notum eſt. Hoc tamen obſervare debebis, quòd ſi punctum H cadit inter puncta omnia, A, C & B, E, vel extra ea omnia, punctum I cadere debebit vel inter puncta omnia, A, C & F, D, vel extra ea omnia; & punctum K inter omnia, D, F & E, G, vel extra ea omnia. At ſi punctum H cadit inter duo puncta A, C, & extra alia duo B, E, vel inter illa duo B, E, & extra altera duo A, C; debebit punctum I cadere inter duo punctorum A, C & F, D, & extra alia duo eorum; & ſimiliter punctum K debebit cadere inter duo punctorum D, F & E, G, & extra alia duo eorum: id quod fiet capiendo IF, KG, ad hanc vel illam partem punctorum I, K, pro exigentiâ problematis. Inventis punctis F ac G, biſeca AC, EG in N & O; item BE, FD, in L & M. Junge NO, LM ſe mutuò ſecantes in R; & erunt LM & NO diametri conicæ ſectionis, R centrum ejus, & BL, FM ordinatim applicatæ ad diametrum LM. Produc LM hinc inde, ſi opus eſt, ad P & Q, ita ut ſit BLq. FMq : : PLQ. PMQ; & erunt P & Q vertices Conicæ ſectionis & PQ latus tranſverſum. Fac PLQ. LBq : : PQ. T : et erit T latus rectum; quibus cognitis cognoſcitur Figura.

Problemata Geometrica.

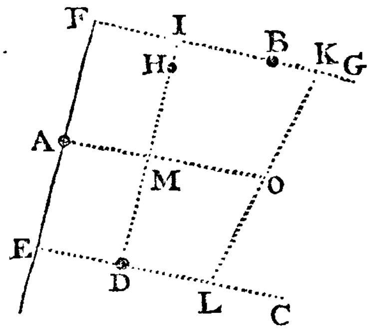
Reſtat tantùm ut doceamus, quomodo LM hinc inde producenda ſit ad P & Q, ita ut fiat BLq FMq : : PLQ. PMQ. Nempe PLQ, ſive PL × LQ, eſt $\overline{PR - LR} \times \overline{PR + LR}$; nam PL eſt PR − LR, & LQ eſt RQ + LR, ſeu PR + LR. Porro $\overline{PR - LR} \times \overline{PR + LR}$ multiplicando fit PRq − LRq. Et ad eundem modum PMQ eſt $\overline{PR + RM} \times \overline{PR - RM}$, ſeu PR$q$ − RMq. Ergo BLq. FMq : : PRq − LRq. PRq − RMq; & dividendo BLq − FMq. FMq : : RMq − LRq. PRq − RMq. Quamobrem cùm dentur BLq − FMq, FMq, & RMq − LRq, dabitur PRq − RMq. Adde datum RMq, & dabitur ſumma PRq, adeoque & latus ejus PR, cui QR æqualis eſt.

PROB.

Caput XIV.

PROB. LX.

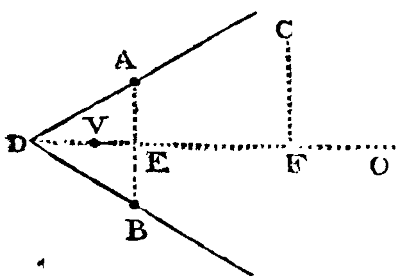
Conicam ſectionem deſcribere quæ tranſibit per quatuor data puncta, & in uno iſtorum punctorum continget rectam poſitione datam.



Sint puncta quatuor data, A, B, C, D, & recta poſitione data AE, quam conica ſectio contingat in puncto A. Junge duo quævis puncta DC; & DC, producta ſi opus eſt, occurrat tangenti in E. Per quartum punctum, B, ipſi DC age parallelam BF; quæ occurrat eidem tangenti in F. Item tangenti parallelam age DI, quæ occurrat ipſi BF in I. In FB, DI, ſi opus eſt productis, cape FG, HI ejus longitudinis, ut ſit AE*q*. CED :: AF*q*. BFG :: DIH BIG. Et erunt puncta G & H in Conicâ ſectione, ut notum eſt: ſi modò capias FG, IH ad legitimas partes punctorum F & I, juxta regulam in ſuperiore Problemate traditam. Biſeca BG, DC, DH in K, L & M; junge KL, AM ſe mutuò ſecantes in O, & erit O centrum, A vertex, & HM ordinatim applicata ad ſemidiametrum AO. Quibus cognitis cognoſcitur figura.

PROB. LXI.

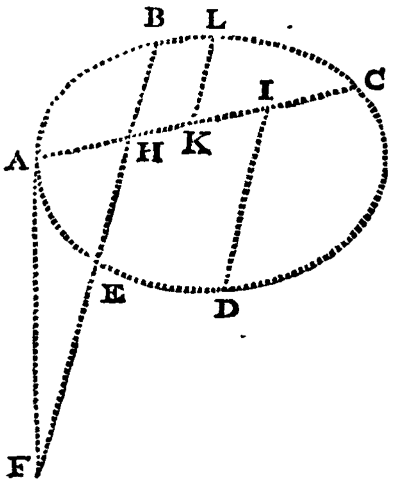
Conicam ſectionem deſcribere quæ tranſibit per tria data puncta, & in duobus iſtorum punctorum continget rectas poſitione datas.



Sint puncta illa data A, B, C; tangentes AD, BD ad puncta A & B; D communis interſectio tangentium. Biſeca AB in E. Age DE, & produc eam donec in F occurrat CF actæ parallelæ AB: & erit DF diameter, & AE, CF ordinatim applicatæ ad diametrum. Produc DF ad O; & in DO cape OV mediam proportionalem inter DO & EO, eâ lege ut ſit etiam AE*q*. CF*q* :: $VE \times \overline{VO + OE}$. $VF \times \overline{VO + OF}$; & erit V vertex, & O centrum Figuræ. Quibus cognitis Figura ſimul

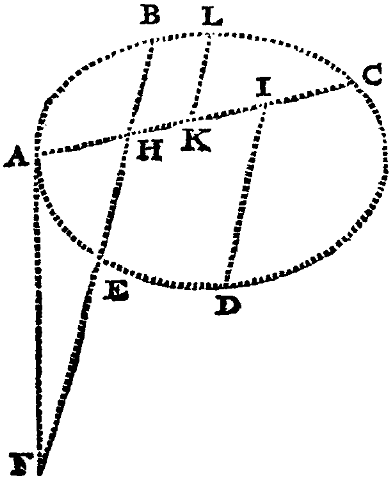
ſimul cognoſcitur. Eſt autem $VE = VO - OE$; adeoque $VE \times \overline{VO+OE} = \overline{VO-OE} \times \overline{VO+OE} = VOq - OEq$. Præterea, quia VO media proportionalis eſt inter DO & EO, erit $VOq = DOE$; adeoque $VOq - OEq = DOE - EOq = DEO$. Et ſimili argumento erit $VF \times \overline{VO+OF} = VOq - OFq = DOE - OFq$. Ergo $AEq.\ CFq :: DEO.\ DOE - OFq$. Eſt $OFq = EOq - 2FEO + FEq$. Adeoque $DOE - OFq = DOE - OEq + 2FEO - FEq = DEO + 2FEO - FEq$. Et $AEq.\ CFq :: DEO.\ DEO + 2FEO - FEq :: DE.\ DE + 2FE - \frac{FEq}{EO}$. Datur ergo $DE + 2FE - \frac{FEq}{EO}$. Aufer hoc de dato $DE + 2FE$, & reſtabit $\frac{FEq}{EO}$ datum. Sit illud N; & erit $\frac{FEq}{N} = EO$, adeoque dabitur EO. Dato autem EO, ſimul datur VO medium proportionale inter DO & EO.

Problemata Geometrica.



Hoc modo per Theoremata quædam Apollonii ſatis expeditè reſolvuntur hæc problemata: quæ tamen, ſine iſtis Theorematibus, per Algebram ſolam reſolvi poſſent. Ut ſi proponatur primum trium noviſſimorum Problematum: Sint puncta quinque data, A, B, C, D, E, per quæ Conica ſectio tranſire debet. Junge duo quævis, A, C, & alia duo, B, E, rectis ſe ſecantibus in H. Ipſi BE parallelam age DI, occurrentem AC in I; ut & aliam quamvis rectam, KL, occurrentem AC in K, & conicæ ſectioni in L. Et finge Conicam ſectionem datam eſſe; ita ut cognito puncto K ſimul cognoſcatur punctum L. Et poſito $AK = x$, & $KL = y$, ad exprimendam relationem inter x & y, aſſume quamvis æquationem, quæ Conicas ſectiones generaliter exprimit; puta hanc, $a + bx + cxx + dy + exy + yy = 0$; ubi a, b, c, d, e, denotant quantitates determinatas cum ſignis ſuis, x verò & y quantitates indeterminatas. Si jam quantitates determinatas a, b, c, d, e invenire poſſumus, habebimus Conicam ſectionem. Fingamus ergo punctum L ſucceſſivè incidere in puncta, A, C, B, E, D, & videamus quid inde ſequetur. Si ergo punctum L incidit in punctum A, erit in eo caſu AK & KL, hoc eſt, x & y, nihil. Proinde æquationis omnes termini præter a evaneſcent, & reſtabit $a = 0$. Quare delendum eſt a in æquatione illâ, & cæteri termini $bx + cxx + dy + cxy + yy$ erunt $= 0$. Porro ſi L

CAPUT XIV. incidit in C, erit AK, ſeu x, $=$ AC, & KL ſeu $y=0$. Pone ergo AC $=f$; & ſubſtituendo f pro x, & 0 pro y, æquatio ad curvam, $bx+cxx+dy+exy+yy=0$, evadet $bf+cff=0$, ſeu $b=-cf$. Et in æquatione



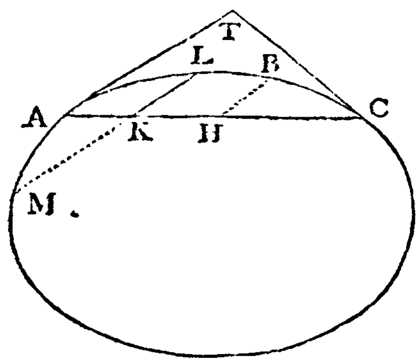
illâ ſcripto $-cf$ pro b, evadet $-cfx+cxx+dy+exy+yy=0$. Adhæc ſi punctum L incidit in punctum B, erit AK, ſeu x, $=$ AH; & KL, ſeu y, $=$ BH. Pone ergo AH $=g$, & BH $=h$, & perinde ſcribe g pro x, & h pro y, & æquatio $-cfx+cxx$, &c. evadet $-cfg+cgg+dh+egh+hh=0$. Quòd ſi punctum L incidit in E, erit AK $=$ AH, ſeu $x=g$; & KL, ſeu y, $=$ HE. Pro HE ergo ſcribe $-k$ cum ſigno negativo, quia HE jacet ad contrarias partes lineæ AC, & ſubſtituendo g pro x, & $-k$ pro y, æquatio $-cfx+cxx$, &c. evadet $-cfg+cgg-dk-egk+kk=0$. Aufer hoc de ſuperiori æquatione, $-cfg+cgg+dh+egh+hh$, & reſtabit $dh+egh+hh+dk+egk-kk=0$. Divide hoc per $h+k$, & fiet $d+eg+h-k=0$. Hoc ductum in h aufer de $-cfg+cgg+dh+egh+hh=0$, & reſtabit $-cfg+cgg+hk=0$, ſeu $\frac{hk}{-gg+fg}=c$. Denique, ſi punctum L incidit in punctum D, erit AK, ſeu x, $=$ AI, & KL, ſeu y, $=$ ID. Quare pro AI ſcribe m, & pro ID, n; & perinde pro x & y ſubſtitue m & n, & æquatio $-cfx+cxx$, &c. evadet $-cfm+cmm+dn+emn+nn=0$. Hoc divide per n, & fiet $\frac{-cfm+cmm}{n}+d+em+n=0$. Aufer $d+eg+h-k=0$; & reſtabit $\frac{-cfm+cmm}{n}+em-eg+n-h+k=0$. Sive $\frac{cmm-cfm}{n}+n-h+k=eg-em$. Jam verò ob data puncta A, B, C, D, E, dantur AC, AH, AI, BH, EH, DI; hoc eſt f, g, m, h, k, n. Atque adeo per æquationem, $\frac{hk}{fg-gg}=c$, datur c. Dato autem c, per æquationem, $\frac{cmm-cfm}{n}+n-h+k=eg-em$, datur $eg-em$. Divide hoc datum per datum $g-m$, & emerget datum e. Quibus inventis, æquatio $d+eg+h-k=0$, ſeu $d=k-h-eg$, dabit d. Et his cognitis, ſimul determinatur æquatio ad quæſitam Conicam ſectionem, $cfx=cxx+dy+exy+yy$. Et ex eâ æquatione per methodum Carteſii determinabitur Conica ſectio.

Quod ſi quatuor, A, B, C, E, & p[illegible]o rectæ, AF, quæ tangit Conicam ſectionem ad unum iſtor[illegible]um, A, daretur; poſſet

Conica ſectio ſic faciliùs determinari. Inventis, ut ſuprà, æquationibus $cfx=cxx+dy+exy+yy$, $d=k-b-eg$, & $c=\frac{bk}{fg-gg}$, concipe tangentem, AF, occurrere rectæ EH in F; dein punctum L moveri per perimetrum figuræ CDE, donec incidat in punctum A: & ultima ratio ipſius LK ad AK erit ratio FH ad AH; ut contemplanti figuram conſtare poteſt. Dic verò $FH=p$; & in hoc caſu, ubi LK eſt ad AK in ultimâ ratione, erit $p.g::y.x$, ſive $\frac{gy}{p}=x$. Quare pro x in æquatione $cfx=cxx+dy+exy+yy$, ſcribe $\frac{gy}{p}$; & orietur $\frac{cfgy}{p}=\frac{cggyy}{pp}+dy+\frac{egyy}{p}+yy$. Divide omnia per y, & emerget $\frac{cfg}{p}=\frac{cggy}{pp}+d+\frac{egy}{p}+y$. Jam quia ſupponitur punctum L incidere in punctum A, adeoque KL, ſeu y, infinitè parvum vel nihil eſſe, dele terminos qui per y multiplicantur, & reſtabit $\frac{cfg}{p}=d$. Quare fac $\frac{bk}{fg-gg}=c$; dein $\frac{cfg}{p}=d$; denique $\frac{k-b-d}{g}=e$; &, inventis c, d & e, æquatio, $cfx=cxx+dy+exy+yy$, determinabit conicam ſectionem.

PROBLEMATA GEOMETRICA.

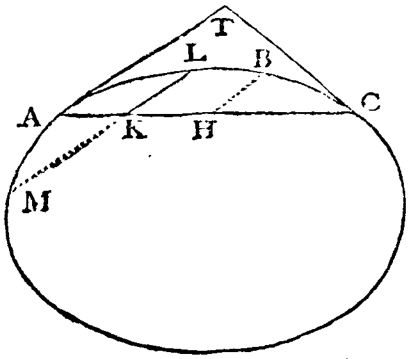
Si denique tria tantùm puncta, A, B, C, dentur, unà cum poſitione duarum rectarum, AT, CT, quæ tangunt Conicam ſectionem in duobus iſtorum punctorum, A & C; obtinebitur ut ſuprà ad Conicam ſectionem æquatio hæc, $cfx=cxx+dy+exy+yy$. Deinde, ſi ſupponatur ordinatam KL parallelam eſſe tangenti AT, & concipiatur



eam produci, donec rurſus occurrat Conicæ ſectioni in M, & lineam illam LM accedere ad tangentem, AT, donec cum eâ conveniat ad A; ultima ratio linearum KL & KM ad invicem erit ratio æqualitatis; ut contemplanti figuram conſtare poteſt. Quamobrem in illo caſu exiſtentibus KL & KM ſibi invicem æqualibus, hoc eſt duobus valoribus ipſius y (affirmativo ſcilicet KL, & negativo KM) æqualibus, debent æquationis, $cfx=cxx+dy+exy+yy$, termini illi in quibus y eſt imparis dimenſionis, hoc eſt termini $+dy+exy$, reſpectu termini yy, in quo y eſt paris dimenſionis, evaneſcere. Aliter enim duo valores ipſius y, affirmativus & negativus, æquales eſſe non poſſunt. Et in illo quidem caſu AK infinitè minor erit quàm LK; hoc eſt x quàm y; proinde & terminus exy quàm terminus yy. Atque adeo infinitè

 minor

CAPUT XVI. minor exiſtens pro nihilo habendus erit. At terminus dy, reſpectu termini yy, non evaneſcet, ut oportet, ſed eo major erit, niſi d ſupponatur eſſe nihil. Delendus eſt itaque terminus dy; & ſic reſtabit $cfx = cxx + exy + yy$, æquatio ad conicam ſectionem. Concipiantur jam tangentes AT, CT ſibi mutuò occurrere in T, & punctum L accedere ad punctum C, donec in illud incidat. Et ultima ratio ipſius KL ad KC erit AT ad AC. KL erat y; AK, x; & AC, f; atque adeo KC, $f-x$. Dic AT$=g$; & ultima ratio y ad $f-x$ erit ea quæ eſt g ad f. Æquatio $cfx=cxx+exy+yy$, ſubducto utrobique cxx, fit $cfx-cxx=exy+yy$; hoc eſt, $\overline{f-x}$ in $cx=y$ in $\overline{ex+y}$. Ergo eſt $y.f-x::cx.ex+y$; adeoque $g.f::cx.ex+y$. At, puncto L incidente in C, fit y nihil. Ergo $g.f::cx.ex$. Divide poſteriorem rationem per x, & evadet $g.f::c.e$, & $\frac{cf}{g}=e$. Quare ſi in æquatione $cfx=cxx+exy+yy$, ſcribas $\frac{cf}{g}$ pro e, fiet $cfx=cxx+\frac{cf}{g}xy+yy$, æquatio ad conicam ſectionem. Denique ipſi KL, ſeu AT, à dato puncto, B, per quod Conica ſectio tranſire debet, age parallelam BH occurrentem AC in H; & concipiendo LK accedere ad BH, donec cum eâ coincidat, in eo caſu erit AH$=x$, & BH$=y$. Dic ergo datam AH$=m$, & datam BH$=n$, & perinde pro x & y, in æquatione $cfx=cxx+\frac{cf}{g}xy+yy$, ſcribe m & n; & orietur $cfm=cmm+\frac{cf}{g}mn+nn$. Aufer utrobique $cmm+\frac{cf}{g}mn$, & fiet $cfm-cmm-\frac{cf}{g}mn=nn$. Pone $f-m-\frac{fn}{g}=s$, & erit $csm=nn$. Divide utramque partem æquationis per sm, & orietur $c=\frac{nn}{sm}$. Invento autem c, determinata habetur æquatio ad Conicam ſectionem, $cfx=cxx+\frac{cf}{g}xy+yy$. Et inde, per methodum Carteſii, Conica ſectio datur, & deſcribi poteſt.



Atque hactenus varia evolvi Problemata. In ſcientiis enim addiſcendis proſunt exempla magis quàm præcepta: quâ de cauſâ in his fuſiùs expatiatus ſum. Sed & aliqua, quæ inter ſcribendum occurrebant, immiſcui ſine Algebrâ ſoluta, ut inſinuarem, in problematis, quæ primâ fronte difficilia videantur, non

ſemper

ſemper ad Algebram recurrendum eſſe. Sed tempus eſt jam æquationum reſolutionem docere. Nam poſtquam Problema ad æquationem deductum eſt, radices illius æquationis, quæ quantitates ſunt Problemati ſatisfacientes, extrahere oportebit. De Æquatione Reſolvenda.

CAPUT XV.

Quomodo æquationes reſolvendæ ſunt.

POſtquam igitur in Quæſtionis alicujus ſolutione ad æquationem perventum eſt, & æquatio illa debitè ordinata eſt & reducta; ubi quantitates, quæ per ſpecies deſignantur & pro datis habentur, revera dantur in numeris, pro ipſis ſubſtituendi ſunt numeri illi in æquatione, & habebitur æquatio numeralis, cujus radix extracta tandem ſatisfaciet Quæſtioni. Ut ſi in ſectione anguli in quinque partes æquales, ſumendo r pro radio circuli, q pro ſubtenſâ complementi anguli propoſiti ad duos rectos, & x pro ſubtenſâ complementi quintæ partis anguli illius, perveniſſem ad hanc æquationem, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Ubi in caſu aliquo particulari dantur in numeris radius, r, & linea dati anguli complementum ſubtendens, q; ut quòd radius ſit 10, & ſubtenſa 3; ſubſtituo numeros illos in æquatione pro r & q; & provenit æquatio numeralis, $x^5 - 500x^3 + 50000x - 30000 = 0$: cujus radix tandem extracta erit x, ſeu linea complementum quintæ partis anguli illius dati ſubtendens.

CAPUT XVI.

De naturâ radicum Æquationis.

R*ADIX verò numerus eſt, qui, ſi in æquatione pro literâ vel ſpecie radicem ſignificante ſubſtituatur, efficiet omnes terminos evaneſcere.*

Sic æquationis $x^4 - x^3 - 19xx + 49x - 30 = 0$, unitas eſt radix, quoniam ſcripta pro x producit $1 - 1 - 19 + 49 - 30$, hoc eſt nihil. Sed æquationis ejuſdem plures eſſe poſſunt radices. Nam ſi in hâc eâdem æquatione $x^4 - x^3 - 19xx + 49x - 30 = 0$, pro x ſcribas numerum 2, & pro poteſtatibus x ſimiles poteſtates numeri 2, producetur $16 - 8 - 76 + 98 - 30$, hoc eſt nihil. Atque ita ſi pro x ſcribas numerum 3, vel numerum negativum -5, utroque caſu producetur nihil, terminis affirmativis & negativis in hiſce

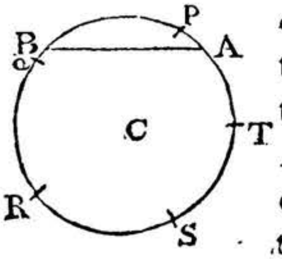
quatuor

CAPUT XVI. quatuor caſibus ſe mutuò deſtruentibus. Proinde cùm numerorum, 1, 2, 3, & − 5, quilibet ſcriptus in æquatione pro x impleat conditionem ipſius x, efficiendo ut termini omnes æquationis conjunctim æquentur nihilo, erit quilibet eorum radix æquationis.

Et ne mireris eandem æquationem habere poſſe *plures radices*, ſciendum eſt *plures eſſe poſſe ſolutiones ejuſdem Problematis*.

Ut ſi circulorum duorum quæreretur *interſectio*; *duæ* ſunt eorum interſectiones, atque adeo quæſtio admittit *duo reſponſa*; & perinde æquatio interſectionem determinans habebit *duas radices*, quibus interſectionem utramque determinet; *ſi modò nihil in datis ſit, quo reſponſum ad unam interſectionem determinetur*.

Sic & ſi arcûs APB pars quinta AP invenienda eſſet, quamvis animum forte advertas tantùm ad arcum APB; tamen æquatio,



quâ quæſtio ſolvetur, determinabit quintam partem arcuum omnium qui terminantur ad puncta A & B; nempe quintam partem arcuum ASB, APBSAPB, ASBPASB, & APBSAPBSAPB, æquè ac quintam partem arcûs APB; quæ quintæ partes, ſi dividas totam circumferentiam in æquales quinque partes, PQ, QR, RS, ST, TP, erunt, AT, AQ, ATS, AQR. Quoniam igitur, quærendo quintas partes arcuum quos recta AB ſubtendit, ad caſus omnes determinandos, circumferentia tota ſecari debet in quinque punctis, P, Q, R, S, T; ideo æquatio ad omnes caſus determinandos habebit radices quinque. Nam quintæ partes horum omnium arcuum pendent ab iiſdem datis, & per ejuſdem generis calculum inveniuntur; ita ut in eandem

(⁹) Geminas æquationibus Quadraticis eſſe radices, item Cubicis certæ cujuſdam formæ vel tergeminas Cardanus (α) monuit; Biquadraticis quibuſdam quadrigeminas; nonnullis Quadrato-cubicis quintuplices (β) Vieta. Id vero generaliter obtinere, ut æquationi cujuſcunque demum gradûs tot poſſint eſſe radices, neque tamen plures, quot ſint in numero gradum æquationis deſignante unitates, primus ni fallor mortalium Albertus Girardus intellexit; cujus in opuſculo Gallicè ſcripto et anno 1629 in lucem Æmſtelædami edito, hæc eſt enuntiatio. "Toutes les equations d'Algebre recoivent autant de ſolutions, que la denomination de la plus haute quantité le demonſtre, excepté les incomplettes." Quæ quidem Latine ſic fere ſonant. "Omnes æquationes algebraicæ, ſi imperfectas exceperis, tot ſolutiones capiunt, quot index poteſtatis elatiſſimæ indicat." Girardo autem hâc in re neque Carteſius neque Harriottus noſtras facem prætuliſſe cenſendi ſunt, quum Harriotti Artis Analyticæ Praxis non niſi biennio poſt Inventa Nova Algebraica Girardi in lucem edita eſt, Carteſii vero Geometria ſerius etiam quinquennio. Quin etiam Harriotus de numero radicum nil plane ſani habet. Vir magnæ quidem diligentiæ, ſed mediocris ingenii, ea ferè in Algebraicis intellexit, quæ a Cardano et a Vietâ acceperat; et radices, ſicut

(α) Arithmet. Lib. 10. c. 1. (β) De emendat. æquat. c. 14.

eandem ſemper æquationem incideris, ſive quæras quintam partem arcûs APB, ſive quintam partem arcûs ASB, ſive alterius cujuſvis ex arcubus quintam partem. Unde ſi æquatio, quâ quinta pars arcûs APB determinatur, non haberet plures radices quàm unam, dùm quærendo quintam partem arcûs ASB incidimus in eandem illam æquationem, ſequeretur, majorem hunc arcum habere eandem quintam partem cum priore qui minor eſt; eo quòd ſubtenſa ejus per eandem æquationis radicem exprimitur. *In omni igitur problemate neceſſe eſt æquationem, quâ reſpondetur, tot habere radices, quot ſunt quæſitæ quantitatis caſus diverſi, ab iiſdem datis pendentes & eâdem argumentandi ratione determinandi.*

De Natura Radicum.

Poteſt verò æquatio tot habere radices quot ſunt dimenſiones ejus, & non plures (q).

Sic æquatio $x^4 - x^3 - 19xx + 49x - 30 = 0$, quatuor habet radices 1, 2, 3, & -5; non autem plures. Nam quilibet ex his numeris ſcriptus in æquatione pro x efficet terminos omnes ſe mutuò deſtruere, ut dictum eſt; præter hos vero nullus eſt numerus cujus ſubſtitutione hoc eveniet.

Cæterùm numerus & natura radicum ex generatione æquationis optimè intelligetur.

Ut ſi ſcire vellemus quomodo generetur æquatio cujus radices ſint 1, 2, 3, & -5, ſupponendum erit x ambiguè ſignificare numeros illos; ſeu eſſe $x = 1$, $x = 2$, $x = 3$, & $x = -5$; vel quòd perinde eſt, $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, & $x + 5 = 0$; Et multiplicando hæc in ſe, prodibit, multiplicatione $x - 1$, in $x - 2$,

ſicut illi fecerant, cuilibet æquationi totidem tribuit, quot illa poſitivas habeat; negativarum ne in ullo quidem caſu ratione habitâ, utpote quas ſemper, ut opinor, inutiles judicavit, cùm earum naturam minimè perſpexiſſet. Sic in ſectione operis ſui quarta prop. 1ª. æquationis quadraticæ, $aa\begin{smallmatrix}-b\\+c\end{smallmatrix}a = bc$, cujus radices ſunt $+b$, $-c$, unicam radicem ſtatuit b; illam c radicem eſſe negat. In propoſitione verò ſecundâ æquationis, $aa\begin{smallmatrix}-b\\-c\end{smallmatrix}a = bc$, radicem utramque, $+b$, $+c$, agnovit. Rurſum in propoſitione 3ª. æquationis cubicæ $aaa\begin{smallmatrix}+b\\+c\\-d\end{smallmatrix}aa\begin{smallmatrix}+bc\\-bd\\-cd\end{smallmatrix}a = bcd$, cujus tres ſunt radices $+d$, $-b$, $-c$ radicem unicam, $+d$, ſtatuit; alteras, b et c, radices eſſe negat. In propoſitione 4ª. æquationis $aaa\begin{smallmatrix}+b\\-c\\-d\end{smallmatrix}aa\begin{smallmatrix}-bc\\-bd\\+cd\end{smallmatrix}a = -bcd$, cujus radices ſunt tres, $+d$, $+c$, $-b$, duas, d, c, agnovit, tertiam $-b$ rejecit. In propoſitione 5ª. æquationis $aaa\begin{smallmatrix}-b\\-c\\-d\end{smallmatrix}aa\begin{smallmatrix}+bc\\+bd\\+cd\end{smallmatrix}a = bcd$ agnovit tres radices poſitivas b, c, d.

hæc

CAPUT XVI. hæc æquatio, $xx-3x+2=0$; quæ duarum eſt dimenſionum, ac duas habet radices 1 & 2. Et hujus multiplicatione in $x-3$ prodibit $x^3-6xx+11x-6=0$, æquatio trium dimenſionum totidemque radicum; quæ iterum multiplicata per $x+5$ fit $x^4-x^3-19xx+49x-30=0$, ut ſuprà. Cùm igitur hæc æquatio generetur ex quatuor factoribus, $x-1$, $x-2$, $x-3$, & $x+5$, in ſe continuo ductis, ubi factorum aliquis nihil eſt, quod ſub omnibus ſit nihil erit; ubi verò horum nullus nihil eſt, quod ſub omnibus continetur nihil eſſe non poteſt. Hoc eſt, non poteſt $x^4-x^3-19xx+49x-30$, eſſe nihilo æquale, ut oportet, niſi his quatuor caſibus, ubi eſt $x-1=0$, vel $x-2=0$, vel $x-3=0$, vel denique $x+5=0$: proinde ſoli numeri, 1, 2, 3, & -5, valere poſſunt x, ſeu radices eſſe æquationis. Et ſimile eſt ratiocinium de omnibus æquationibus. Nam tali multiplicatione imaginari poſſumus omnes generari, quamvis factores ab invicem ſecernere ſolet eſſe difficillimum, & ipſum eſt quod æquationem reſolvere & radices extrahere. Habitis enim radicibus habentur factores.

Radices verò ſunt duplices; affirmativæ *ut in allato exemplo* 1, 2, & 3, *&* negativæ *ut* -5. *Ex his verò aliquæ non rarò evadunt* impoſſibiles (r).

Sic æquationis, $xx-2ax+bb=0$, radices duæ, quæ ſunt $a+\sqrt{aa-bb}$, & $a-\sqrt{aa-bb}$ reales quidem ſunt, ubi aa majus eſt quàm bb; at ubi aa minus eſt quàm bb, evadunt impoſſibiles; eò quòd $aa-bb$ tunc evadet negativa quantitas, & negativæ quantitatis radix quadratica eſt impoſſibilis. Omnis enim radix poſſibilis ſive affirmativa ſit, ſive negativa, ſi per ſeipſam multiplicetur, producet quadratum affirmativum; proinde impoſſibilis erit quæ quadratum negativum producere debet. Eodem argumento

(r) Ainſi qu'on peut donner trois noms aux ſolutions, veu qu'il y en a qui ſont plus que rien, d'autres moins que rien; & d'autres envelopées, comme celles qui ont des $\sqrt{-}$, comme des $\sqrt{-3}$, ou autres nombres ſemblables. *Albert Girard, Invent. nouv. en Alg.*

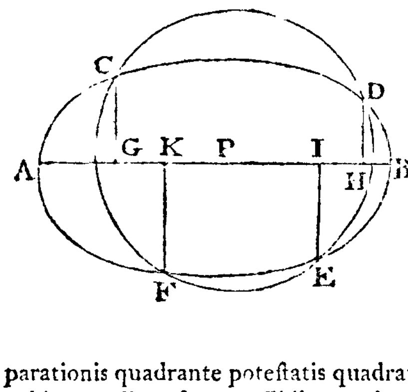
"Tribus igitur nominibus radices diſtinguendæ ſunt. Sunt enim nihilo majores, ſunt minores nihilo, ſunt etiam involutæ; nimirum hoc ſigno implicitæ, $\sqrt{-}$, ut $\sqrt{-3}$, aut aliis cujuſvis numeri negati radix quadratica." Quas vero hoc loco *involutas* alibi aptiore vocabulo *impoſſibiles* dixit.

( ) Id verò notandum, in æquationibus quibus membra nulla deſunt, non ex formâ æquationis ſed ex magnitudinibus coefficientium et homogenei comparationis radices impoſſibiles fieri. Nam in qualibet æquatione radices ſunt poſſibiles, quamdiu coefficientium et homogenei comparationis magnitudines, ac mutuæ proportiones, intra limites quoſdam conſtiterint certis legibus præfiniendos. Sic omnis æquationis quadraticæ radices ſunt poſſibiles, cujus homogeneum comparationis

gumento colligitur æquationem, $x^3-4xx+7x-6=0$, unam quidem realem radicem habere, quæ eſt 2; duas verò impoſſibiles, $1+\sqrt{-2}$, & $1-\sqrt{-2}$. Nam quælibet ex his, 2, $1+\sqrt{-2}$, & $1-\sqrt{-2}$, ſcripta in æquatione pro x, efficiet omnes ejus terminos ſe mutuò deſtruere; ſunt verò $1+\sqrt{-2}$, & $1-\sqrt{-2}$, numeri impoſſibiles, eò quòd extractionem radicis quadraticæ ex numero negativo -2 præſupponant (f). De Natura Radicum.

Æquationum verò radices ſæpe impoſſibiles eſſe æquum eſt, ne caſus problematum, qui ſæpe impoſſibiles ſunt, exhibeant poſſibiles.

Ut ſi rectæ & circuli interſectio determinanda eſſet, & pro circuli radio, & rectæ à centro ejus diſtantiâ, ponantur literæ duæ; ubi æquatio interſectionem definiens habetur, ſi pro literâ deſignante diſtantiam rectæ à centro ponatur numerus minor radio, interſectio poſſibilis erit; ſin major, fiet impoſſibilis; & æquationis radices duæ quæ interſectiones duas determinant, debent eſſe perinde poſſibiles vel impoſſibiles, ut rem ipſam verè exprimant. Atque ita ſi circulus CDEF, & Ellipſis ACBF ſe mutuò ſecent in punctis, C, D, E, F, & ad rectam aliquam poſitione datam, AB, demittantur perpendicula, CG, DH, EI, FK, & quærendo longitudinem alicujus è perpendiculis, perveniatur tandem ad æquationem; æquatio illa, ubi circulus ſecat Ellipſin in quatuor punctis, habebit quatuor radices reales; quæ erunt quatuor illa perpendicula.



Quòd ſi circuli radius, manente centro ejus, minuatur, donec, punctis E & F coaleſcentibus, circulus tandem tangat Ellipſin, ex radicibus duæ illæ, quæ perpendicula, EI & FK, jam coincidentia, exprimunt, evadent æquales. Et ſi circulus adhuc minuatur,

parationis quadrante poteſtatis quadraticæ coefficientis lateris non ſit majus. Et omnis æquationis cubicæ radices ſunt poſſibiles, cujus deleto membro ſecundo (id quod viâ mox tradendâ in omnibus facere licet) homogenei comparationis æquationis transformatæ poteſtas cubica ad poteſtatem quadraticam coefficientis membri ejuſdem transformatæ tertii proportionem habeat minorem quàm numerus viceniſeptenarius ad quaternarium. Erravit igitur Girardus cùm regulæ generalis, quam de radicum numero veriſſimam allegavit, in æquationibus imperfectis exceptionem conſtituit, quaſi verò aut ſolis aut ſemper illis radices impoſſibiles eſſent. Sunt quidem perfectæ, quibus nulla radix eſt poſſibilis; ſunt imperfectæ, quæ juſto radicum numero gaudent. Æquationi biquadraticæ $x^4-5x^3+18x^{-}-31x^2+35=0$, perfectæ licèt, radices omnes ſunt impoſſibiles: cubicæ, $y^3-19y+30=0$, veræ ſunt omnes. Nempe prioris radices ſunt $1+\sqrt{-6}$, $1-\sqrt{-6}$, $\frac{3}{2}+\sqrt{-2\frac{3}{4}}$, $\frac{3}{2}-\sqrt{-2\frac{3}{4}}$; poſterioris 2, 3, -5.

 ut Ellipſin in puncto EF ne quidem tangat, ſed ſecet tantùm in alteris duobus punctis, C, D; tunc ex quatuor radicibus duæ illæ, quæ perpendicula, EI, FK, jam facta impoſſibilia exprimebant, fient unà cum perpendiculis illis impoſſibiles. Et hoc modo in omnibus æquationibus, augendo vel minuendo terminos earum, ex inæqualibus radicibus duæ primò æquales, deinde impoſſibiles, evadere ſolent. (t) Et inde fit quòd radicum impoſſibilium numerus ſemper ſit par.

Sunt tamen radices æquationum aliquandò poſſibiles, ubi Schema impoſſibiles exhibet. Sed hoc fit, ob limitationem aliquam in Schemate, quod ad æquationem nil ſpectat.

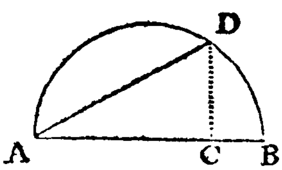
Ut ſi in ſemicirculo, ADB, datis diametro, AB, & lineâ inſcriptâ, AD, demiſſoque perpendiculo, DC, quærerem diametri ſegmentum, AC, foret $\frac{ADq}{AB}$ = AC. Et per hanc æquationem AC realis exhibetur quantitas, ubi linea inſcripta AD major eſt quàm diameter

(s) Vides igitur radices ex eo fieri impoſſibiles, quòd datarum magnitudines eæ ſint, quæ cum conditionibus Problematis conſiſtere nequeant: unde talis neceſſariò naſcitur æquatio, cujus coefficientes et homogeneum comparationis conditiones recuſant, quas ad rei propoſitæ effectionem omninò illis ſubeundas ſigna æquationis indicant. Vel ut breviùs dicam, cùm id aggreſſus ſis quod fieri omninò rerum Natura vetuit, exinde eveniet ut in ſymbola incidas quorum nulla eſt interpretatio, quæ intelligi nequeunt. (confer Maclaurin. Algebr. part. 3. cap. 1.)

Reverâ Radix impoſſibilis illud eſt, quod in rerum Naturâ nuſquam extat. Cautè igitur accipiendum eſt, quod à Maclaurino, credo, omnium primo dictum, nunc omnibus eſt in ore, quantitatis cujuſvis, atque ipſius adeo unitatis, triplicem eſſe radicem cubicam Algebraicam; unam veram, duas impoſſibiles: unitatis quidem, præter ipſam unitatem quæ ſola vera eſt, impoſſibiles duas, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$: alìus autem cujuſvis quantitatis cubicæ, puta a^3, præter veram a, duas illas $-\frac{1}{2}a+\frac{1}{2}a\sqrt{-3}$, $-\frac{1}{2}a-\frac{1}{2}a\sqrt{-3}$: atque hæc ita eſſe ex eo conficitur, quòd æquationi cubicæ, $x^3-a^3=0$, radices illæ tres ſint, a, $-\frac{1}{2}+\frac{1}{2}a\sqrt{-3}$, $-\frac{1}{2}a-\frac{1}{2}a\sqrt{-3}$. Hæc verò minimè ita accipienda ſunt, ac ſi quantitas ulla eſſet, quæ in ſe cubicè multiplicata unitatem redderet, præter ipſam unitatem; qualis certè in rerum Naturâ nulla extat: ſicut nulla eſt, quæ in ſe cubicè multiplicata quantitatem cubicam ſymbolo a^3 deſignatam reddat præter ipſam a. Neque profectò linea recta eſt ulla, ex quâ ſi cubus ſit extructus cubum è rectâ datæ longitudinis adæquabit, præter ipſam illam datæ longitudinis rectam. Symbola autem illa, $-\frac{1}{2}a+\frac{1}{2}a\sqrt{-3}$, $-\frac{1}{2}a-\frac{1}{2}a\sqrt{-3}$, quantitatis a^3 radices cubicæ dicuntur, ſenſu merè algebraico. Nimirum ſi horum binomiorum utrumvis, nullâ ſignificantiæ ejus ratione habitâ, perinde ac ſi veram aliquam quantitatem deſignaret, ſecundum leges multiplicationis algebraicæ in ſe cubicè multiplicaveris, exitu operis, quantitas a^3 exſiſtet; vel ſi binomium $x-a$ cum trinomiis $x+\frac{1}{2}a-\frac{1}{2}a\sqrt{-3}$, $x+\frac{1}{2}a+\frac{1}{2}a\sqrt{-3}$ algebraicè multiplicaveris, binomium y^3-a^3 effeceris; cujus propterea radices binomia $-\frac{1}{2}a+\frac{1}{2}a\sqrt{-3}$, $-\frac{1}{2}a-\frac{1}{2}a\sqrt{-3}$ jure optimo vocanda ſunt, cùm ad ejus compoſitionem algebraicam eodem planè modo eorum unumquodque contulerit, ac quantitas a quæ verè radix eſt. Similis ſemper interpretatio adhibenda eſt, quando mathematici radices æquationum impoſſibiles certis tamen ſymbolis repræſentari volunt. Sunt utique iſtiuſmodi ſymbola æquationum, quarum radices dicuntur, elementa merè algebraica; ex quibus, certa lege compoſitis, aſſumptis etiam quas poſſibiles quæque radices habeat, æquationes exſiſtunt, haud aliter ac ſyllabæ et voces è literis certo ordine conjunctis, aſcitiſque vocalibus, quarum ſeparatim et ſine illis neque ſonus neque ſenſus eſt.

(t) Idem planè eodemque modo Albertus Girardus dixit, cujus hæc ſunt verba.

Juſques



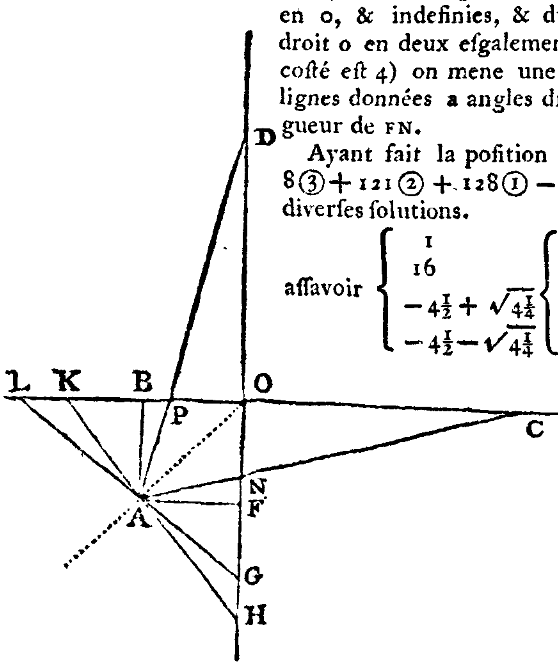
meter AB; per Schema verò AC tunc evadit impoſſibilis. Nimirum, in ſchemate, linea AD ſupponitur inſcribi in circulo, atque adeo diametro circuli major eſſe non poteſt; in æquatione verò nihil eſt, quod à conditione illâ pendeat. Ex hâc ſolâ linearum conditione colligitur æquatio, quòd ſint AB, AD, & AC continuè proportionales. Et quoniam æquatio non complectitur omnes conditiones ſchematis, non neceſſe eſt, ut omnium conditionum teneatur limitibus. Quicquid amplius eſt in ſchemate, quam in æquatione, poteſt illud limitibus arctare, hanc non item. Quâ de cauſâ ubi æquationes ſunt imparium dimenſionum, adeoque radices omnes impoſſibiles habere non poſſunt; ſchemata quantitatibus, à quibus radices omnes pendent, ſæpè limites imponunt, quos tranſgredi, ſervatis ſchematum conditionibus, impoſſible eſt (u). De Natura Radicum.

Ex

Juſques icy nous n'avons encor expliqué a quoy ſervent les ſolutions par moins, quand il y en a. La ſolution par moins s'explique en Geometrie en retrogradant, et le moins recule là ou le + avance.

BROBLEME D'INCLINAISON.

Soyent deux lignes droictes, DG, BC, ſe couppant en angles droits en O, & indefinies, & du poinct A (dans la ligne qui couppe l'angle droit O en deux eſgalement, ainſi que ABOF eſt quarré dont chacun coſté eſt 4) on mene une ANC, ainſi que l'intercepté, NC, (entre les lignes données a angles droit, DG, BC) ſoit $\sqrt{153}$, on demande la longueur de FN.



Ayant fait la poſition de FN 1① on trouvera que 1④ ſera eſgale a 8③ + 121② + 128① − 256, & ainſi la valeur de 1① recevra quatre diverſes ſolutions.

aſſavoir:

1	FN
16	FD
$-4\frac{1}{2}+\sqrt{4\frac{1}{4}}$	monſtrant le poinct G } du poinct F.
$-4\frac{1}{2}-\sqrt{4\frac{1}{4}}$	monſtrant le poinct H } du poinct F.

Aſſavoir, monſtrant leſdits poincts G & H, comme ſi les diſtances FG, FH eſtoyent moins que rien, en retrogradant; prenant que FN, FD avancent, & FG, FH reculent en arriere, tellement donc que les interceptes CN, DP, GL, HK, tendent & s'enclinent au poinct A, faiſant chacune $\sqrt{153}$, ſelon le requis.

Et pour l'interpreter encor mieux, les deux ſolutions qui ſont moins que 0, ſe doivent changer, aſſavoir les ſignes.

viendra $\begin{cases} 4\frac{1}{2}-\sqrt{4\frac{1}{4}} & \text{pour FG} \\ 4\frac{1}{2}+\sqrt{4\frac{1}{4}} & \text{pour FH} \end{cases}$

Leſquels il faut poſer au contraire de FN, FD, comme il eſt exprimé en la figure precedente: & ainſi

CAPUT XVI. *Ex radicibus verò quæ reales ſunt, affirmativæ & negativæ ad plagas oppoſitas ſolent tendere.*

Sic, in ſchemate penultimo, quærendo perpendiculum CG, incidetur in æquationem, cujus duæ erunt affirmativæ radices, CG ac DH, à punctis C & D tendentes verſus unam plagam, & duæ negativæ, EI & FK, tendentes à punctis E & F verſus plagam oppoſitam. Aut ſi in lineâ AB, ad quam perpendicula demittuntur, detur aliquod punctum P, & pars ejus PG, à puncto illo dato ad perpendiculorum aliquod CG extendens, quæratur, incidemus in æquationem quatuor radicum PG, PH, PI, PK; quarum quæſita PG, & quæ à puncto P ad eaſdem partes cum PG tendunt (ut PK) affirmativæ erunt, quæ verò tendunt ad partes contrarias (ut PH, PI) negativæ.

Ubi æquationis radices nullæ impoſſibiles ſunt, numerus radicum *affirmativarum & negativarum ex* ſignis *terminorum æquationis cognoſci poteſt.* Tot *enim ſunt radices affirmativæ quot ſignorum in continuâ ſerie* mutationes *de* + *in* − & − *in* +; *cæteræ* negativæ *ſunt* (x).

Ut in æquatione $x^4-x^3-19xx+49x-30=0$, ubi terminorum ſigna ſe ſequuntur hoc ordine, + − − + −, variationes ſecundi − à primo +, quarti + à tertio −, & quinti −, à quarto +, indicant tres

ainſi le faudra-t-il entendre de toutes ſolutions par moins, qui eſt une choſe de conſequence en Geometrie, incogneuë auparavant.

Hæc quidem de radicibus negativis Girardus, quorum ſenſum vocibus Latinis, & notis Algebraicis hodiè uſitatis ſic effero.

Radices negativæ, ſi quando obveniant, nondum diximus ad quid ſint utiles. Radicis igitur negatio in Geometriâ ſic accipienda eſt, ut per eam ſignificetur contrarius quidam linearum ac quaſi retrorſum ductus. Etenim ubi + progreditur ibi − recedit.

PROBLEMA EX INCLINATIONIBUS.

Sint duæ rectæ indefinitæ, DG, BC, quarum decuſſatio ſiat in O ad angulos rectos. A puncto autem A (ad rectam ſcilicet quæ rectum angulum ad O medium dividit, ut ſit ABOF quadratum cujus latus ſit 4) ductam puta rectam ANC, cujus pars NC rectis primò poſitis, BC, DG, intercepta, ſit $\sqrt{153}$. Quæritur longitudo rectæ FN.

Si rectam FN litterâ x deſignaveris, ſubductis calculis invenies, $x^4 = 8x^3 + 121x^2 + 128x - 256$. Hinc literæ x æſtimationes ſunt quatuor.

Nimirum
- 1 — FN
- 16 — FD
- $-4\frac{1}{2}+\sqrt{4\frac{1}{4}}$ — diſtantia puncti G a puncto F.
- $-4\frac{1}{4}-\sqrt{4\frac{1}{4}}$ — diſtantia puncti H a puncto F.

Quibus

tres affirmativas eſſe radices, adeoque quartam negativam eſſe. At, ubi radices aliquæ *impoſſibiles* ſunt, regula non valet; niſi quatenus impoſſibiles illæ, quæ nec negativæ ſunt nec affirmativæ, pro *ambiguis* habeantur. Sic in æquatione, $x^3 + pxx + 3ppx - q = 0$; ſigna indicant unam eſſe affirmativam radicem, & duas negativas. Finge $x = 2p$, ſeu $x - 2p = 0$; & multiplica æquationem priorem per hanc $x - 2p = 0$, ut una adhuc radix affirmativa addatur prioribus; & prodibit hæc æquatio $x^4 - px^3 + ppxx \begin{smallmatrix} -6p^3 \\ -q \end{smallmatrix} x + 2pq = 0$, quæ habere deberet duas affirmativas ac duas negativas radices: habet tamen, ſi mutationem ſignorum ſpectes, affirmativas quatuor. Sunt ergo *duæ impoſſibiles*; quæ, pro ambiguitate ſuâ, priori caſu negativæ, poſteriori affirmativæ, eſſe videntur. DE NATURA RADICUM.

Verùm quot radices impoſſibiles ſunt cognoſci ferè poteſt per hanc regulam.

Conſtitue ſeriem fractionum, quarum denominatores ſunt numeri in hâc progreſſione, 1, 2, 3, 4, 5, &c. *pergendo ad numerum uſque qui eſt dimenſionum æquationis; numeratores vero eadem ſeries numerorum in ordine contrario. Divide unamquamque fractionem poſteriorem per priorem. Fractiones prodeuntes colloca ſuper terminis mediis æquationis. Et ſub quolibet mediorum terminorum, ſi quadratum ejus ductum in fractionem capiti imminentem ſit majus* (y) *quàm rectangulum terminorum utrinque conſiſtentium,*

Quibus punctorum G, H ſitus ita indicatur, ac ſi diſtentiæ FG, FH retrorſum emenſæ nihilo minores eſſent. Etenim ſi linearum FN, FD verſus puncta N, D ductum *progreſſum* dixeris, contrarium rectarum FG, FD ductum neceſſe eſt *regreſſum* dicas. Quatuor autem, CN, DP, GL, HK, rectis DG, EC, interceptæ, quantitati $\sqrt{153}$ ſingulatim ſunt æquales; et, ad punctum A vergendo, problematis requiſitis ſinguli quidem ſatisfaciunt.

Vel ut rem etiam clariùs dicam, radicum duarum negativarum ſigna [partium] mutari debent, ut ex [duarum $4\frac{1}{2}$, $\sqrt{4\frac{1}{4}}$ differentiâ] $4\frac{1}{2} - \sqrt{4\frac{1}{4}}$, efficiatur diſtantia FG; ex [earumdem ſummâ] $4\frac{1}{2} + \sqrt{4\frac{1}{4}}$ diſtantia FH. Tum duæ illæ FH, FG, ad partes puncti F ponendæ ſunt, earum ad quas poſitæ ſunt FN, FD contrarias; quemadmodum figuræ appoſitæ lineationes indicant. Atque omnium quidem radicum negativarum ſimilis eſt interpretatio, quæ res licèt hactenus incognita maximas tamen habet in Geometriâ utilitates.

(x) Hæc regula magnum illum *Carteſium* inventorem habuit, quem graviter ſanè nonnulli incuſarunt quaſi in Algebra.cis totus alienus eſſet. Mihi verò crimen non probant.

(y) Intellige, *algebraice* majus; licet enim quantitas fracta, quæ ſuper coefficientem aliquem ſcripta eſt, coefficientis illius poteſtatem quadraticam multiplicans quantitatem fecerit facto ex coefficientibus utrinque proximis minorem, tamen ſi coefficientibus illis proximis contraria ſigna ſint, ut factum ex illis negativum ſit, ſub coefficiente intermedio collocandum eſt ſignum +. Vel ſi membro aliquo æquationis deficiente, membra utrinque proxima ſignis prædita ſint contrariis: ſub membro deficiente collocandum eſt ſignum +.

colloca

CAPUT XVI. *colloca ſignum* +; *ſin minus, ſignum* −. *Sub primo verò & ultimo termino colloca ſignum* +. *Et tot erunt radices impoſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes, de* + *in* − & − *in* +.

Ut ſi habeatur æquatio, $x^3+pxx+3ppx-q=0$: Divido ſeriei hujus $\frac{3}{1}.\frac{2}{2}.\frac{1}{3}$ fractionum ſecundam, $\frac{2}{2}$, per primam, $\frac{3}{1}$; & tertiam, $\frac{1}{3}$, per ſecundam, $\frac{2}{2}$; & fractiones prodeuntes, $\frac{1}{3}$, & $\frac{1}{3}$ colloco ſuper mediis terminis æquationis, ut ſequitur. Dein quoniam quadratum ſecundi termini, pxx, ductum in imminentem fractionem, $\frac{1}{3}$, nimirum $\frac{ppx^4}{3}$, minus eſt, quàm primi termini x^3 & tertii $3ppx$ rectangulum, $3ppx^4$; ſub termino pxx, colloco ſignum −. At quia tertii termini, $3ppx$, quadratum, $9p^4xx$, ductum in imminentem fractionem, $\frac{1}{3}$, majus eſt quàm nihil, atque adeo multo majus quàm ſecundi termini, pxx, & quarti, $-q$, rectangulum negativum, colloco ſub tertio illo termino ſignum +. Dein ſub primo termino, x^3, & ultimo, $-q$, colloco ſigna +. Et ſignorum ſubſcriptorum, quæ in hâc ſunt ſerie, + − + +, mutationes duæ, una de + in −, alia de − in +, indicant duas eſſe radices impoſſibiles.

$$\begin{array}{cccc} & \frac{1}{3}. & \frac{1}{3}. & \\ x^3 & +pxx & +3ppx & -q=0 \\ + & - & + & + \end{array}$$

Sic & æquatio, $x^3-4xx+4x-6=0$, duas habet radices impoſſibiles.

$$\begin{array}{cccc} & \frac{1}{3} & \frac{1}{3} & \\ x^3 & -4xx & +4x & -6=0 \\ + & + & - & + \end{array}$$

Æquatio item, $x^4-6xx-3x-2=0$, duas habet. Nam hæc fractionum ſeries, $\frac{4}{1}.\frac{3}{2}.\frac{2}{3}.\frac{1}{4}$, dividendo ſecundam per primam, tertiam per ſecundam, & quartam per tertiam, dat hanc ſeriem, $\frac{3}{8}.\frac{4}{9}.\frac{3}{8}$, ſuper mediis æquationis terminis collocandam.

$$\begin{array}{ccccc} & \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & \\ x^4 & * & -6xx & -3x & -2=0 \\ + & + & + & - & + \end{array}$$

Dein ſecundi termini, qui hic nihil eſt, quadratum, ductum in fractionem imminentem $\frac{3}{8}$, producit, nihil; quod tamen majus eſt quàm rectangulum negativum, $-6x^6$, ſub terminis utrinque poſitis, x^4 & $-6xx$, contentum. Quare ſub termino illo deficiente ſcribo +. In cæteris pergo ut in exemplo ſuperiori; & ſignorum ſubſcriptorum prodit hæc ſeries, + + + − +; ubi duæ mutationes indicant duas radices impoſſibiles. Et ad eundem modum in æquatione $x^5-4x^4+4x^3-2xx-5x-4=0$, deteguntur impoſſibiles duæ.

$$\begin{array}{cccccc} & \frac{2}{5}. & \frac{1}{2}. & \frac{1}{2}. & \frac{2}{5}. & \\ x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$$

Ubi

Ubi termini duo vel plures ſimul deſunt, ſub primo termi- DE NATURA RADICUM.
norum deficientium collocandum eſt ſignum −, ſub ſecundo ſignum +, ſub tertio ſignum −, & ſic deinceps, ſemper variando ſigna; niſi quòd ſub ultimo terminorum ſimul deficientium ſemper collocandum eſt ſignum +, ubi deficientibus utrinque proximi habent ſigna contraria. Ut in æquationibus

$$\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & +a^5=0, \\ + & + & - & + & - & + \end{array}$$

$$\&\ \begin{array}{cccccc} x^5 & +ax^4 & * & * & * & -a^5=0, \\ + & + & - & + & + & + \end{array}$$

quarum prior quatuor, poſterior duas, habet impoſſibiles radices. Sic & æquatio

$$\begin{array}{cccccccc} & \frac{3}{7}. & \frac{5}{9}. & \frac{3}{5}. & \frac{3}{5}. & \frac{5}{9}. & \frac{3}{7}. & \\ x^7 & -2x^6 & +3x^5 & -2x^4 & +x^3 & * & * & -3=0 \\ + & - & + & - & + & - & + & + \end{array}$$

ſex habet impoſſibiles.

Hinc etiam cognoſci poteſt, utrùm radices impoſſibiles inter affirmativas radices latent, an inter negativas. Nam ſigna terminorum, ſignis ſubſcriptis variantibus imminentium, indicant tot affirmativas eſſe impoſſibiles, quot ſunt ipſorum variationes; & tot negativas, quot ſunt ipſorum ſucceſſiones, ſine variatione. Sic in æquatione

$$\begin{array}{cccccc} x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$$

quoniam ſignis infrà ſcriptis variantibus, + − +, quibus radices duæ impoſſibiles indicantur, imminentes termini, $-4x^4 + 4x^3 - 2xx$, ſigna habent − + −, quæ, per duas variationes, indicant duas affirmativas radices; ideo radices duæ impoſſibiles inter affirmativas latebunt. Cùm itaque omnium æquationis terminorum ſigna, + − + − − −, per tres variationes indicant tres eſſe affirmativas radices, & reliquas duas negativas eſſe, & inter affirmativas lateant duæ impoſſibiles, ſequitur, æquationis unam eſſe radicem verè affirmativam, duas negativas ac duas impoſſibiles. Quòd ſi æquatio fuiſſet

$$\begin{array}{cccccc} x^5 & -4x^4 & -4x^3 & -2xx & -5x & -4=0 \\ + & + & - & - & + & + \end{array}$$

tunc termini ſubſcriptis ſignis prioribus variantibus, + −, imminentes, nimirum $-4x^4 - 4x^3$, per ſigna ſua non variantia − & −, indicant unam, ex negativis radicibus, impoſſibilem eſſe; & termini ſignis ſubſcriptis poſterioribus variantibus, − +, imminentes, nimirum $-2xx - 5x$, per ſigna ſua non variantia, − & −, indicant aliam, ex negativis radicibus, impoſſibilem eſſe.

CAP. XVI. eſſe. Quamobrem, cùm æquationis ſigna, + − − − − −, per unam variationem, indicent unam affirmativam radicem, cæteras quatuor negativas eſſe; ſequitur, unam eſſe affirmativam, duas negativas, ac duas impoſſibiles. Atque hæc ita ſe habent, ubi non ſunt plures impoſſibiles radices, quàm per regulam allatam deteguntur. Poſſunt enim plures eſſe, licet id perrarò eveniat.

CAPUT XVII.

De tranſmutationibus Æquationum (z).

CÆterùm æquationis cujuſvis radices omnes affirmativæ *in* negativis, *&* negativæ *in* affirmativas, *mutari poſſunt, idque mutando tantùm ſigna terminorum alternorum* (a).

Sic æquationis, $x^5-4x^4+4x^3-2xx-5x-4=0$, radices tres affirmativæ mutabuntur in negativas, & duæ negativæ in affirmativas, mutando tantùm ſigna ſecundi quarti & ſexti termini ut hîc fit, $x^5+4x^4+4x^3+2xx-5x+4=0$. Eaſdem habet hæc æquatio radices cum priore, niſi quòd hîc affirmativæ ſunt, quæ ibi erant negativæ, & hîc negativæ, quæ ibi erant affirmativæ; & radices duæ impoſſibiles, quæ ibi inter affirmativas latebant, hîc latent inter negativas; ita ut his deductis reſtet unica tantùm radix verè negativa.

Sunt & aliæ æquationum tranſmutationes quæ diverſis uſibus inſerviunt. *Poſſumus enim ſupponere, radicem æquationis ex cognitâ & incognitâ aliquâ quantitate utcunque componi, & perinde pro eâ ſubſtituere, quod æquipollens eſſe fingitur.* Ut ſi ſupponamus radicem æqualem eſſe ſummæ vel differentiæ cognitæ alicujus & incognitæ quantitatis. Nam poſſumus hoc pacto radices æquationis cognitâ illâ quantitate augere vel diminuere, vel de cognitâ quantitate ſubducere; atque ita efficere, ut earum aliquæ, quæ priùs erant negativæ, jam fiant affirmativæ, vel ut aliquæ ex affirmativis evadant negativæ; vel etiam ut omnes evadant affirmativæ aut omnes negativæ (b). Sic in æquatione, $x^4-x^3-19xx+49x-30=0$, ſi radices unitate augeri vellem, fingo $x+1=y$, ſeu $x=y-1$; & perinde pro x ſcribo in æquatione

(z) Hujuſce doctrinæ fundamenta jecit Franciſcus Vieta, cujus librum vide de Recognitione Æquationum, c. 7.

(a) Anaſtrophe Vietæa De Emendat. Æquationum, c. 3. Vide etiam Carteſii Geometr. Lib. 3.

(b) Vid.

tione $y-1$, & pro quadrato, cubo, quadrato-quadrato de x ſimilem poteſtatem de $y-1$, ad hunc modum. *De Transmut. Æquat.*

$$\begin{array}{l|rrrrr} x^4 & y^4 - 4y^3 & + 6yy & - 4y & + 1 \\ -x^3 & - y^3 & + 3yy & - 3y & + 1 \\ -19xx & & -19yy & +38y & -19 \\ +49x & & & +49y & -49 \\ -30 & & & & -30 \\ \hline \text{Summa} & y^4 - 5y^3 & -10yy & +80y & -96 = 0. \end{array}$$

Et æquationis prodeuntis $y^4 - 5y^3 - 10yy + 80y - 96 = 0$, radices erunt, 2, 3, 4, -4; quæ priùs erant, 1, 2, 3, -5, unitate jam factæ majores. Quòd ſi pro x ſcripſiſſem $y + 1\frac{1}{2}$, prodiiſſet æquatio $y^4 + 5y^3 - 10yy - \frac{5}{4}y + \frac{39}{16} = 0$; cujus duæ fuiſſent radices affirmativæ, $\frac{1}{2}$ & $1\frac{1}{2}$, ac duæ negativæ, $-\frac{1}{2}$ & $-6\frac{1}{2}$. Pro x vero ſcribendo $y-6$, prodiiſſet æquatio, cujus radices fuiſſent, 7, 8, 9, 1, omnes nimirum affirmativæ; & pro eodem ſcribendo $y+4$, radices, jam numero quaternario diminutæ, evaſiſſent -3, -2, -1, -9; negativæ omnes.

Et hoc modo augendo vel diminuendo radices, ſiquæ impoſſibiles ſunt, hæ aliquando faciliùs detegentur quàm priùs. Sic in æquatione, $x^3 - 3aax - 3a^3 = 0$, radices nullæ per præcedentem regulam apparent impoſſibiles. At ſi augeas radices quantitate a, ſcribendo $y-a$ pro x, in æquatione reſultante, $y^3 - 3ayy - a^3 = 0$, radices duæ impoſſibiles jam per regulam illam detegi poſſunt.

Eâdem operatione poſſumus etiam ſecundos terminos æquationum tollere (c). Hoc enim fiet, ſi cognitam quantitatem ſecundi termini æquationis propoſitæ, per numerum dimenſionum æquationis diviſam, ſubducamus de quantitate, quæ pro novæ æquationis radice ſignificandâ aſſumitur, & reſiduum ſubſtituamus pro radice æquationis propoſitæ. Ut ſi proponatur æquatio, $x^3 - 4xx + 4x - 6 = 0$, cognitam quantitatem ſecundi termini, quæ eſt -4, diviſam per numerum dimenſionum æquationis, 3, ſubduco de ſpecie, quæ pro novâ radice ſignificandâ aſſumitur, puta de y; & reſiduum $y + \frac{4}{3}$ ſubſtituo pro x, & provenit,

$$\begin{array}{rrrr} y^3 + 4yy & + \frac{16}{3}y & + \frac{64}{27} \\ - 4yy & - \frac{32}{3}y & - \frac{64}{9} \\ & + 4y & + \frac{16}{3} \\ & & - 6 \\ \hline y^3 \quad * & - \frac{4}{3}y & - \frac{146}{27} = 0. \end{array}$$

Eâdem methodo poteſt & tertius æquationis terminus tolli. Proponatur æquatio $x^4 - 3x^3 + 3xx - 5x - 2 = 0$, & finge $x = y - e$; & ſubſtituendo $y - e$ pro x orietur hæc æquatio.

$$\left.\begin{array}{l} y^4 \begin{array}{l} -4e \\ -3 \end{array} y^3 \begin{array}{l} +6ee \\ +9e \\ +3 \end{array} yy \begin{array}{l} -4e^3 \\ -9ee \\ -6e \\ -5 \end{array} y \begin{array}{l} +e^4 \\ +3e^3 \\ +3ee \\ +5e \\ -2 \end{array} \end{array}\right\} = 0.$$

Hujus æquationis tertius terminus eſt $6ee + 9e + 3$ ductum in yy. Ubi ſi $6ee + 9e + 3$ nullum

( ) Vid. Carteſii Geomet. Lib. 3.

( ) Expurgatio per Uncias Vietæ. Vid. Librum de emendatione æquationum, c. 1.

eſſet,

Cap. XVII. eſſet, eveniret ipſum quod volumus. Fingamus itaque nullum eſſe, ut inde colligamus quinam numerus ad hunc effectum ſubſtitui debet pro e, & habebimus æquationem quadraticam $6ee+9e+3=0$; quæ diviſa per 6 fiet $ee+\frac{3}{2}e+\frac{1}{2}=0$, ſeu $ee=-\frac{3}{2}e-\frac{1}{2}$; &, extractâ radice, $e=-\frac{3}{4}\pm\sqrt{\frac{9}{16}-\frac{1}{2}}$, ſeu $=-\frac{3}{4}\pm\sqrt{\frac{1}{16}}$, hoc eſt $=-\frac{3}{4}\pm\frac{1}{4}$, atque adeo vel $=-\frac{1}{2}$ vel $=-1$. Unde $y-e$ erit vel $y+\frac{1}{2}$, vel $y+1$. Quamobrem, cùm $y-e$ ſcriptum fuit pro x, vice $y-e$ debet $y+\frac{1}{2}$, vel $y+1$, ſcribi pro x, ut tertius æquationis reſultantis terminus nullus ſit. Et in utroque quidem caſu id eveniet. Nam ſi pro x ſcribatur $y+\frac{1}{2}$, orietur hæc æquatio, $y^4-y^3-\frac{15}{4}y-\frac{65}{16}=0$; ſin ſcribatur $y+1$, orietur hæc, $y^4+y^3-4y-6=0$.

Poſſunt & radices æquationis per datos numeros multiplicari vel dividi; & hoc pacto termini æquationum diminui, fractioneſque & radicales quantitates aliquandò tolli (d).

Ut ſi æquatio ſit $y^3-\frac{4}{3}y-\frac{146}{27}=0$, ad tollendas fractiones fingo eſſe $y=\frac{1}{3}z$; & perinde pro y ſubſtituendo $\frac{1}{3}z$ provenit æquatio nova $\frac{z^3}{27}-\frac{12z}{27}-\frac{146}{27}=0$; & rejecto terminorum communi denominatore, $z^3-12z-146=0$, cujus æquationis radices ſunt triplo majores quàm antè. Et rurſus, ad diminuendos terminos æquationis hujus, ſi ſcribatur $2v$ pro z, prodibit $8v^3-24v-146=0$; &, diviſis omnibus per 8, fiet $v^3-3v-18\frac{1}{4}=0$; cujus æquationis radices dimidiæ ſunt radicum prioris. Et hîc, ſi tandem inveniatur v, ponendum erit $2v=z$, $\frac{1}{3}z=y$, & $y+\frac{4}{3}=x$; & æquationis primo propoſitæ, $x^3-4xx+4x-6=0$, habebitur radix x.

Sic & in æquatione $x^3-2x+\sqrt{3}=0$, ad tollendam quantitatem radicalem $\sqrt{3}$, pro x ſcribo $y\sqrt{3}$, & provenit æquatio $3y^3\sqrt{3}-2y\sqrt{3}+\sqrt{3}=0$; quæ, diviſis omnibus terminis per $\sqrt{3}$, fit $3y^3-2y+1=0$.

Rurſus

(d) *Multiplicatio & Diviſio iſomerica Vietæ* (De Emendat. Æquationum, c. 14.) quarum præcepta Carteſius breviter ac luculenter tradidit. "Multiplicetur vel dividatur coefficiens membri ſecundi æquationis propoſitæ per quantitatem illam, quæ multiplicare aut dividere debet radices. Ejuſdem poteſtas quadratica multiplicet vel dividat coefficientem membri tertii; cubica coefficientem quarti, eodemque deinceps modo." Cæterùm harum tranſmutationum in numerali æquationum exegeſi maximæ ſunt utilitates; utpote quarum ope non modò fractiones & ſurda tollantur, ſed numeri ipſâ mole intractabiles ad minores revocentur, calculis faciliùs ſubjiciendos. Sic ſi æquatio fuerit $x^3 * -203125x-23437500=0$, divido ἰσομοιρικῶς per numerum

Rurſus æquationis radices in earum reciprocas tranſmutari poſſunt, & hoc pacto æquatio aliquandò ad formam commodiorem reduci (e). De Transmut. Æquat.

Sic æquatio noviſſima, $3y^3-2y+1=0$, ſcribendo $\frac{1}{z}$ pro y, evadit $\frac{3}{z^3}-\frac{2}{z}+1=0$, ſeu terminis omnibus multiplicatis per z^3, & ordine terminorum mutato, $z^3-2zz+3=0$. Poteſt etiam æquationis terminus penultimus hoc pacto tolli, ſi modò ſecundus priùs tollatur; ut factum vides in exemplo præcedente. Aut ſi antepenultimum tolli cupias, id fiet, ſi modò tertium priùs tollas. Sed & radix minima hoc pacto in maximam convertitur, & maxima in minimam; quod uſum nonnullum habere poteſt in ſequentibus. Sic in æquatione $x^4-x^3-19xx+49x-30=0$, cujus radices ſunt; 3, 2, 1, −5; ſi ſcribatur $\frac{1}{y}$ pro x, reſultabit æquatio $\frac{1}{y^4}-\frac{1}{y^3}-\frac{19}{yy}+\frac{49}{y}-30=0$; quæ, terminis omnibus multiplicatis per y^4, ac diviſis per 30, ſignisque mutatis, fiet $y^4-\frac{49}{30}y^3+\frac{19}{30}yy+\frac{1}{30}y-\frac{1}{30}=0$; cujus radices ſunt, $\frac{1}{3}$, $\frac{1}{2}$, 1, $-\frac{1}{5}$; radicum affirmativarum maximâ, 3, jam converſâ in minimam, $\frac{1}{3}$, & minimâ, 1, jam factâ maximâ, & radice negativâ, −5, quæ omnium maximè diſtabat à nihilo, jam omnium maximè accedente ad nihil.

Sunt & aliæ æquationum tranſmutationes, (f) ſed quæ omnes ad exemplum tranſmutationis illius, ubi tertium æquationis terminum ſuſtulimus confici poſſunt; ut non opus ſit hâc de re plura dicere. Addamus potiùs aliqua de limitibus æquationum.

CAPUT XVIII.

De Limitibus æquationum.

EX *Æquationum generatione conſtat, quod cognita quantitas* ſecundi *termini æquationis, ſi ſignum ejus mutetur, æqualis*

numerum 125. $\overset{1}{x^3}\ \overset{125}{*}\ \overset{15625}{-\ 203125x}\ \overset{1953125}{-\ 23437500}=0$. Ex diviſione provenit nova, $y^3-13y-12=0$; cujus radicibus 4, −3, −1, cum numero 125 multiplicatis, exſiſtunt propoſitæ radices, 500, −375, −125.

(e) Tranſmutatio Πρῶτον ἔσχατον Vietæ, cui nomen inditum ex inverſo, in æquatione tranſmutatâ coefficientium ordine. De Emendat. Æquationum, c. 2.

(f) Harum, ni fallor, præcipua eſt *Canonica æquationum tranſmutatio*, ut coefficientes ſint quæ præſcribantur: de quâ conſule Vietam in Libro de Emendatione Æquationum, c. 8. & Carteſium in Libro tertio Geometriæ.

CAP. XVIII. *ſit aggregato omnium radicum ſub ſignis propriis*; *ea* tertii *æqualis aggregato rectangulorum ſub ſingulis binis radicibus*; *ea* quarti, *ſi ſignum ejus mutetur, æqualis aggregato contentorum ſub ſingulis ternis radicibus*; *ea* quinti *æqualis aggregato contentorum ſub ſingulis quaternis*; *& ſic in infinitum* (g).

Aſſumamus $x=a$, $x=b$, $x=-c$, $x=d$, &c. ſeu $x-a=0$, $x-b=0$, $x+c=0$, $x-d=0$, & ex horum continuâ multiplicatione generemus æquationes, ut ſuprà. Jam multiplicando $x-a$ per $x-b$, producetur æquatio $xx \begin{array}{c}-a\\-b\end{array} x + ab = 0$; ubi cognita quantitas ſecundi termini, ſi ſigna ejus mutentur, nimirum $a+b$, eſt ſumma duarum radicum, a & b; & cognita tertii, ab, illud unicum quod ſub utràque continetur rectangulum. Rurſus, multiplicando hanc æquationem per $x+c$, producetur æquatio cubica,

$$x^3 \begin{array}{c}-a\\-b\\+c\end{array} xx \begin{array}{c}+ab\\-ac\\-bc\end{array} x + abc = 0;$$

ubi cognita quantitas ſecundi, ſub ſignis

(g) Hæc eſt primaria æquationum proprietas, quam omnibus generaliter ineſſe primus affirmare auſus eſt Albertus Girardus, cùm de illis, quarum radices omnes poſſibiles ac poſitivæ eſſent, prior pronuntiaverat Franciſcus Vieta. Harriottus autem veritatem hàc in re, quam primus inveniſſe dicitur atque demonſtraſſe, ne intellexit quidem, licèt ea ſcripſiſſet, ex quibus aliis intelligere facile eſſet, ſi non priùs patefacta eſſet à Girardo. Nimirum Harriottus æquationem quadraticam $aa \begin{array}{c}-b\\+c\end{array} a = bc$ procreari docet è multiplicatione æquationis ſimplicis, $a-b$, in quantitatem $a+c$; æquationem vero cubicam $aaa \begin{array}{c}-b\\+c\\+d\end{array} aa \begin{array}{c}-bc\\-bd\\+cd\end{array} a = bcd$ è multiplicatione æquationis ſimplicis $a-b=0$ in quantitatem $a+c$, æquationiſque quadraticæ, ex primâ illâ multiplicatione effectæ, in quantitatem $a+d$. Hæc ſanè ac plura hujuſcemodi exempla, quorum quidem affatim congeſſit, tantundem valent, ac ſi generaliter dixiſſet, coefficientes in æquationibus compoſitis ex quantitatibus illis, b, c, d, generari, quæ cum quantitate ignotâ ſignis, +, −, connexæ binomia conſtituunt, quorum multiplicatione continuâ æquatio quæque generata eſt; legemque præterea procreationis eam attuliſſet, ut quantitatum illarum ſumma, ſignis contrariis, coefficientem ſecundi membri æquationis efficeret: factorum è binis ſumma coefficientem tertii: factorum è ternis ſignis contrariis ſumma coefficientem quartæ, eodemque uſque modo. Hæc inquam, exempla illa Harriotto congeſta ſatis apertè ſignificant. Hæc ipſe Harriottus tam enuntiaviſſe cenſendus eſt, ac ſi verbis diſertiſſimè enuntiâſſet. Ex hiſce verò Girardi Theorema procudere Carteſio quidem facile fuiſſet. Quid dicam Carteſio, ſummo plane Arithmetico? Cùm idem vel è populo cuilibet licuiſſet, cui id modò perceptum eſſet, æquationum ex multiplicationibus provenientium quantitates negativas, $-c$, $-d$, æquè ac poſitivam b, habendas eſſe radices. Harriottus autem, cui id nunquam planè perceptum fuit, qui arctè adeo ad lumen connivebat, ut quantitates $-c$, $-d$, æquationum, quæ ex illis procreatæ eſſent, radices eſſe conſtantiſſimè negaret, qui fieri potuit ut ille intelligeret coefficientium geneſin è *radicum* multiplicatione? Girardi autem, ut ad illum redeam, hæc ſunt verba. "La premiere faction des ſolutions eſt eſgale au nombre du premier meſlé, la ſeconde faction des meſmes eſt eſgale au nombre du deuxieſme meſlé, la troiſieſme au troiſieſme, & tousjours ainſi, tellement que la derniere faction eſt eſgale a la fermeture, & ce ſelon les ſignes qui ſe peuvent remarquer en l' ordre alternatif."

Quæ

ſignis mutatis, nimirum $a+b-c$, eſt ſumma radicum, a, b & $-c$; cognita tertii, $ab-ac-bc$, ſumma rectangulorum ſub ſingulis binis, a & b, a & $-c$, b & $-c$; & cognita quarti, ſub ſigno mutato, $-abc$, illud unicum contentum eſt quod omnium continuâ multiplicatione generatur, a in b in $-c$. Adhæc, multiplicando cubicam illam æquationem per $x-d$, producetur hæcce quadrato-quadratica, ubi cognita quantitas ſecundi termini, ſub ſignis mutatis, $a+b-c+d$, eſt ſumma omnium radicum; ea tertii, $ab-ac-bc+ad+bd-cd$, ſumma rectangulorum ſub ſingulis binis; ea quarti, ſub ſignis mutatis, $-abc+abd-bcd-acd$, ſumma contentorum ſub ſingulis ternis; ea quinti, $-abcd$, contentum unicum ſub omnibus. Et hinc primò colligimus omnes æquationis cujuſcunque terminos nec fractos nec ſurdos habentis radices non ſurdas, & radicum binarum rectangula, ternarumque aut plurium. DE LIMITIBUS ÆQUAT.

$$x^4 \begin{matrix} -a \\ -b \\ +c \\ -d \end{matrix} x^3 \begin{matrix} +ab \\ -ac \\ -bc \\ +ad \\ +bd \\ -cd \end{matrix} xx \begin{matrix} +abc \\ -abd \\ +bcd \\ +acd \end{matrix} x - abcd = 0 :$$

Quæ Latinè ſic ſonant:

Radicum effectio prima coefficienti primi compoſiti æqualis eſt, effectio earundem ſecunda compoſiti ſecundi coefficienti, tertia tertii, atque ſic continuò donec ad effectionem ultimam perventum fuerit, quæ homogeneo comparationis eſt æqualis. Signa autem coefficientium ea adhibenda ſunt, quæ reperiuntur in ordine æquationis alterno.

Quòd ſi vocibus illis minùs uſitatis, *effectio*, *compoſitum*, *alternus ordo* (effection, meſlè, ordre alternatif) obſcuritas aliqua inſit, eam omnem amolientur auctoris definitiones. "Es equations meſlées la plus haute quantité eſt dite Maxime, ou haute extremité; celle qui eſt un degré plus bas, eſt dite premier meſlé; celle qui eſt encor un degre plus bas eſt dite ſecond meſlé, & ainſi conſequemment, tellement que le ⓞ eſt la fermeture ou baſſe extremité."

"Quant pluſieurs nombres ſont propoſez, la ſomme totale ſoit dite premiere faction; la ſomme de tous les produits de deux à deux ſoit dite deuxieſme faction: la ſomme de tous les produits de 3 à 3 ſoit dite la troiſieſme faction, & touſjours ainſi juſques à la fin; mais le produit de tous les nombre ſoit la derniere faction: or il y a autant de factions que de nombres propoſez."

"Ordre alterne des equations eſt quand les maxime ou haute extremité n'a autre nombre que l'unite avec le ſigne +, & que les denominateurs, ou caracteres, impairs ſont d'un coſté & les pairs de l'autre."

Quæ Latinè ſic dici poſſunt. "Æquationis cujuſvis compoſitæ membrum illud, quod ex quantitatis incognitæ poteſtate elatiſſimâ efficitur, maximum dicitur, ſive extremum ſuperius; quod ex poteſtate uno tantùm gradu inferiore, compoſitum primum; quod ex uno ruſum gradu inferiore, compoſitum ſecundum; eodemque ſimiliter progreſſu uſquedum ad membrum perventum fuerit incognitâ quantitate prorſus vacuum, quod *clauſula* dicatur, ſive extremum inferius. [Membrum illud ultimum quod *clauſulam* Girardus, ſive *poſitionem*, dixit, Vieta *homogeneum comparationis* appellabat.]

Quotcunque numeris propoſitis, ſumma quæ ex omnibus Algebraicè conficitur, effectio prima dicatur. Summa factorum è binis effectio ſecunda: ſumma factorum è ternis effectio tertia: ac ſimili deinceps modo. Factum verò ex omnibus dicatur effectio ultima. Cæterùm tot ſemper effectiones erunt quot numeri ex quibus procreatæ fuerint.

Alternus eſt æquationum ordo, cùm membra omnia, indicibus poteſtatum incognitæ imparibus prædita, ab alterâ parte conſtituantur, ab alterâ verò ſeorſim quot indicibus paribus gaudent: eâ verò lege, ut poteſtati elatiſſimæ ſignum, ſit +, neque coefficiens alius præter ipſam unitatem."

Id:

CAP. XVIII. rium contenta, eſſe aliquos ex diviſoribus integris ultimi termini; atque adeo ubi conſtiterit, nullum ultimi termini diviſorem eſſe aut radicem æquationis, aut duarum radicum rectangulum, pluriumve contentum, ſimul conſtabit, nullam eſſe radicem, radicumve rectangulum, aut contentum, niſi quod ſit ſurdum.

Ponamus jam cognitas quantitates terminorum æquationis ſub ſignis mutatis eſſe, p, q, r, s, t, v, &c. eam nempe ſecundi p, tertii q, quarti r, quinti s, & ſic deinceps. Et, ſignis terminorum probè obſervatis, ſiat $p=a$. $pa+2q=b$. $pb+qa+3r=c$. $pc+qb+ra+4s=d$. $pd+qc+rb+sa+5t=e$. $pe+qd+rc+sb+ta+6v=f$. & ſic in infinitum, obſervatâ ſerie progreſſionis. Et erit a ſumma radicum; b ſumma quadratorum ex ſingulis radicibus; c ſumma cuborum; d ſumma quadrato-quadratorum; e ſumma quadrato-cuborum; f ſumma cubo-cuborum; & ſic in reliquis ([h]). Ut in æquatione $x^4-x^3-19xx+49x-30=0$, ubi cognita quantitas ſecundi termini eſt -1, tertii -19, quarti $+49$, quinti -30; ponendum erit $1=p$, $19=q$, $-49=r$, $30=s$. Et inde orientur $a=(p=)\ 1$. $b=(pa+2q=1+38=)\ 39$. $c=(pb+qa+3r=39+19-147=)-89$. $d=(pc+qb+ra+4s=-89+741-49+120=)723$. Quare ſumma radicum erit 1; ſumma quadratorum radicum 39; ſumma cuborum -89; & ſumma quadrato-quadratorum 723. Nimirum æquationis illius radices ſunt, 1, 2, 3, & -5; & harum ſumma, $1+2+3-5$, eſt 1; ſumma quadratorum, $1+4+9+25$, eſt 39; ſumma cuborum, $1+8+27-125$, eſt -89; & ſumma quadrato-quadratorum, $1+16+81+625$, eſt 723.

Et hinc colliguntur *limites*, inter quos conſiſtent radices æquationis, ubi nulla earum impoſſibilis eſt. Nam cùm radicum omnium quadrata ſunt affirmativa, quadratorum ſumma affirmativa erit, ideoque quadrato maximæ radicis major. Et eodem argumento,

Id nimirum efficit illa alternatio, ut membri ſecundi, quarti, ſexti, aliorumque paribus in locis æquationis ordinatæ conſiſtentium, ſigna mutentur, reliquorum non mutatis.

([h]) Poteſtatum è radicibus incognitis conſummationem idem vir ille magnus Albertus Girardus omnium primus, credo, ante Newtonum aggreſſus eſt; ex cujus formulis, inter nova inventa Algebraica editis, Newtonianæ facilè deducendæ ſunt

Soit	A premier meſlé B ſecond C troiſieſme D quatrieſme &c.	ſit	A coefficiens primus; intellige membri ſecundi coefficientem B ſecundus C tertius D quartus &c.

Alors

argumento, ſumma quadrato-quadratorum radicum omnium major erit quàm quadrato-quadratum radicis maximæ; & ſumma cubo-cuborum major quàm cubo-cubus radicis maximæ. *De Limitibus Æquat.*

Regula I.

Quamobrem ſi limitem deſideres, quem radices nullæ tranſgrediuntur, quære ſummam quadratorum radicum, & extrahe ejus radicem quadraticam. Hæc enim radix major *erit quam radix maxima æquationis.* Sed ad radicem maximam propius accedes, ſi quæras *ſummam quadrato-quadratorum, & extrahas ejus radicem quadrato-quadraticam : & adbuc* magis, *ſi quæras ſummam cuborum, & extrahas ejus radicem cubo-cubicam : et ita in infinitum.*

Sic in æquatione præcedente, radix quadratica ſummæ quadratorum radicum, ſeu $\sqrt{39}$, eſt $6\frac{1}{2}$ quàm proximè ; & $6\frac{1}{2}$ magis diſtat à nihilo quam ulla radicum, 1, 2, 3, − 5. At radix quadrato-quadratica ſummæ quadrato-quadratorum radicum, nempe $\sqrt[4]{723}$, quæ eſt $5\frac{1}{2}$ circiter, propiùs accedit ad radicem à nihilo remotiſſimam, − 5.

Regula II.

Si inter ſummam quadratorum & ſummam quadrato-quadratorum radicum, inveniatur media proportionalis, erit ea paulo major quàm ſumma cuborum radicum ſub ſignis affirmativis connexorum. Et inde hujus mediæ proportionalis & ſummæ cuborum ſub propriis ſignis, ut priùs inventæ, ſemiſumma erit major, quàm ſumma cuborum radicum affirmativarum, & ſemidifferentia major quàm ſumma cuborum radicum negativarum.

Atque adeo maxima radicum affirmativarum minor erit quàm radix cubica illius ſemiſummæ, & maxima radicum negativarum minor quàm radix cubica illius ſemidifferentiæ.

Alors en toute ſorte d'equations	A	ſera la forme de	ſolutions
	Aq − B2		quarez
	A cub. − AB3 + C3		cubes
	Aqq − AqB4 + AC4 + Bq2 − D4		quare-quarez.

Tum in æquatione cujuſcunque formæ	A	ſumma erit	radicum
	$A^2 - 2B$		quadratorum è radicibus
	$A^3 - 3A \times B + 3C$		cuborum ex iiſdem
	$A^4 - 4B \times A^2 + 4A \times C + 2B^2 - 4D$		quadrato-quadratorum.

Sic

CAP. XVIII. Sic in æquatione præcedente, media proportionalis inter ſummam quadratorum radicum, 39, & ſummam quadrato-quadratorum, 723, eſt 168 circiter. Summa cuborum ſub propriis ſignis ſupra erat − 89. Hujus & 168 ſemiſumma eſt $39\frac{1}{2}$, ſemidifferentia $128\frac{1}{2}$. Prioris radix cubica, quæ eſt $3\frac{1}{2}$ circiter, major eſt quàm maxima radicum affirmativarum, 3. Poſterioris radix cubica, quæ eſt $5\frac{1}{21}$ proximè, tranſcendit radicem negativam, − 5. Quo exemplo videre eſt, quàm propè ad radicem hâc methodo acceditur, ubi unica tantùm radix negativa eſt, vel unica affirmativa.

R E G U L A III.

Et tamen propiùs adhuc accederetur; ſi inter ſummam quadrato-quadratorum radicum & ſummam cubo-cuborum media proportionalis inveniretur, atque ex hujus, & ſummæ quadrato-cuborum radicum ſemiſummâ, & ſemidifferentiâ, radices quadrato-cubicæ extraherentur.

Nam radix quadrato-cubica ſemiſummæ tranſcenderet maximam radicem affirmativam; & radix quadrato-cubica ſemidifferentiæ maximam ſeu extimam negativam, ſed exceſſu multo minore quàm antè. Cùm igitur radix quælibet, augendo vel diminuendo radices omnes fieri poteſt minima, dein minima in maximam converti, & poſtea omnes præter maximam fieri negativæ, conſtat quomodo radix imperata, quàm proximè, poteſt obtineri.

R E G U L A IV.

Si radices omnes præter duas negativæ ſunt, poſſunt illæ duæ ſimul hoc modo erui.

Inventâ juxta methodum præcedentem ſummâ cuborum duarum illarum radicum, ut & ſummâ quadrato-cuborum, & ſummâ quadrato-quadrato cuborum radicum omnium; inter poſteriores duas ſummas quære mediam proportionalem; & ea erit differentia inter ſummam cubo-cuborum radicum affirmativarum, & ſummam cubo-cuborum radicum negativarum, quàm proximè; adeoque hujus mediæ proportionalis, & ſummæ

 cubo-

cubo-cuborum radicum omnium ſemiſumma erit ſumma cubo- De Limiti-
cuborum radicum affirmativarum, & ſemidifferentia erit ſumma busÆquat.
cubo-cuborum radicum negativarum. Habitâ igitur tùm ſummâ cuborum, tum ſummâ cubo-cuborum radicum duarum affirmativarum, de duplo ſummæ poſterioris aufer quadratum ſummæ prioris, & reliqui radix quadratica erit differentia cuborum duarum radicum. Habitâ vero tum ſummâ tum differentiâ cuborum, habentur cubi ipſi. Extrahe eorum radices cubicas, & habebuntur æquationis radices duæ affirmativæ quàm proximè. Et ſi in altioribus poteſtatibus opus conſimile inſtitueretur, magis adhuc accederetur ad radices. Sed hæ limitationes, ob difficilem calculum, minùs uſui ſunt, & ad æquationes tantùm extendunt, quæ nullas habent radices imaginarias. Quapropter limites aliâ ratione invenire jam docebo, quæ & facilior ſit, & ad omnes æquationes extendat.

Regula V.

Multiplicetur æquationis terminus unuſquiſque per numerum dimenſionum ejus, & dividatur factum per radicem æquationis. Dein rurſus multiplicetur unuſquiſque terminorum prodeuntium per numerum unitate minorem quàm priùs; & factum dividatur per radicem æquationis. Et ſic pergatur, ſemper multiplicando per numeros unitate minores quàm priùs, & factum dividendo per radicem, donec tandem termini omnes deſtruantur, quorum ſigna diverſa ſunt à ſigno primi, ſeu altiſſimi termini, præter ultimum. Et numerus ille erit omni affirmativâ radice major; qui in terminis prodeuntibus ſcriptus pro radice, efficit, eorum, qui ſingulis vicibus per multiplicationem producebantur, aggregatum ejuſdem ſemper eſſe ſigni cum primo, ſeu altiſſimo, termino æquationis.

Ut ſi proponatur æquatio $x^5 - 2x^4 - 10x^3 + 30xx + 63x - 120 = 0$. Hanc primum ſic multiplico, $\overset{5}{x^5} - \overset{4}{2x^4} - \overset{3}{10x^3} + \overset{2}{30xx} + \overset{1}{63x} - \overset{0}{120}$. Dein terminos prodeuntes, diviſos per x, rurſum multiplico ſic, $\overset{4}{5x^4} - \overset{3}{8x^3} - \overset{2}{30xx} + \overset{1}{60x} + \overset{0}{63}$; & terminos prodeuntes rurſum dividendo per x prodeunt $20x^3 - 24xx - 60x + 60$; quos, minuendi gratiâ, divido per maximum diviſorem 4, & fiunt $5x^3 - 6xx - 15x + 15$.

Cap. XVIII. Hi itidem multiplicati per progreſſionem 3. 2. 1. 0, & diviſi per x, fiunt $15xx-12x-15$; & rurſum diviſi per 3 fiunt $5xx-4x-5$. Et hi multiplicati per progreſſionem 2. 1. 0, & diviſi per $2x$, fiunt $5x-2$. Jam cùm terminus æquationis altiſſimus, x^5, affirmativus ſit; tento, quinam numerus ſcriptus in his productis pro x efficiet ea omnia affirmativa eſſe. Et quidem tentando 1, fit $5x-2=3$, affirmativum; ſed $5xx-4x-5$, fit -4, negativum. Quare limes erit major quàm 1. Tento itaque numerum aliquem majorem, puta 2. Et in ſingulis ſubſtituendo 2 pro x, evadunt,

$$\begin{array}{l} 5x-2=8 \\ 5xx-4x-5=7 \\ 5x^3-6xx-15x+15=1 \\ 5x^4-8x^3-30xx+60x+63=79 \\ x^5-2x^4-10x^3+30xx+63x-120=46. \end{array}$$

Quare cùm numeri prodeuntes 8. 7. 1. 79. 46, ſint omnes affirmativi, erit numerus 2 major quàm radicum affirmativarum maxima. Similiter ſi limitem negativarum radicum invenire vellem, tento numeros negativos. Vel, quod perinde eſt, muto ſigna terminorum alternorum, & tento affirmativos. Mutatis autem terminorum alternorum ſignis, quantitates, in quibus numeri ſubſtituendi ſunt, fient

$$\begin{array}{l} 5x+2 \\ 5xx+4x-5 \\ 5x^3+6xx-15x-15 \\ 5x^4+8x^3-30xx-60x+63 \\ x^5+2x^4-10x^3-30xx+63x+120 \end{array}$$

Ex his ſeligo quantitatem aliquam, ubi termini negativi maximè prævalere videntur; puta $5x^4+8x^3-30xx-60x+63$: & hîc, ſubſtituendo pro x numeros 1 & 2, prodeunt numeri negativi -14 & -33. Unde limes erit major quàm -2. Subſtituendo autem numerum 3, prodit numerus affirmativus 234. Et ſimiliter in cæteris quantitatibus ſubſtituendo numerum 3 pro x prodit ſemper numerus affirmativus. Id quod ex inſpectione ſolâ colligere licet. Quare numerus -3 tranſcendit omnes radices negativas. Atque ita habentur limites 2 & -3, inter quos radices omnes conſiſtunt.

Horum verò limitum inventio uſui eſt, tum in reductione æquationum per radices rationales, tum in extractione radicum ſurdarum ex ipſis; ne fortè radicem extra hos limites aliquando quæramus. Sic in æquatione noviſſimâ, ſi radices rationales, ſiquas fortè habeat, invenire vellem; ex ſuperioribus certum eſt, has non alias eſſe poſſe, quàm diviſores ultimi termini æquationis, qui hìc eſt 120. Proin tentando omnes ejus diviſores, ſi nullus earum, ſcriptus in æquatione pro radice x, efficeret omnes

nes terminos evaneſcere; certum eſt æquationem non admittere radicem, niſi quæ ſit ſurda. At ultimi termini 120, diviſores permulti ſunt, nimirum 1. −1.2. −2.3. −3.4. −4.5. −5.6. −6.8. −8.10. −10.12. −12.15. −15.20. −20.24. −24.30. −30.40. −40.60. −60.120. & −120. Et hos omnes diviſores tentare, tædio eſſet. Cognito autem quòd radices inter limites 2 & −3 conſiſtunt, liberamur à tanto labore. Jam enim non opus erit, diviſores tentare, niſi qui ſunt inter hos limites; nimirum diviſores 1, −1, & −2. Nam ſi horum nullus radix eſt, certum eſt æquationem non habere radicem, niſi quæ ſit ſurda.

De Surdis Diviſoribus.

CAPUT XIX.

Æquationum reductio per diviſores ſurdos.

HActenus reductionem æquationum tradidi, quæ rationales diviſores admittunt. Sed antequam æquationem quatuor, ſex, aut plurium dimenſionum irreducibilem eſſe concludere poſſumus, tentandum erit etiam, annon per ſurdum aliquem diviſorem reduci queat; vel, quod perinde eſt, tentandum erit, annon æquatio ita in duas æquales partes dividi poſſit, ut ex utrâque radix extrahatur. Id autem fiet per ſequentem methodum.

Diſpone æquationem ſecundum dimenſiones literæ alicujus, ita ut omnes ejus termini ſub ſignis ſuis conjunctim æquales ſint nibilo, & terminus altiſſimus affirmativo ſigno afficiatur. Deinde, ſi æquatio quadratica *ſit, (nam & hunc caſum ob rei analogiam adjicere lubet) aufer utrobique terminum infimum, & adde quartam partem quadrati cognitæ quantitatis termini medii.*

Ut ſi æquatio ſit $xx-ax-b=0$, aufer utrobique $-b$, & adde $\frac{1}{4}aa$, & emerget $xx-ax+\frac{1}{4}aa=b+\frac{1}{4}aa$, *& extractâ utrobique radice, fiet* $x-\frac{1}{2}a=\pm\sqrt{b+\frac{1}{4}aa}$; ſive $x=\frac{1}{2}a\pm\sqrt{b+\frac{1}{4}aa}$.

Quòd ſi æquatio ſit quatuor dimenſionum, *ſit ea* $x^4+px^3+qxx+rx+s=0$, ubi p, q, r, & s, *denotant cognitas quantitates terminorum æquationis ſignis propriis adfectas. Fac* $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}\alpha p=\beta$. $s-\frac{1}{4}\alpha\alpha=\zeta$. *Dein pone pro n communem aliquem terminorum β & 2ζ divi-*

CAPUT XIX. *ſorem integrum, & non quadratum; qui & impar eſſe debet, & per 4 diviſus unitatem relinquere, ſi terminorum p & r alteruter ſit impar. Pone etiam pro k diviſorem aliquem quantitatis $\frac{\beta}{n}$, ſi p ſit par; vel imparis diviſoris dimidium, ſi p ſit impar; vel nihil, ſi dividuum β ſit nihil. Aufer Quotum de $\frac{1}{2}pk$, & reliqui dimidium dic l. Dein pro Q pone $\frac{\alpha+nkk}{2}$, & tenta ſi n dividat $QQ-s$, & Quoti radix ſit rationalis & æqualis l. Si hoc contigerit ad utramque partem æquationis adde $nkkxx+2nklx+nll$, & radicem extrahes utrobique; prodeunte $xx+\frac{1}{2}px+Q=\sqrt{n}$ in $kx+l$.*

Exempli gratiâ, proponatur æquatio $x^4+12x-17=0$; & quia p & q hîc deſunt, & r eſt 12, & s eſt -17, ſubſtitutis hiſce numeris fiet $\alpha=0$, $\beta=12$, & $\zeta=-17$; & ipſorum β & 2ζ, ſeu 12 & -34, communis diviſor unicus, nimirum 2, erit n. Porro $\frac{\beta}{n}$ eſt 6, & ejus diviſores 1, 2, 3, & 6 ſucceſſivè tentandi ſunt pro k, & -3, $-\frac{3}{2}$, -1, $-\frac{1}{2}$ pro l reſpectivè. Eſt autem $\frac{\alpha+nkk}{2}$, id eſt kk, æquale Q. Eſt & $\sqrt{\frac{QQ-s}{n}}$, id eſt, $\sqrt{\frac{QQ+17}{2}}=l$. Ubi numeri pares 2 & 6 ſcribuntur pro k, Q fit 4 & 36, & $QQ-s$ numerus erit impar, adeoque dividi non poteſt per n ſeu 2. Quare numeri illi 2 & 6 rejiciendi ſunt. Ubi verò 1 & 3 ſcribuntur pro k, Q fit 1 & 9, & $QQ-s$ fit 18 & 98, qui numeri dividi poſſunt per n, & quotorum radices extrahi. Sunt enim ± 3 & ± 7: quarum tamen ſola -3 congruit cum l. Pono itaque $k=1$, $l=-3$, & $Q=1$; & quantitatem $nkkxx+2nklx+nll$, id eſt $2xx-12x+18$, addo ad utramque partem æquationis; & prodit $x^4+2xx+1=2xx-12x+18$; & extractâ utrobique radice, $xx+1=x\sqrt{2}-3\sqrt{2}$. Quòd ſi radicis extractionem effugere malueris, pone $xx+\frac{1}{2}px+Q=\sqrt{n}\times\overline{kx+l}$, & invenietur, ut antè, $xx+1=\pm\sqrt{2}\times\overline{x-3}$. Et ex hâc æquatione, ſi radices iterum extrahas, proveniet $x=\pm\frac{1}{2}\sqrt{2}\pm\sqrt{\frac{-1}{2}\mp 3\sqrt{2}}$, *h. e.* ſecundùm ſignorum variationes, $x=-\frac{1}{2}\sqrt{2}+\sqrt{3\sqrt{2}-\frac{1}{2}}$, & $x=-\frac{1}{2}\sqrt{2}-\sqrt{3\sqrt{2}-\frac{1}{2}}$. Item $x=\frac{1}{2}\sqrt{2}+\sqrt{-3\sqrt{2}-\frac{1}{2}}$; & $x=\frac{1}{2}\sqrt{2}-\sqrt{-3\sqrt{2}-\frac{1}{2}}$. Quæ quidem quatuor ſunt radices æquationis ſub initio propoſitæ, $x^4+12x-17=0$. Sed earum ultimæ duæ ſunt impoſſibiles.

Proponamus

Proponamus jam æquationem $x^4-6x^3-58xx-114x-11=0$, & ſcribendo -6, -58, -114, & -11 pro p, q, r, & s reſpectivè, orietur $-67=\alpha$, $-315=\beta$, & $-1133\frac{1}{4}=\zeta$. Numerorum β & 2ζ, ſeu -315 & $-\frac{4533}{2}$, communis diviſor eſt unicus 3, adeoque hic erit n; & ipſius $\frac{\beta}{n}$, ſeu -105, diviſores ſunt 3, 5, 7, 15, 21, 35, & 105, qui itaque tentandi ſunt pro k. Quare tento primum 3, & quotum -35, qui prodit dividendo $\frac{\beta}{n}$ per k, ſeu -105 per 3, ſubduco de $\frac{1}{2}pk$, ſeu -3×3, & reſtat 26; cujus dimidium, 13, eſſe debet l. Sed $\frac{\alpha+nkk}{2}$, ſeu $\frac{-67+27}{2}$, id eſt -20, erit Q, & QQ$-s$ erit 411, qui dividi poteſt per n ſeu 3; ſed quoti 137 radix non poteſt extrahi. Quamobrem rejicio 3, & tento 5 pro k. Quotus qui jam prodit dividendo $\frac{\beta}{n}$ per k, ſeu -105 per 5, eſt -21, & hunc ſubducendo de $\frac{1}{2}pk$ ſeu -3×5, reſtat 6, cujus dimidium 3 erit l. Eſt & Q, ſeu $\frac{\alpha+nkk}{2}$, id eſt $\frac{-67+75}{2}$, numerus 4. Et QQ$-s$, ſeu $16+11$ dividi poteſt per n; & Quoti, qui eſt 9, radix extracta 3 congruit cum l. Quamobrem concludo eſſe $l=3$, $k=5$, Q$=4$, & $n=3$; & ſi $nkkxx+2nklx+nll$, id eſt $75xx+90x+27$ ad utramque partem æquationis addatur, radicem utrobique extrahi poſſe, & prodire $xx+\frac{1}{2}px+$Q$=\sqrt{n}\times\overline{kx+l}$; ſeu $xx-3x+4=\pm\sqrt{3}\times\overline{5x+3}$: & extractâ iterum radice $x=\frac{3\pm5\sqrt{3}}{2}\pm\sqrt{17\pm\frac{21\sqrt{3}}{2}}$. De Surdis Divisoribus.

Haud ſecus ſi proponatur æquatio hæcce, $x^4-9x^3+15xx-27x+9=0$, ſcribendo -9, $+15$, -27, & $+9$, pro p, q, r, & s reſpectivè, emerget $-5\frac{1}{4}=\alpha$, $-50\frac{5}{8}=\beta$, & $2\frac{7}{64}=\zeta$. Ipſorum β & 2ζ, ſeu $-\frac{405}{8}$ & $\frac{135}{32}$ communes diviſores ſunt 3, 5, 9, 15, 27, 45, & 135; ſed 9 quadratus eſt, & 3, 15, 27, 135 diviſi per numerum 4 non relinquunt unitatem, ut ob imparem terminum p oporteret. His itaque rejectis reſtant ſoli 5 & 45 tentandi pro n. Ponamus primò $n=5$, & ipſius $\frac{\beta}{n}$, ſeu $-\frac{81}{8}$, diviſores impares dimidiati, nempe $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, tentandi erunt pro k. Si k ponatur $\frac{1}{2}$, quotus $-\frac{81}{4}$, qui prodit dividendo $\frac{\beta}{n}$ per k, ſubductus de $\frac{1}{2}pk$, ſeu $-\frac{9}{4}$, relinquit 18 pro $2l$; & $\frac{\alpha+nkk}{2}$, ſeu -2 eſt, Q; &

QQ$-s$,

CAPUT XIX. $QQ - s$, ſeu -5, dividi quidem poteſt per n, ſeu 5; ſed Quoti negativi, -1, radix impoſſibilis eſt, quæ tamen deberet eſſe 9. Quare concludo k non eſſe $\frac{1}{2}$, & tento jam ſi ſit $\frac{3}{2}$. Quotum, qui oritur dividendo $\frac{\beta}{n}$ per k, ſeu $-\frac{81}{8}$ per $\frac{3}{2}$, nempe Quotum $-\frac{27}{4}$, ſubduco de $\frac{1}{2}pk$, ſeu $-\frac{27}{4}$, & reſtat o. Unde l jam nihil erit. Eſt autem $\frac{\alpha+nkk}{2}$, ſeu 3, æqualis Q, & $QQ - s$ nihil eſt; unde rurſus l, qui hujus $QQ - s$ diviſi per n radix eſt, invenitur nihil. Quamobrem, his ita quadrantibus, concludo eſſe $n=5$, $k=\frac{3}{2}$, $l=0$, & $Q=3$: adeoque addendo ad utramque partem æquationis propoſitæ terminos $nkkxx + 2nlkx + nll$ id eſt $\frac{45}{4}xx$, & radicem quadraticam utrobique extrahendo, prodire $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx+l}$; id eſt $xx - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

Eâdem methodo reducuntur etiam æquationes literales. Ut ſi fuerit $x^4 - 2ax^3 \begin{smallmatrix}+2aa\\-cc\end{smallmatrix} xx - 2a^3x + a^4 = 0$, ſubſtituendo $-2a$, $2aa - cc$, $-2a^3$ & $+a^4$ pro p, q, r, & s reſpectivè, obtinebuntur $aa - cc = \alpha$, $-acc - a^3 = \beta$, & $\frac{3}{4}a^4 + \frac{1}{2}aacc - \frac{1}{4}c^4 = \zeta$. Quantitatum β & 2ζ diviſor communis eſt $aa + cc$, qui proinde erit n; & $\frac{\beta}{n}$, ſeu $-a$, diviſores habet 1 & a. Sed quia n duarum eſt dimenſionum, & $k\sqrt{n}$ non niſi unius eſſe debet, ideo k nullius erit, adeoque non poteſt eſſe a. Sit ergo $k = 1$, & diviſo $\frac{\beta}{n}$ per k, aufer quotum $-a$ de $\frac{1}{2}pk$, ſeu $-a$, & reſtabit nihil pro l. Porro $\frac{\alpha+nkk}{2}$, ſeu aa, eſt Q, & $QQ - s$, ſeu $a^4 - a^4$, nihil eſt; & inde rurſus prodit nihil pro l. Quod arguit quantitates n, k, l, & Q rectè inventas eſſe; & additis ad utramque partem æquationis propoſitæ terminis $nkkxx + 2nklx + nll$, id eſt $aaxx + ccxx$, radicem utrobique extrahi poſſe, & extractione illâ prodire $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx+l}$; id eſt $xx - ax + aa = \pm x\sqrt{aa+cc}$. Et, extractâ iterum radice, $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{aa+cc} +$ vel $- \sqrt{\frac{1}{4}cc - \frac{1}{2}aa \pm \frac{1}{2}a\sqrt{aa+cc}}$.

Hactenus regulam applicui ad extractionem *radicum ſurdarum*: poteſt tamen eadem ad extractionem etiam *rationalium* applicari, ſi modò pro quantitate n uſurpetur unitas; eoque pacto unâ vice examinare poſſumus, utrùm æquatio, fractis & ſurdis terminis carens, diviſorem aliquem duarum dimenſionum, aut

4 rationalem

rationalem aut ſurdum, admittat. Ut ſi æquatio $x^4-x^3-5xx+12x-6=0$ proponatur, ſubſtituendo -1, -5, $+12$, & -6, pro p, q, r, & s reſpectivè invenientur $-5\frac{1}{4}=\alpha$, $9\frac{3}{8}=\beta$, & ponendo $n=1$, Quantitatis $\frac{\beta}{n}$, ſeu $\frac{75}{8}$, diviſores ſunt 1, 3, 5, 15, 25, 75: quorum dimidia (ſiquidem p ſit impar) tentanda ſunt pro k. Et ſi pro k tentemus $\frac{5}{2}$, fiet $\frac{1}{2}pk-\frac{\beta}{nk}=-5$, & ejus dimidium $-\frac{5}{2}=l$. Item $\frac{\alpha+nkk}{2}=\frac{1}{2}=Q$, & $\frac{QQ-s}{n}=6\frac{1}{4}$, cujus radix congruit cum l. De Surdis Diviſoribus.

Concludo itaque quantitates n, k, l, Q rectè inventas eſſe; & additis ad utramque partem æquationis terminis $nkkxx+2nklx+nll$, id eſt $6\frac{1}{4}xx-12\frac{1}{2}x+6\frac{1}{4}$, radicem utrobique extrahi poſſe; & extractione illâ prodire $xx+\frac{1}{2}px+Q=\pm\sqrt{n}\times\overline{kx+l}$: id eſt, $xx-\frac{1}{2}x+\frac{1}{2}=\pm1\times\overline{2\frac{1}{2}x-2\frac{1}{2}}$, ſeu $xx-3x+3=0$, & $xx+2x-2=0$; adeoque per haſce duas æquationes quadraticas, æquationem propoſitam quadrato-quadraticam dividi poſſe. Sed hujuſmodi diviſores rationales expeditiùs inveniuntur per aliam methodum ſuprà traditam.

Siquando quantitatis $\frac{\beta}{n}$ multi ſunt diviſores, ita ut omnes pro k tentare moleſtum fuerit, poteſt eorum numerus citò minui, quærendo omnes diviſores quantitatis $\alpha s-\frac{1}{4}rr$. Nam horum alicui, aut imparis alicujus dimidio, debet quantitas Q æqualis eſſe. Sic in exemplo noviſſimo $\alpha s-\frac{1}{4}rr$ eſt $-\frac{9}{2}$, è cujus diviſoribus 1, 3, 9, aut iiſdem dimidiatis, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, aliquis debet eſſe Q. Quare ſigillatim tentando quantitatis $\frac{\beta}{n}$ diviſores dimidiatos $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, & $\frac{75}{2}$ pro k, rejicio omnes qui non efficiunt $\frac{1}{2}\alpha+\frac{1}{2}nkk$, ſeu $-\frac{21}{8}+\frac{1}{2}kk$, id eſt Q, eſſe aliquem è numeris 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$. Scribendo autem $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, &c. pro k, prodeunt reſpectivè $-\frac{5}{2}$, $-\frac{3}{2}$, $+\frac{1}{2}$, $+\frac{51}{2}$, &c. pro Q, è quibus ſoli $-\frac{3}{2}$ & $\frac{1}{2}$ reperiuntur in prædictis numeris, 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, adeoque, cæteris rejectis, aut erit $k=\frac{3}{2}$, & $Q=-\frac{3}{2}$ aut $k=\frac{5}{2}$, & $Q=\frac{1}{2}$. Qui duo caſus examinentur. Atque hactenus de æquationibus *quatuor* dimenſionum.

Si æquatio ſex *dimenſionum reducenda eſt*, ſit ea $x^6+px^5+qx^4+rx^3+sxx+tx+v=0$, & fac $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}p\alpha=\beta$. $s-\frac{1}{2}p\beta=\gamma$.

$$\gamma-\frac{1}{4}\alpha\alpha=\zeta. \quad t-\frac{1}{2}\alpha\beta=\eta. \quad v-\frac{1}{4}\beta\beta=\vartheta.$$

$$\zeta\vartheta-\frac{1}{4}\eta\eta=\lambda.$$

Dein ſumatur pro n, communis aliquis terminorum 2ζ, η, 2ϑ, diviſor

 diviſor integer, & non quadratus, nec per numerum quadratum, diviſibilis, qui etiam per numerum 4 diviſus relinquit unitatem; ſi modò terminorum p, r, t aliquis ſit impar. Pro k ſumatur diviſor aliquis integer quantitatis $\frac{\lambda}{2nn}$, ſi p ſit par, vel diviſoris imparis dimidium, ſi p ſit impar, vel nihil ſi λ nihil ſit. Pro Q, quantitas $\frac{1}{2}\alpha+\frac{1}{2}nkk$. Pro l diviſor aliquis quantitatis $\frac{Qr-QQp-t}{n}$, ſi Q ſit integer; vel diviſoris imparis dimidium, ſi Q ſit fractus denominatorem habens numerum 2; vel nihil, ſi dividuum iſtud $\frac{Qr-QQp-t}{n}$ ſit nihil. Et pro R quantitas $\frac{1}{2}r-\frac{1}{2}Qp+nkl$. Dein tenta ſi $RR-v$ dividi poſſit per n, & Quoti radix extrahi; & præterea ſi radix iſta æqualis ſit tam quantitati $\frac{QR-\frac{1}{2}t}{nl}$ quàm quantitati $\frac{QQ+pR-nll-s}{2nk}$. Si hæc omnia evenerint, dic radicem illam m; & vice æquationis propoſitæ, ſcribe hanc $x^3+\frac{1}{2}pxx+Qx+R=\pm\sqrt{n}\times\overline{kxx+lx+m}$. Etenim hæc æquatio, quadrando partes & auferendo utrobique terminos ad dextram, producet æquationem propoſitam. Quòd ſi ea omnia in nullo caſu evenerint, reductio erit impoſſibilis, ſi modò priùs conſtet, æquationem per diviſorem rationalem reduci non poſſe.

Exempli gratiâ, proponatur æquatio $x^6-2ax^5+2bbx^4+2abbx^3 \begin{matrix}-2aabb\\+2a^3b\\-4ab^3\end{matrix}\,xx\begin{matrix}+3aab^4\\-a^4bb\end{matrix}=0$, & ſcribendo $-2a$, $+2bb$, $+2abb$, $-2aabb+2a^3b-4ab^3$, 0, & $3aab^4-a^4bb$ pro p, q, r, s, t, & v reſpectivè, prodibunt $2bb-aa=\alpha$. $4abb-a^3=\beta$. $2a^3b+2aabb-4ab^3-a^4=\gamma$. $-b^4+2a^3b+3aabb-4ab^3-\frac{5}{4}a^4=\zeta$. $-\frac{1}{2}a^5+3a^3bb-4ab^4=\eta$. & $-aab^4+a^4bb-\frac{1}{4}a^6=\theta$. Et terminorum 2ζ, η, & 2ϑ communis diviſor eſt $aa-2bb$, ſeu $2bb-aa$, perinde ut aa vel $2bb$ majus ſit. Sed eſto aa majus quàm $2bb$, & $aa-2bb$ erit n. Debet enim n ſemper affirmativum eſſe. Porro $\frac{\zeta}{n}$ eſt $-\frac{5}{4}aa+2ab+\frac{1}{2}bb$; $\frac{\eta}{n}$ eſt $-\frac{1}{2}a^3+2abb$; & $\frac{\theta}{n}$ eſt $-\frac{1}{4}a^4+\frac{1}{2}aabb$: adeoque $\frac{\zeta}{2n}\times\frac{\theta}{n}-\frac{\eta\eta}{8nn}$ ſeu, $\frac{\lambda}{2nn}$, eſt $\frac{1}{8}a^6-\frac{1}{4}a^5b-\frac{1}{8}a^4bb+\frac{1}{2}a^3b^3-\frac{3}{8}aab^4$; cujus diviſores ſunt 1, a, aa; ſed quia $\sqrt{n}\times k$ non niſi unius dimenſionis eſſe poteſt, & $\sqrt{n}$ unius eſt, ideo k nullius erit; proinde non niſi numerus eſſe poteſt. Quare rejectis a & aa, reſtat ſolum 1 pro k. Præterea $\frac{1}{2}\alpha+\frac{1}{2}nkk$ dat nihil pro Q, & $\frac{Qr-QQp-t}{n}$ etiam nihil eſt; adeoque l, qui ejus diviſor eſſe debet, erit

erit nihil. Denique $\frac{1}{2}r-pQ+nkl$ dat abb pro R. Et RR$-v$ eſt $-2aab^4+a^4bb$; quod dividi poteſt per n, ſeu $aa-2bb$, & quoti $aabb$ radix extrahi, & radix illa negativè ſumpta, nempe $-ab$, indefinitæ quantitati $\frac{QR-\frac{1}{2}t}{nl}$, ſeu $\frac{0}{0}$, non eſt inæqualis; quantitati verò definitæ $\frac{QQ+pR-nll-s}{2nk}$ æqualis eſt. Quamobrem radix illa $-ab$ erit m; &, loco æquationis propoſitæ, ſcribi poteſt $x^3+\frac{1}{2}pxx+Qx+R=\sqrt{n}\times\overline{kxx+lx+m}$, i. e. $x^3-axx+abb=\sqrt{aa-2bb}\times\overline{xx-ab}$. Cujus concluſionis veritatem probare potes, quadrando partes æquationis inventæ, & auferendo terminos ad dextram ex utrâque pârte. Eâ enim operatione producetur æquatio $x^6-2ax^5+2bbx^4+2abbx^3-2aabbxx+2a^3bxx-4ab^3xx+3aab^4-a^4bb=0$, quæ reducenda proponebatur.

De Surdis Diviſoribus.

Si æquatio eſt octo *dimenſionum, ſit ea* $x^8+px^7+qx^6+rx^5+sx^4+tx^3+vxx+wx+z=0$, & fiat $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}p\alpha=\beta$. $s-\frac{1}{2}p\beta-\frac{1}{2}\alpha\alpha=\gamma$. $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$. $v-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$. $w-\frac{1}{2}\beta\gamma=\zeta$. & $z-\frac{1}{4}\gamma\gamma=\eta$. Et terminorum 2δ, 2ε, 2ζ, 8η, quære communem diviſorem, qui integer ſit, & non quadratus, nec per quadratum diviſibilis; quique etiam per 4 diviſus relinquat unitatem, ſi modò terminorum alternorum p, r, t, w, aliquis ſit impar. Si nullus eſt ejuſmodi diviſor communis, certum eſt æquationem per extractionem ſurdæ radicis quadraticæ reduci non poſſe; & ſi non poteſt ea ita reduci, vix occurret illarum omnium quatuor quantitatum diviſor communis. Opuſculum igitur hactenus inſtitutum examinatio quædam eſt, utrùm æquatio reducibilis ſit necne, adeoque, cùm ejuſmodi reductiones raró poſſibiles ſint, finem operi ut plurimùm imponet.

Et ſimili ratione ſi æquatio ſit decem, duodecim, vel plurium dimenſionum, impoſſibilitas reductionis cognoſci poteſt.

Ut ſi ea ſit $x^{10}+px^9+qx^8+rx^7+sx^6+tx^5+vx^4+ax^3+bxx+cx+d=0$, faciendum erit $q-\frac{1}{4}pp=\alpha$, $r-\frac{1}{2}p\alpha=\beta$, $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$, $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$, $v-\frac{1}{2}p\delta-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$, $a-\frac{1}{2}\alpha\delta-\frac{1}{2}\beta\gamma=\zeta$, $b-\frac{1}{2}\beta\delta-\frac{1}{4}\gamma\gamma=\eta$, $c-\frac{1}{2}\gamma\delta=\theta$, $d-\frac{1}{4}\delta\delta=\varkappa$; & quærendus communis diviſor terminorum quinque 2ε, 2ζ, 8η, 4θ, $8\varkappa$, qui integer ſit & non quadratus; quique etiam per 4 diviſus relinquat unitatem, ſi modò terminorum alternorum p, r, t, a, c aliquis ſit impar.

CAPUT XIX. Sic ſi *duodecim* dimenſionum æquatio ſit $x^{12}+px^{11}+qx^{10}+rx^9+sx^8+tx^7+vx^6+ax^5+bx^4+cx^3+dxx+ex+f=0$, faciendum erit $q-\frac{1}{4}pp=\alpha$, $r-\frac{1}{2}p\alpha=\beta$, $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$, $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$, $v-\frac{1}{2}p\delta-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$, $a-\frac{1}{2}p\varepsilon-\frac{1}{2}\alpha\delta-\frac{1}{2}\beta\gamma=\zeta$, $b-\frac{1}{2}\alpha\varepsilon-\frac{1}{2}\beta\delta-\frac{1}{4}\gamma\gamma=\eta$, $c-\frac{1}{2}\beta\varepsilon-\frac{1}{2}\gamma\delta=\theta$, $d-\frac{1}{2}\gamma\varepsilon-\frac{1}{4}\delta\delta=\varkappa$, $e-\frac{1}{2}\delta\varepsilon=\lambda$, $f-\frac{1}{4}\varepsilon\varepsilon=\mu$; & quærendus communis diviſor integer, & non quadratus, terminorum ſex 2ζ, 8η, 4θ, $8\varkappa$, 4λ, 8μ, qui per 4 diviſus relinquat unitatem, ſi modò terminorum alternorum p, r, t, a, c, e aliquis ſit impar.

Atque ita in infinitum progredi licebit, & æquatio propoſita ſemper per extractionem ſurdæ radicis quadraticæ irreducibilis erit, ubi ejuſmodi diviſor communis nullus eſt. Siquando verò ejuſmodi diviſor n inventus ſpem faciat futuræ reductionis, poteſt ea inſtituti inſiſtendo veſtigiis operis, quod in æquatione octo dimenſionum ſubjungimus.

Quære numerum quadratum, cui, per n multiplicato, ultimus æquationis terminus z, ſub ſigno proprio adnexus, quadratum numerum efficit. Id autem expeditè fiet, ſi ad z, ubi n eſt par, vel ad $4z$ ubi n eſt impar, ſucceſſivè addantur n, $3n$, $5n$, $7n$, $9n$, $11n$, & deinceps, donec ſumma æqualis fiat numero alicui in tabulâ numerorum quadratorum, quam ad manus eſſe ſuppono. Et ſi nullus ejuſmodi quadratus numerus priùs occurrit, quàm ſummæ illius radix quadratica aucta radice quadraticâ exceſſûs illius ſummæ ſupra ultimum æquationis terminum, quadruplo major ſit quàm maximus terminorum æquationis propoſitæ p, q, r, s, t, v, &c. non opus erit rem ultrà tentare. Æquatio enim reduci non poteſt. Sed ſi ejuſmodi numerus quadratus priùs occurrit, ſit ejus radix s, ſi n eſt par, vel 2s ſi n eſt impar; & $\sqrt{\frac{ss-z}{n}}$ dic b. Debent autem s & b eſſe numeri integri, ſi n eſt par; at ſi n impar eſt, poſſunt eſſe fracti denominatorem habentes numerum binarium. Et ſi unus eorum fractus eſt, alter fractus eſſe debet. Quod idem de numeris R & m, Q & l, p & k, poſt inveniendis, obſervandum eſt. Et omnes numeri s & b, qui intra præfatum limitem inveniri poſſunt, in catalogum referendi ſunt.

Poſtea pro k tentandi ſunt omnes numeri ſucceſſivè, qui non efficiunt $nk\pm\frac{1}{2}p$ quadruplo majus quàm maximus terminus æquationis;

quationis; & ponendum eſt in omni caſu $\frac{nkk+\alpha}{2}=\mathrm{Q}$. Dein pro l tentandi ſunt ſucceſſivè numeri omnes, qui non efficiunt $nl \pm \mathrm{Q}$ quadruplo majus quàm maximus terminus æquationis: & in omni tentamine ponendum $\frac{-npkk+2\beta}{4}+nkl=\mathrm{R}$. Denique pro m tentandi ſunt ſucceſſivè omnes numeri, qui non efficiunt $nm \pm \mathrm{R}$ quadruplo majus quàm maximus terminorum æquationis; & videndum, an in caſu quovis, ſi fiat $s-\mathrm{QQ}-p\mathrm{R}+nll=2\mathrm{H}$, & $\mathrm{H}+nkm=\mathrm{S}$, ſit S aliquis numerorum, qui priùs pro S in catalogum relati erant; & præterea ſi alter numerus ei S reſpondens, qui pro b in eundem catalogum relatus erat, ſit his tribus $\frac{2\mathrm{RS}-w}{2nm}$, $\frac{2\mathrm{QS}+\mathrm{RR}-v-nmm}{2nl}$ & $\frac{p\mathrm{S}+2\mathrm{QR}-t-2nlm}{2nk}$ æqualis. Si hæc omnia in aliquo caſu evenerint, vice æquationis propoſitæ, ſcribenda erit hæcce $x^4+\frac{1}{2}px^3+\mathrm{Q}xx+\mathrm{R}x+\mathrm{S}=\sqrt{n}\times\overline{kx^3+lxx+mx+b}$.

De Surdis Divisoribus.

Exempli gratiâ proponatur æquatio $x^8+4x^7-x^6-10x^5+5x^4-5x^3-10xx-10x-5=0$. Et erit $q-\frac{1}{4}pp=-1-4=-5=\alpha$. $r-\frac{1}{2}p\alpha=-10+10=0=\beta$. $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=5-\frac{25}{4}=-\frac{5}{4}=\gamma$. $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=-5+\frac{5}{2}=-\frac{5}{2}=\delta$. $v-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=-10-\frac{25}{8}=-\frac{105}{8}$. $w-\frac{1}{2}\beta\gamma=-10=\zeta$. $z-\frac{1}{4}\gamma\gamma=-5-\frac{25}{64}=-\frac{345}{64}=\eta$. Ergo 2δ, 2ε, 2ζ, 8η, reſpectivè, ſunt -5, $-\frac{105}{4}$, -20, & $-\frac{345}{8}$, & earum diviſor communis 5; qui per 4 diviſus relinquit 1, perinde ut ob terminum imparem s oportuit. Cùm itaque inventus ſit diviſor communis n, ſeu 5, qui ſpem facit futuræ reductionis, quoniam iſte impar eſt, ad 4ε, ſeu -20, ſucceſſivè addo n, $3n$, $5n$, $7n$, $9n$, &c. ſeu 5, 15, 25, 35, 45, &c. & prodeunt -15. 0. 25. 60. 105. 160. 225. 300. 385. 480. 585. 700. 825. 960. 1105. 1260. 1425. 1600. Ex quibus ſolùm 0. 25. 225, & 1600 quadrati ſunt. Quare horum radices dimidiatæ 0, $\frac{5}{2}$, $\frac{15}{2}$, 20, in catalogum referendæ ſunt pro S, & $\sqrt{\frac{\mathrm{SS}-z}{n}}$, id eſt 1, $\frac{3}{2}$, $\frac{7}{2}$, 9, reſpectivè pro b. Sed quia $\mathrm{S}+nb$, ſi ſcribatur 20 pro S & 9 pro b, ſit 65, numerus major quadruplo maximi terminorum æquationis, ideo rejicio 20 & 9, & reliquos ſolùm refero in tabulam, ut ſequitur.

b	1. $\frac{3}{2}$. $\frac{7}{2}$.
S	0. $\frac{5}{2}$. $\frac{15}{2}$.

His ita diſpoſitis, tento pro k numeros omnes, qui non efficiunt $\frac{1}{2}p \pm nk$, ſeu $2 \pm 5k$, majus quadruplo maximi termini æquationis

CAPUT XIX. tionis 40; id eſt numeros, $-8.-7.-6.-5.-4.-3.-2.-1.0.1.2.3.4.5.6.7$, ponendo $\frac{nkk+\alpha}{2}$, ſeu $\frac{5kk-5}{2}$, id eſt numeros $\frac{315}{2}.120.\frac{175}{2}.60.\frac{75}{2}.20.\frac{15}{2}.0.-\frac{5}{2}.0.\frac{15}{2}.20.\frac{75}{2}.60.\frac{175}{2}.120.$ reſpectivè pro Q. Imo verò cum $Q\pm nl$, & multò magis Q, non debeat majus eſſe quàm 40, rejiciendos eſſe ſentio $\frac{315}{2}.120.\frac{175}{2}$ & 60, & qui his reſpondent $-8.-7.-6.-5.\ 5.6.7.$ adeoque ſolos $-4.-3.-2.-1.0.1.2.3.4$ pro k, & $\frac{75}{2}.20.\frac{15}{2}.0.-\frac{5}{2}.0.\frac{15}{2}.20.\frac{75}{2}$ pro Q reſpectivè tentandos. Tentemus autem -1 pro k, & 0 pro Q; & in hoc caſu, pro l tentandi deinceps erunt ſucceſſivè omnes numeri, qui non efficiunt $Q\pm nl$ majus quàm 40, id eſt omnes numeri inter 10 & -10, & pro R reſpectivè numeri $\frac{2\beta-npkk}{4}+nkl$, ſeu $-5-5l$ id eſt $-55.-50.-45.-40.-35.-30.-25.-20.-15.-10.-5.0.5.10.15.20.25.30.35.40.45$, quorum tamen tres priores & ultimum, quia majores quàm 40, negligere licebit. Tentemus autem -2 pro l & 5 pro R; & in hoc caſu pro m tentandi præterea erunt omnes numeri, qui non efficiunt $R\pm nm$, ſeu $5\pm 5m$, majus quàm 40; id eſt numeri omnes inter 7 & -9; & videndum, an ſi ponendo $s-QQ-pR+nll$, id eſt $5-20+20$, ſeu $5,=2H$, ſit $H+nkm$, ſeu $\frac{5}{2}-5m,=S$: id eſt, ſi ex his numeris $\frac{-65}{2}.\frac{-55}{2}.\frac{-45}{2}.\frac{-35}{2}.\frac{-25}{2}.\frac{-15}{2}.\frac{-5}{2}.\frac{5}{2}.\frac{15}{2}.\frac{25}{2}.\frac{35}{2}.\frac{45}{2}.\frac{55}{2}.\frac{65}{2}\ \frac{75}{2}.\frac{85}{2}$, aliquis æqualis ſit alicui numerorum $0.\pm\frac{5}{2}.\pm\frac{15}{2}$, qui priùs in tabulam pro S relati erant. Et hujuſmodi quatuor occurrunt, $-\frac{15}{2}.-\frac{5}{2}.\frac{5}{2}.\frac{15}{2}$, quibus reſpondent $\pm\frac{7}{2}.\pm\frac{3}{2}.\pm\frac{3}{2}.\pm\frac{7}{2}$ pro b in eâdem tabulâ ſcripti, ut & $2.1.0.-1$ pro m ſubſtituti. Verùm tentemus $-\frac{5}{2}$ pro S, 1 pro m, & $+\frac{3}{2}$ pro b; & fiet $\frac{2RS-v}{2nm}=\frac{-25+10}{10}=-\frac{3}{2}$, & $\frac{2QS+RR-\varsigma-nmm}{2nl}=\frac{25+10-5}{-20}=-\frac{3}{2}$, & $\frac{pS+2QR-t-2nlm}{2nk}=\frac{-10+5+20}{-10}=-\frac{3}{2}$. Quare cùm prodeat omni caſu $-\frac{3}{2}$ ſeu b, concludo numeros omnes rectè inventos eſſe, adeoque vice æquationis propoſitæ ſcribendum eſſe $x^4+\frac{1}{2}px^3+Qxx+Rx+S=\sqrt{n}\times\overline{kx^3+lxx+mx+b}$, id eſt $x^4+2x^3+5x-2\frac{1}{2}=\sqrt{5}\times\overline{-x^3-2xx+x-1\frac{1}{2}}$. Etenim quadrando partes hujus, producetur æquatio illa octo dimenſionum, quæ ſub initio proponebatur.

Quòd ſi tentando caſus omnes numerorum, prædicti valores omnes ipſius b nullo in caſu inter ſe conſenſiſſent, argumento fuiſſet, æquationem, per extractionem ſurdæ radicis quadraticæ, reduci non potuiſſe.

Deberent

Deberent autem aliqua hîc in operis abbreviationem annotari; ſed quæ brevitatis cauſâ prætereo, cùm tantarum reductionum perexiguus ſit uſus, & rei poſſibilitatem potiùs quàm praxin commodiſſimam voluerim exponere. Sunt igitur hæ reductiones æquationum per extractionem *ſurdæ radicis quadraticæ*. De Surdis Diviſoribus.

Adjungere jam liceret reductiones æquationum per *extractionem ſurdæ radicis cubicæ*, ſed & has, ut quæ perrarò utiles ſint, brevitatis gratiâ prætereo.

Sunt tamen reductiones quædam *cubicarum æquationum* vulgò notæ, quas, ſi penitus præterirem, Lector fortaſſe deſideraret. Proponatur æquatio cubica $x^3 * + qx + r = 0$, cujus ſecundus terminus deeſt. Ad hanc enim formam æquationem omnem cubicam reduci poſſe conſtat ex precedentibus. Et ſupponatur x eſſe $= a + b$. Erit $a^3 + 3aab + 3abb + b^3$ (id eſt x^3) $+ qx + r = 0$. Sit $3aab + 3abb$ (id eſt $3abx$) $+ qx = 0$; & erit $a^3 + b^3 + r = 0$. Per priorem æquationem eſt $b = -\frac{q}{3a}$, & cubicè, $b^3 = -\frac{q^3}{27a^3}$. Ergo, per poſteriorem, eſt $a^3 - \frac{q^3}{27a^3} + r = 0$, ſeu $a^6 + ra^3 = \frac{q^3}{27}$; & per extractionem affectæ radicis quadraticæ $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Extrahe radicem cubicam, & habebitur a. Et ſupra erat $-\frac{q}{3a} = b$, & $a + b = x$. Ergo $a - \frac{q}{3a}$ radix eſt æquationis propoſitæ.

Exempli gratiâ, proponatur æquatio $y^3 - 6yy + 6y + 12 = 0$. Ad tollendum ſecundum æquationis hujus terminum, ponatur $x + 2 = y$, & orietur $x^3 * - 6x + 8 = 0$; ubi eſt $q = -6$; $r = 8$; $\frac{1}{4}rr = 16$; $\frac{q^3}{27} = -8$; $a^3 = -4 \pm \sqrt{8}$; $a - \frac{q}{3a} = x$; & $x + 2 = y$; id eſt $2 + \sqrt[3]{-4 \pm \sqrt{8}} + \frac{2}{\sqrt[3]{-4 \pm \sqrt{8}}} = y$.

Et hoc modo erui poſſunt radices omnium cubicarum æquationum, ubi q affirmativum eſt; vel etiam ubi q negativum eſt, & $\frac{q^3}{27}$ non majus quàm $\frac{1}{4}rr$; *id eſt ubi duæ ex radicibus æquationis ſunt impoſſibiles*. At ubi q negativum eſt, & $\frac{q^3}{27}$ ſimul majus quàm $\frac{1}{4}rr$, fit $\sqrt{\frac{1}{4}rr - \frac{q^3}{27}}$ quantitas impoſſibilis; atque adeo æquationis radix x vel y, hoc caſu impoſſibilis erit. Scilicet hoc caſu *tres ſunt radices poſſibiles*, quæ omnes eodem modo ſe habent ad

æquationis

CAPUT XIX. æquationis terminos q & r, & indifferenter deſignantur per literam x vel y, adeoque omnes eâdem deberent lege erui & exprimi, quâ una aliqua eruitur & exprimitur: *ſed omnes tres lege præfatâ exprimere impoſſibile eſt.* Quantitas $a - \frac{q}{3a}$, quâ x deſignatur, multiplex eſſe non poteſt, eâque de cauſâ Hypotheſis quòd x, hoc in caſu ubi triplex eſt, æqualis eſſe poteſt binomio $a - \frac{q}{3a}$, ſeu $a + b$, cujus nominum cubi, $a^3 + b^3$, conjunctim æquentur r, & triplum rectangulum $3ab$ æquetur q, plane impoſſibilis eſt; & ex hypotheſi impoſſibili concluſionem impoſſibilem colligi, mirum eſſe non debet.

Eſt & alius modus has radices exprimendi. Nimirum de $a^3 + b^3 + r$ id eſt de nihilo, aufer $a^3 + r$, ſeu $\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$, & reſtabit $b^3 = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Eſt itaque $a = \sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, & $b = \sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$; vel $a = \sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, & $b = \sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$; adeoque horum ſumma $\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$ erit $= x$.

Poſſunt etiam æquationum biquadraticarum radices mediantibus cubicis erui & exprimi.

Tollendus eſt autem primùm ſecundus æquationis terminus. Sit æquatio reſultans $x^4 + qxx + rx + s = 0$. Pone hanc multiplicatione duarum, $xx + ex + f = 0$, & $xx - ex + g = 0$, generari; id eſt eandem eſſe cum hâc, $x^4 * \begin{matrix} +f \\ +g \\ -ee \end{matrix} xx \begin{matrix} +eg \\ -ef \end{matrix} x + fg = 0$; &, collatis terminis, fiet $f + g - ee = q$; $eg - ef = r$; & $fg = s$. Quare $q + ee = f + g$; $\frac{r}{e} = g - f$; $\frac{q + ee + \frac{r}{e}}{2} = g$; $\frac{q + ee - \frac{r}{e}}{2} = f$; $\frac{qq + 2eeq + e^4 - \frac{rr}{ee}}{4} (= fg) = s$; &, per reductionem, $e^6 + 2qe^4 \begin{matrix} +qq \\ -4s \end{matrix} ee - rr = 0$. Pro ee ſcribe y; & fiet $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, æquatio cubica, cujus terminus ſecundus tolli poteſt, & radix deinceps per regulam præcedentem, vel ſecus, extrahi. Dein, habitâ illâ radice, regrediendum erit, ponendo $\sqrt{y} = e$; $\frac{q + ee - \frac{r}{e}}{2} = f$; $\frac{q + ee + \frac{r}{e}}{2} = g$; & æquationes duæ, $xx + ex + f = 0$, & $xx - ex + g = 0$,

$+g=0$, extractis earum radicibus, dabunt quatuor radices æqua- De Surdis Diviſoribus.
tionis biquadraticæ, $x^4+qxx+rx+s=0$; nimirum $x=-\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee-f}$, & $x=\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee-g}$. Ubi notandum eſt, quòd ſi æquationis biquadraticæ radices quatuor poſſibiles ſunt, æquationis cubicæ, $y^3+2qyy\,{+qq \atop -4s}\,y-rr=0$, radices tres poſſibiles erunt; atque adeo per regulam præcedentem extrahi nequeunt. *Sic & ſi æquationis quinque, vel plurium, dimenſionum radices affectæ in radices non affectas, mediis æquationis terminis quoque pacto ſublatis, convertantur; illa radicum expreſſio ſemper erit impoſſibilis, ubi plures quàm una radix, in æquatione imparium dimenſionum, poſſibiles ſunt, aut plures quàm duæ, in æquatione parium dimenſionum, quæ per extractionem ſurdæ radicis quadraticæ, methodo ſuprà expoſitâ, reduci nequeunt.*

Docuit *Carteſius* æquationem biquadraticam per regulas ultimò traditas reducere. *E. g.* proponatur æquatio à nobis ſuprà reducta, $x^4-x^3-5xx+12x-6=0$. Tolle ſecundum terminum, ſcribendo $v+\frac{1}{4}$ pro x, & orietur $v^4-\frac{43}{8}vv+\frac{75}{8}v-\frac{851}{256}=0$. Ad tollendas fractiones ſcribe $\frac{1}{4}z$ pro v, & orietur $z^4-86zz+600z-851=0$. Hìc eſt $-86=q$; $600=r$; & $-851=s$; adeoque $y^3+2qyy\,{+qq \atop -4s}\,y-rr=0$, ſubſtitutis æquipollentibus, fiet $y^3-172yy+10800y-360000=0$. Ubi tentando omnes ultimi termini diviſores 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, & deinceps uſque ad 100, invenietur tandem $y=100$. Quod idem multo expeditiùs, per methodum à nobis ſuprà expoſitam, inveniri potuit. Dein habito y, radix ejus 10 erit e, & $\frac{q+ee-\frac{r}{e}}{2}$, id eſt $\frac{-86+100-60}{2}$, ſeu -23, erit f; & $\frac{q+ee+\frac{r}{e}}{2}$, ſeu 37, erit g; adeoque æquationes, $xx+ex+f=0$, & $xx-ex+g=0$, ſcripto z pro x, & ſubſtitutis æquipollentibus, evadent $zz+10z-23=0$, & $zz-10z+37=0$. Reſtitue v pro $\frac{1}{4}z$, & orientur $vv+2\frac{1}{2}v-\frac{23}{16}=0$, & $vv-2\frac{1}{2}v+\frac{37}{16}=0$ Reſtitue inſuper $x-\frac{1}{4}$ pro v, & emergent $xx+2x-2=0$, & $xx-3x+3=0$; æquationes duæ, quarum radices quatuor, $x=-1\pm\sqrt{3}$, & $x=1\frac{1}{2}\pm\sqrt{-\frac{3}{4}}$, eædem ſunt cum radicibus quatuor æquationis biquadraticæ ſub initio propoſitæ, $x^4-x^3-5xx+12x-6=0$. Sed hæ faciliùs, per methodum inveniendi diviſores à nobis ſuprà explicatam, inveniri potuerunt.

ÆQUA-

APPENDIX.

ÆQUATIONUM
CONSTRUCTIO LINEARIS.

HActenus æquationum proprietates, tranſmutationes, limites & omnis generis reductiones, docui. Demonſtrationes non ſemper adjunxi, quoniam ſatis faciles mihi viſæ ſunt, & nonnunquam, abſque nimiis ambagibus, tradi non poſſent. Reſtat jam tantùm, ut æquationum, poſtquam ad formam commodiſſimam reductæ ſunt, radices in numeris extrahere doceam. Et hîc præcipua difficultas eſt, in figuris duabus vel tribus prioribus obtinendis. Id quod commodiſſimè per æquationis conſtructionem aliquam, ſeu Geometricam, ſive Mechanicam, conſit. Quâ de cauſâ non pigebit hujuſmodi conſtructiones aliquas ſubjungere.

Veteres, ut ex Pappo diſcimus, triſectionem anguli, & inventionem duarum mediè proportionalium, ſub initio per rectam lineam & circulum, fruſtrà aggreſſi ſunt. Poſtea conſiderare cœperunt alias permultas lineas, ut Conchoidem, Ciſſoidem, & Conicas ſectiones, & per harum aliquas ſolverunt Problemata. Tandem re penitius examinatâ, & Conicis ſectionibus in Geometriam receptis, Problemata diſtinxerunt in tria genera: *Plana* quæ per lineas ortum à ſolidi, id eſt Coni, conſideratione derivantes, ſolvebantur; & *Linearia*, ad quorum ſolutionem requirebantur lineæ magis compoſitæ. Et juxta hanc diſtinctionem, problemata ſolida per alias lineas quàm Conicas ſectiones ſolvere à Geometriâ alienum eſt; præſertim ſi nullæ aliæ lineæ præter rectam, circulum, & Conicas ſectiones in Geometriam recipiantur. At Recentiores longiùs progreſſi receperunt lineas omnes in Geometriam, quæ per æquationes exprimi poſſunt, & pro dimenſionibus æquationum diſtinxerunt lineas illas in genera, legemque tulerunt, non licere Problema per lineam ſuperioris generis conſtruere, quod conſtrui poteſt per lineam inferioris. In lineis contemplandis, & eruendis earum proprietatibus, diſtinctionem earum in genera, juxta dimenſiones æquationum, per

quas definiuntur, laudo. At æquatio non eſt, ſed deſcriptio, quæ curvam Geometricam efficit. Circulus linea Geometrica eſt, non quòd per æquationem exprimi poteſt; ſed quòd deſcriptio ejus poſtulatur. Æquationis ſimplicitas non eſt, ſed deſcriptionis facilitas, quæ lineam ad conſtructiones Problematum priùs admittendam eſſe indicat. Nam æquatio ad Parabolam ſimplicior eſt, quàm æquatio ad circulum; & tamen circulus, ob ſimpliciorem deſcriptionem, priùs admittitur. Circulus & Coni ſectiones, ſi æquationum dimenſiones ſpectentur, ejuſdem ſunt ordinis; & tamen circulus, in conſtructione problematum, non connumeratur cum his, ſed ob ſimplicem deſcriptionem deprimitur ad ordinem inferiorem lineæ rectæ; ita ut per circulum conſtruere, quod per rectas conſtrui poteſt, non ſit illicitum; per Conicas verò ſectiones conſtruere, quod per circulum conſtrui poteſt, vitio vertatur. Aut igitur legem à dimenſionibus æquationum in circulo obſervandam eſſe ſtatue, & ſic diſtinctionem inter problemata plana & ſolida ut vitioſam tolle; aut concede, legem illam in lineis ſuperiorem generum non ita obſervandam eſſe, quin aliquæ, ob ſimpliciorem deſcriptionem, præferantur aliis ejuſdem ordinis, &, in conſtructione Problematum, cum lineis inferiorum ordinum connumerentur. In conſtructionibus quæ ſunt æquè Geometricæ, præferendæ ſemper ſunt ſimpliciores. Hæc lex omni exceptione major eſt. Ad ſimplicitatem verò conſtructionis expreſſiones Algebraicæ nil conferunt. Solæ deſcriptiones linearum hìc in cenſum veniunt. Has ſolas conſiderabant Geometræ, qui circulum conjungebant cum rectâ. Prout hæ ſunt faciles, vel difficiles, conſtructio facilis vel difficilis redditur. Adeòque à rei naturâ alienum eſt, leges conſtructionibus aliunde præſcribere. Aut igitur lineas omnes præter rectam & circulum, & fortè Conicas ſectiones, è Geometriâ cum Veteribus excludamus, aut admittamus omnes ſecundum deſcriptionis ſimplicitatem. Si Trochoides in Geometriam reciperetur, liceret ejus beneficio angulum in datâ ratione ſecare. Numquid ergo reprehenderes, ſiquis hâc lineâ ad dividendum angulum in ratione numeri ad numerum uteretur, & contenderes, hanc lineam per æquationem non definiri, lineas verò quæ per æquationes definiuntur adhibendas eſſe? Igitur ſi an-

Æquationum Conſtructiones.

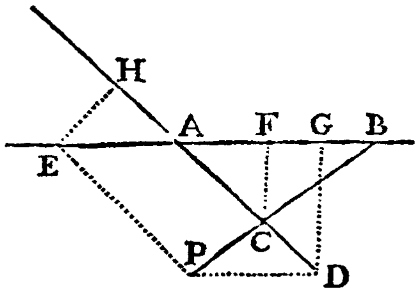
Appendix. gulus *e.g.* in 10001 partes dividendus eſſet, teneremur curvam lineam æquatione pluſquam centum dimenſionum definitam in medium afferre, quam tamen nemo mortalium deſcribere, nedum intelligere, valeret; & hanc anteponere Trochoidi, quæ linea notiſſima eſt, & per motum rotæ, vel circuli, facillimè deſcribitur. Quod quàm abſurdum ſit quis non videt? Aut igitur Trochoides in Geometriam non eſt admittenda, aut, in conſtructione Problematum, curvis omnibus difficilioris deſcriptionis anteferenda. Et eadem eſt ratio de reliquis curvis. Quo nomine triſectiones anguli per Conchoidem, quas *Archimedes* in Lemmatis & *Pappus* in collectionibus poſuere, præ aliorum hâc de re inventis omnibus laudamus; ſiquidem lineas omnes præter rectam & circulum è Geometriâ excludere debeamus, aut ſecundum deſcriptionis ſimplicitatem admittere, & Conchoides, ſimplicitate deſcriptionis, nulli curvæ præter circulum cedit. Æquationes ſunt expreſſiones computi Arithmetici, & in Geometriâ locum propriè non habent, niſi quatenus quantitates verè Geometricæ (id eſt lineæ, ſuperficies, ſolida & proportiones) aliquæ aliis æquales enunciantur. Multiplicationes, Diviſiones, & ejuſmodi computa in Geometriam recèns introducta ſunt; idque inconſultò, & contra primum inſtitutum ſcientiæ hujus. Nam qui conſtructiones Problematum per rectam & circulum, à primis Geometris adinventas, conſiderabit, facilè ſentiet, Geometriam excogitatam eſſe, ut expedito linearum ductu effugeremus computandi tædium. Proinde hæ duæ ſcientiæ confundi non debent. Veteres tam ſedulò diſtinguebant eas ab invicem, ut in Geometriam terminos Arithmeticos nunquam introduxerint. Et recentes, utramque confundendo, amiſerunt ſimplicitatem, in quâ Geometriæ elegantia omnis conſiſtit. Eſt itaque *Arithmeticè* quidem ſimplicius quod per ſimpliciores æquationes determinantur; at *Geometricè* ſimplicius eſt, quod per ſimpliciorem ductum linearum colligitur; & in Geometriâ prius & præſtantius eſſe debet, quod eſt ratione Geometricâ ſimplicius. Mihi igitur vitio vertendum non erit, ſi cum Mathematicorum Principe *Archimede*, aliiſque Veteribus, Conchoidem ad Solidorum problematum conſtructionem adhibeam. Attamen ſiquis aliter ſenſerit, ſciat me hic de conſtructione non Geometricâ, ſed qualicunque, ſollicitum eſſe, quâ

 radices

radices æquationum in numeris proximè aſſequar. Cujus rei gratià præmitto hoc Problema Lemmaticum. ÆQUATIONES CONSTRUCTIONES.

Inter datas duas lineas, AB, AC, *rectam datæ longitudinis*, BC, *ponere quæ producta tranſeat per datum punctum* P.

SI circa polum P gyret linea BC, & ſimul termino ejus C incedat ſuper rectâ AC, ejus alter terminus B deſcribet Conchoidem Veterum. Secet hæc lineam AB in puncto B. Junge PB, & ejus pars BC erit recta quam ducere oportuit. Et eâdem lege linea BC duci poteſt, ubi vice rectæ AC linea aliqua curva adhibetur.

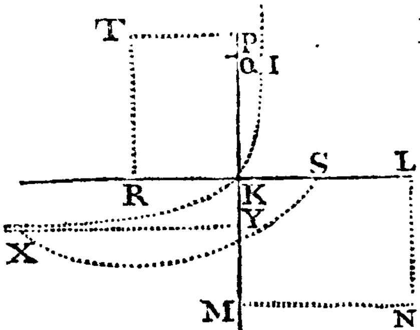


Sicui conſtructio hæcce per Conchoidem minùs placeat, poteſt alia per Conicam ſectionem ejus vice ſubſtitui. A puncto P ad rectas AD, AE age PD, PE conſtituentes parallelogrammum EADP, & à punctis C ac D ad rectam AB demitte perpendicula CF, DG, ut & à puncto E ad rectam AC verſus A productam perpendiculum EH; & dictis AD $= a$. PD $= b$. BC $= c$. AG $= d$. AB $= x$, & AC $= y$. Erit AD. AG :: AC. AF, adeoque AF $= \frac{dy}{a}$. Erit & AB.AC :: PD.CD, ſeu $x.y :: b.a-y$. Ergo $by = ax - yx$, quæ æquatio eſt ad Hyperbolam. Rurſus per 13. II. *Elem.* erit BCq $=$ ACq $+$ ABq $-$ 2FAB, id eſt $cc = yy + xx - \frac{2dxy}{a}$. Prioris æquationis partes ductas in $\frac{2d}{a}$ aufer de partibus hujus, & reſtabit $cc - \frac{2bdy}{a} = yy + xx - 2dx$, æquatio ad circulum, ubi x & y ad rectos ſunt angulos. Quare ſi haſce duas lineas, Hyperbolam & Circulum, ope harum æquationum componas, earum interſectione habebis x & y, ſeu AB & AC, quæ poſitionem rectæ BC determinant. Componentur autem lineæ illæ ad hunc modum.

Duc rectas duas quaſvis KL æqualem AD, & KM æqualem PD continentes angulum rectum MKL. Comple parallelogrammum KLMN, & aſymptotis LN, MN per punctum K deſcribe Hyperbolam IKX.

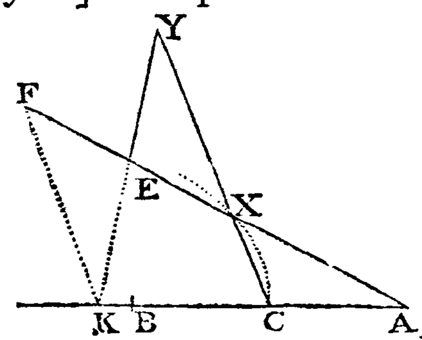
APPENDIX.

In KM verſus K productâ cape KP æqualem AG & KQ æqualem BC. Et in KL productâ verſus K cape KR æqualem AH, & RS æqualem RQ. Comple parallelogrammum PKRT, & centro T intervallo TS deſcribe circulum. Secet hic Hyperbolam in puncto X. Ad KP demitte perpendiculum XY, & erit XY æqualis AC & KY æqualis AB. Quæ duæ lineæ AC & AB, vel una earum cum



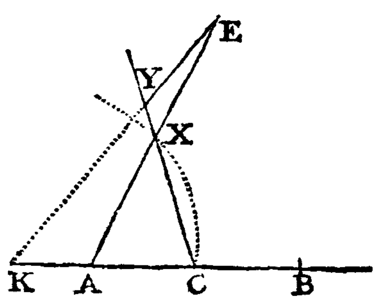
puncto P, determinant poſitionem quæſitam rectæ BC. Cui conſtructioni demonſtrandæ, & ejus caſibus ſecundum caſus Problematis determinandis non immoror.

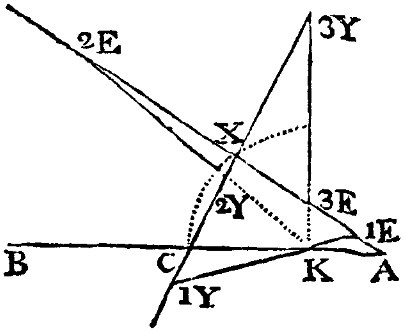
Hâc, inquam, conſtructione ſolvi poteſt Problema ſicui ita viſum ſit. Sed hæc ſolutio magis compoſita eſt quàm ut uſibus ullis inſervire poſſit. Nuda ſpeculatio eſt, & ſpeculationes Geometricæ tantum habent elegantiæ quantum ſimplicitatis, tantumque laudis merentur quantum utilitatis ſecum afferunt. Eâ de cauſâ conſtructionem per Conchoidem præfero ut multo ſimpliciorem, & non minùs Geometricam; & quæ reſolutioni æquationum à nobis propoſitæ optimè conducit. Præmiſſo igitur præcedente Lemmate conſtruimus Geometricè Problemata cubica, & quadrato-quadratica [*utpote quæ ad cubica reduci poſſunt*] ut ſequitur.



Proponatur æquatio cubica $x^3 * + qx + r = 0$. *cujus terminus ſecundus deeſt, tertius verò ſub ſigno ſuo deſignatur per* + q & *quartus per* + r.

Duc quamlibet KA quam dic n. In KA utrinque productâ cape $KB = \frac{q}{n}$, ad eaſdem partes cum KA ſi habeatur $+q$, aliter ad contrarias. Biſeca BA in C, & centro K radio KC fac circulum CX, cui inſcribe rectam CX æqualem $\frac{r}{nn}$, & produc eam utrinque. Dein junge AX, & produc eam utrinque. Denique inter has lineas CX & AX inſcribe EY ejuſdem longitudinis





gitudinis cum CA, quæque producta tranſeat per punctum K, & XY erit radix æquationis. Et ex his radicibus affirmativæ erunt quæ cadunt ad partes X verſus C, & negativæ quæ cadunt ad partes contrarias, ſi habeatur $+r$, & contrò ſi habeatur $-r$. ÆQUATIONUM CONSTRUCTIONES.

Demonſtratio.

Ad demonſtrationem præmittimus Lemmata ſequentia.

LEM. I. *Eſt* YX *ad* AK *ut* CX *ad* KE. Etenim age KF parallelam CX, & ob ſimilia triangula ACX, AKF, & EYX, EKF, erit AC ad AK ut CX ad KF; & YX ad YE, ſeu AC, ut KF ad KE; adeóque ex æquo perturbatè YX ad AK ut CX ad KE. Q. E. D.

LEM. II. *Eſt* YX *ad* AK *ut* CY *ad* AK + KE. Nam componendo eſt YX ad AK ut YX + CX, id eſt CY ad AK + KE. Q. E. D.

LEM. III. *Eſt* KE − BK *ad* YX *ut* YX *ad* AK.

Nam per 12 II. *Elem.* eſt $YKq - CKq = CYq - CY \times CX = CY \times YX$; hoc eſt, ſi Theorema reſolvatur in proportionem, CY ad YK − CK ut YK + CK ad YX. Sed eſt $YK - CK = YK - YE + CA - CK = KE - BK$. Et $YK + CK = YK - YE + CA + CK = KE + AK$. Adeoque eſt CY ad KE − BK ut KE + AK ad YX. Sed per *Lemma ſecundum* erat CY ad KE + AK ut YX ad AK. Ergo ex æquo eſt YX ad KE − BK ut AK ad YX. Seu KE − BK ad YX ut YX ad AK. Q. E. D.

His præmiſſis demonſtrabitur Theorema ut ſequitur. In *primo Lemmate* erat YX ad AK ut CX ad KE, ſeu $KE \times YX = AK \times CX$. In *tertio* erat KE − BK ad YX ut YX ad AK. Unde, ſi prioris rationis termini ducantur in YX, fiet $KE \times YX - BK \times YX$ ad YXq ut YX ad AK, id eſt $AK \times CX - BK \times YX$ ad YXq ut YX ad AK; & ductis extremis & mediis in ſe $AKq \times CX - AK \times BK \times Yx = YX$ *cub.* Denique pro YX, AK, BK, & CX reſtitutis x, n, $\frac{q}{n}$, & $\frac{q}{nn}$ orietur $r - qx = x^3$. Q. E. D. Quod verò ad ſignorum variationes attinet, iſtis ſecundum caſus Problematum determinandis non immoror.

Proponatur jam æquatio cujus tertius terminus deeſt $x^3 + pxx + r = 0$. Et ad ejus conſtructionem aſſumpto quolibet n, cape in rectâ aliquâ longitudines duas $KA = \frac{r}{nn}$, & $KB = p$; idque ad eaſdem partes ſi

r & p

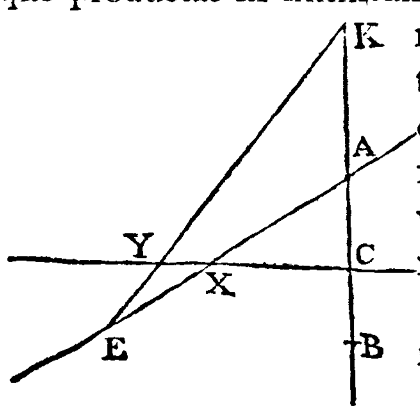
APPENDIX. r & p habeant eadem ſigna, aliter ad contrarias. Biſeca BA in C, & centro K radio KC deſcribe circulum cui, inſcribe CX æqualem n, & produc eam utrinque. Item junge AK, & produc eam utrinque. Denique inter has lineas CX & AX inſcribe EY ejuſdem longitudinis cum CA, ita ut ea ſi producatur tranſeat per K, & KE erit radix æquationis. Radices autem affirmativæ ſunt ubi punctum Y cadit à parte puncti X verſus C, & negativæ ubi punctum Y cadit ad alteras partes puncti X; ſi modò habeatur $+r$, & contrà ſi habeatur $-r$.

Ad hujus Propoſitionis demonſtrationem Schemata & Lemmata de priori propoſitione mutuò ſumantur, & *Demonſtratio* erit ut ſequitur.

Per *Lemma* 1, erat YX ad AK ut CX ad KE, ſeu $YX \times KE = AK \times CX$; & per *Lemma* 3, $KE - KB$ ad YX ut YX ad AK, aut (ſumpto KB ad contrarias partes) $KE + KB$ ad YX ut YX ad AK; adeóque $KE + KB$ in KE ad $YX \times KE$, ſeu $AK \times CX$, ut YX ad AK, ſeu CX ad KE. Quare ductis extremis & mediis in ſe, eſt KE *cub.* $+ KB \times KEq = AK \times CXq$, & ipſarum KE, KB, AK, & CX reſtitutis valoribus ſuprà aſſignatis, $x^3 + pxx = r$.

Proponimus jam æquationem trium dimenſionum $x^3 + pxx + qx + r = 0$, *nullo termino carentem, & cujus tres radices non ſunt omnes affirmativæ, neque omnes negativæ.*

Et primò ſi terminus q *negativus* eſt, in rectâ aliquâ KB capiantur longitudines duæ $KA = \frac{r}{q}$, & $KB = p$, idque ad eaſdem partes puncti K ſi p & $\frac{r}{q}$ habent ſigna diverſa; aliter ad contrarias. Biſeca AB in C, & ad punctum illud C erige perpendiculum CX æquale radici quadraticæ termini q: Et inter lineas rectas AX & CX, utrinque productas in infinitum, inſcribatur recta EY, quæ æqualis ſit rectæ AC, & producta tranſeat per punctum K, atque KE erit radix æquationis; quæ quidem affirmativa erit ſi punctum X cadat inter puncta A & E, negativa verò ſi punctum E cadat ad partes puncti X verſus A.



Quòd ſi terminus q *affirmativus* eſt, in rectâ KB capiantur longitudines illæ duæ

KA =

$KA=\sqrt{\frac{-r}{p}}$, & $KB=\frac{q}{KA}$, idque ad eaſdem partes puncti K, ſi $\sqrt{\frac{-r}{p}}$ & $\frac{q}{KA}$ habent ſigna diverſa; aliter ad contrarias: Biſeca AB in C, & ad punctum illud C erige perpendiculum CX æquale termino p: & inter lineas rectas AX & CX, utrinque productas in infinitum, inſcribatur recta EY quæ æqualis ſit rectæ AC, & producta tranſeat per punctum K, atque XY erit radix æquationis; quæ quidem negativa erit ſi punctum x cadat inter puncta A & F, affirmativa verò ſi punctum Y cadat ad partes puncti X verſus punctum C. Æquationum Conſtructiones.

Demonſtratio caſûs prioris.

Per *Lemma primum* erat KE ad CX ut AK ad YX, & ita (componendo) eſt KE + AK, id eſt KY + KC ad CX + YX, id eſt CY. Sed in triangulo rectangulo KCY eſt YCq æquale YKq − KCq, id eſt æquale KY + KC in KY − KC, & reſolvendo terminos æquales in proportionales, KY + KC ad CY ut CY ad KY − KC, ſeu KE + AK ad CY ut CY ad EK − KB. Quare cùm in hâc proportione fuerit KE ad CX; duplicetur proportio, & erit KEq ad CXq ut KE + AK ad KE − KB; & ductis extremis & mediis in ſe KE *cub.* − KB × KEq = CXq × KE + CXq × AK. Et reſtitutis valoribus ſuprà aſſignatis $x^3-pxx=qx+r$.

Demonſtratio caſûs ſecundi.

Per *Lemma primum* eſt KE ad CX ut AK ad YX, ductiſque extremis & mediis in ſe, fit KE × YX = CX × AK. Scribe ergo in ſuperioribus KE × YX pro CX × AK, & fiet KE *cub.* − KB × KEq = CXq × KE + CX × KE × YX. Et applicatis omnibus ad KE erit KEq − KB × KE = CXq + CX × YX: ductiſque omnibus in AK, habebitur AK × KEq − AK × KB × KE = AK × CXq + AK × CX × YX: Ac rurſus ſcripto KE × YX pro CX × AK, fiet AK × KEq − AK × KB × KE = KE × YX × CX + KE × YXq: & applicatis omnibus ad KE orietur AK × KE − AK × KB = YX × CX + YXq: ductiſque omnibus in YX emerget AK × KE × YX − AK × KB × YX = YXq × CX + YX *cub.* Et pro KE × YX ſcriptis in primo termino CX × AK, fiet CX × AKq − AK × KB × YX = CX × YXq + YX *cub.* ſeu quod perinde eſt YX *cub.* + CX × YXq + AK × KB × YX − CX × AKq = 0. Atque pro YX, CX, AK & KB ſubſtitutis valoribus ſuprà aſſignatis x, p, $\sqrt{\frac{-r}{p}}$, $q\sqrt{\frac{p}{-r}}$ emerget tandem $x^3+pxx+qx+r=0$, æquatio conſtruenda.

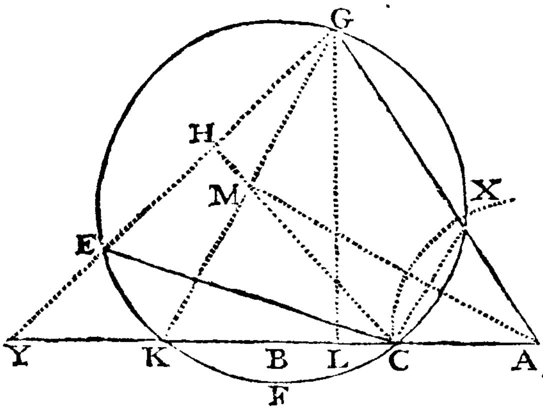
Solvuntur

APPENDIX.

Solvuntur etiam hæ æquationes ducendo rectam lineam datæ longitudinis inter circulum & aliam rectam poſitione datos, eâ lege ut recta illa ducta convergat ad punctum datum.

Proponatur enim æquatio cubica $x^3 * + qx + r = 0$, *cujus terminus ſecundus deeſt.*

Duc rectam KA ad arbitrium. Eam dic n. In KA utrinque productâ cape $KB = \frac{q}{n}$, idque ad eaſdem partes puncti K cum lineâ KA ſi modò habeatur $-q$, aliter ad diverſas. Biſeca BA in C, & centro A intervallo AC deſcribe circulum CX. Ad hunc apta lineam rectam $CX = \frac{r}{nn}$, & per puncta K, C, & X deſcribe circulum KCXG. Junge AX, & junctam produc donec ea iterum ſecet circulum ultimò deſcriptum KCXG in puncto G. Denique inter hunc ultimò deſcriptum circulum & rectam KC utrinque productam inſcribe rectam EY ejuſdem longitudinis cum rectâ AC; ita ut ea convergat ad punctum G. Et acta recta EC erit una ex radicibus æquationis. Radices autem affirmativæ ſunt quæ cadunt in majori circuli ſegmento KGC, & negativæ quæ in minori KFC ſi habeatur $-r$; & contrà ſi habeatur $+r$ affirmativæ in minori ſegmento KFC, negativæ in majori KGC reperientur.



Ad hujus verò conſtructionis demonſtrationem præmittimus Lemmata ſequentia.

LEM. I. *Poſitis quæ in conſtructione ſuperiore, eſt* CE *ad* KA *ut* CE + CX *ad* AY, & CX *ad* KA.

Nam rectâ KG ductâ, eſt AC ad AK ut CX ad KG, idque ob ſimilia triangula ACX, AKG. Sunt etiam triangula YEC, YKG ſimilia: quippe quæ communem habent angulum ad Y, & angulos ad G & C in eodem circuli KCG ſegmento EGCK, atque adeo æquales. Inde fit CE ad EY ut KG ad KY, id eſt CE ad AC ut KG ad KY, eò quòd EY & AC juxta Hypotheſin æquantur. Collata autem hacce cum ſuperiore proportionalitate colligitur ex æquo perturbatè, quòd ſit CE ad KA ut CX ad KY, & viciſſim CE ad CX ut

KA

KA ad KY. Unde componendo fit CE+CX ad CX ut KA+KY ad KY, id eſt ut AY ad KY, & viciſſim CE+CX ad AY ut CX ad KY hoc eſt ut CE ad KA. Q. E. D. Æquationum Conſtructiones.

LEM. II. *Demiſſo ad lineam* GY *perpendiculo* CH, *fiet rectangulum* 2HEY *æquale rectangulo* CE×CX.

Nam demiſſo etiam ad lineam AY perpendiculo GL, triangula KGL, ECH rectos habentia angulos ad L & H, & angulos ad K & E in eodem circuli CGK ſegmento CKEG, adeoque æquales, æquiangula ſunt & proinde ſimilia. Eſt ergo KG ad KL ut EC ad EH. Porro, à puncto A ad lineam KG demiſſo perpendiculo AM, ob æquales AK, AG biſecabitur KG in M, & triangula KAM KGL ob angulum ad K communem, & angulos ad M & L rectos fient ſimilia: & inde eſt AK ad KM ut KG ad KL. Sed ut eſt AK ad KM ita eſt 2AK ad 2KM ſeu KG, & ita (ob ſimilia triangula AKG, ACX) eſt 2AC ad CX; & (ob æquales AC & EY) ita eſt 2EY ad CX. Ergo eſt 2EY ad CX ut KG ad KL. Sed erat KG ad KL ut EC ad EH, ergo eſt 2EY ad CX ut EC ad EH, atque adeo rectangulum 2HEY (ductis nimirum extremis & mediis in ſe) æquale eſt rectangulo EC×CX. Q. E. D.

Aſſumpſimus hic lineas AK, AG æquales eſſe. Nimirum rectangula CAK, XAG (per *Corol. Prop.* 36. *lib.* III. *Elem.*) æqualia ſunt, atque adeo ut CA eſt ad XA ita AG eſt ad AK. Sed CA, XA æquales ſunt per Hypotheſin; ergo & AG, AK.

LEM. III. *Conſtructis omnibus ut ſupra, tres lineæ* BY, CE, KA, *ſunt continue proportionales.*

Nam (per *Prop.* 12. *lib.* II. *Elem.*) eſt $CYq = EYq + CEq + 2EY \times EH$. Et ablato utrinque EYq fit $CYq - EYq = CEq + 2EY \times EH$. Sed $2EY \times EH$ (per *Lem.* 2.) æquale eſt rectangulo $CE \times CX$, & addito utrinque CEq fit $CEq + 2EY \times EH = CEq + CE \times CX$. Ergo $CYq - EYq$ æquale eſt $CEq + CE \times CX$, id eſt $CY + EY$ in $CY - EY$ æquale eſt $CEq + CE \times CX$. Et reſolutis æqualibus rectangulis in latera proportionalia fit $CE + CX$ ad $CY + EY$ ut $CY - EY$ ad CE. Sunt autem tres lineæ EY, CA, CB æquales, & inde $CY + EY = CY + CA = AY$, & $CY - EY = CY - CB = BY$. Scribantur itaque AY pro $CY + EY$, & BY pro $CY - EY$, & fiet $CE + CX$ ad AY ut BY ad CE. Sed (*per Lem.* 1.) eſt CE ad KA ut $CE + CX$ ad AY, ergo eſt CE ad KA ut BY ad CE, hoc eſt lineæ tres BY, CE, KA, ſunt continue proportionales. Q. E. D.

APPENDIX. Tandem ope horum Lemmatum conſtructio ſuperioris Problematis ſic *demonſtratur*.

Per Lemma 1. eſt CE ad KA ut CX ad KY, adeoque KA × CX = KY × CE, & applicatis his æqualibus extremorum & mediorum rectangulis ad CE ſit $\frac{KA \times CX}{CE} = KY$. His lateribus æqualibus adde BK & æqualia erunt BK + $\frac{KA \times CX}{CE}$ & BY. Unde per *Lemma tertium* eſt BK + $\frac{KA \times CX}{CE}$ ad CE ut CE ad KA, & inde, ductis extremis & mediis in ſe provenit CEq æquale BK × KA + $\frac{KAq \times CX}{CE}$, & omnibus præterea ductis in CE fit CE *cub.* æquale BK × KA × CE + KAq × CX. CE erat radix æquationis dicta x, KA erat n, KB $\frac{q}{n}$, & CX $\frac{r}{nn}$. His pro CE, KA, KB, & CX ſubſtitutis oritur $x^3 = qx + r$, ſeu $x^3 - qx - r = 0$, æquatio conſtruenda; ubi q & r negativa prodeunt ſumptis KA & KB ad eaſdem partes puncti K, & radice affirmativa in majori ſegmento CGK exiſtente. Hic *unus caſus* eſt Conſtructionis demonſtrandæ. Ducatur KB ad partes contrarias, id eſt, mutetur ſignum ejus ſeu ſignum ipſius $\frac{q}{n}$, vel quod perinde eſt, ſignum termini q, & habebitur conſtructio æquationis $x^3 + qx - r = 0$: *Qui caſus eſt alter*. In his caſibus CX, & radix affirmativa CE cadunt ad eaſdem partes lineæ AK. Cadant CX & radix negativa ad eaſdem mutato ſigno ipſius CX ſeu $\frac{r}{nn}$ vel (quod perinde eſt) ſigno ipſius r, & habebitur *caſus tertius* $x^3 + qx + r = 0$, ubi radices omnes ſunt negativæ. Et mutato rurſus ſigno ipſius KB ſeu $\frac{q}{n}$ vel ſolius q, incidetur in *caſum quartum* $x^3 - qx + r = 0$. Quorum omnium caſuum conſtructiones percurrere licebit, & ſigillatim demonſtrare ad modum caſus primi. Nos uno caſu demonſtrato cæteros leviter attingere ſatis eſſe putavimus. Hi verbis iiſdem mutato ſolum linearum ſitu demonſtrantur.

Conſtruenda jam ſit æquatio cubica $x^3 + pxx * + r = 0$, *cujus tertius terminus deeſt.*

In figura ſuperiore aſſumpta longitudine quavis n, capias in recta quavis infinita AY, KA, & KB quarum KA valeat $\frac{r}{nn}$, & KB valeat p. Has cape ad eaſdem partes puncti K, ſi modo ſigna terminorum p & r ſint eadem, ſecus ad contrarias. Biſeca BA in C,

& centro

& centro K intervallo KC deſcribe circulum CXG. In eo aptes rectam CX, æqualem longitudini aſſumptæ n. Junge AX & produc junctam ad G ita ut fiat AG æqualis AK, & per puncta K, C, X, G, deſcribe circulum. Denique inter hunc circulum & rectam KC utrinque productam inſcribe rectam EY ejuſdem longitudinis cum recta AC ea lege ut hæc inſcripta recta tranſeat per punctum G ſi modo ipſa producatur: & acta recta KY erit una ex radicibus æquationis. Sunt autem radices affirmativæ quæ cadunt ad partes puncti K verſus punctum A ſi modo habeatur $+r$; ſin habeatur $-r$, affirmativæ ſunt quæ cadunt ad partes contrarias. Et ſi affirmativæ radices jacent ex una parte puncti A, negativæ ſunt quæ jacent ex altera.

Æquationum Constructiones.

Demonſtratur autem hæc conſtructio ope Lemmatum trium noviſſimorum in hunc modum.

Per *Lemma tertium* ſunt BY, CE, KA continue proportionales; & per *Lemma primum* ut eſt CE ad KA ita eſt CX ad KY. Ergo BY eſt ad CE ut CX ad KY. BY idem eſt quod KY−KB. Ergo KY−KB eſt ad CE ut CX ad KY. Sed ut eſt KY−KB ad CE ita eſt KY−KB in KY ad CE in KY, idque per *Prop.* I. *lib.* VI. *Elem.* & ob proportionales CE ad KA ut CX ad KY eſt CE in KY æquale KA in CX. Ergo KY−KB in KY eſt ad KA in CX (ut KY−KB ad CE, hoc eſt) ut CX ad KY. Et ductis extremis & mediis in ſe invicem fit KY−KB in KYq æquale KA in CXq: id eſt KY *cub.* −KB×KY *quad.* æquale KA×CX *quad.* Erat autem in conſtructione, KY radix æquationis dicta x, KB æqualis p, KA æqualis $\frac{r}{nn}$, & CX æqualis n. Scribantur igitur x, p, $\frac{r}{nn}$, & n pro KY, KB, KA, & CX reſpective, & fiet $x^3-pxx=r$, ſeu $x^3-pxx-r=0$.

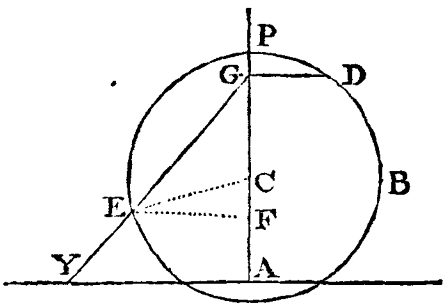
Reſolvi poteſt conſtructio demonſtranda in haſce quatuor æquationum caſus, $x^3-pxx-r=0$, $x^3-pxx+r=0$, $x^3+pxx-r=0$, & $x^3+pxx+r=0$. Caſum primum jam demonſtratum dedi, cæteri tres iiſdem verbis mutato tantum linearum ſitu demonſtrantur. Nimirum uti ſumendo KA & KB ad eaſdem partes puncti K, & radicem affirmativam KY ad contrarias partes, jam prodiit KY *cub.* −KB×KYq = KA×CXq, & inde $x^3-pxx-r=0$: ſic ſumendo KB ad contrarias partes puncti K, prodibit ſimili argumentationis progreſſu KY *cub.* +KB×KYq = KA×CXq, & inde $x^3+pxx-r=0$.

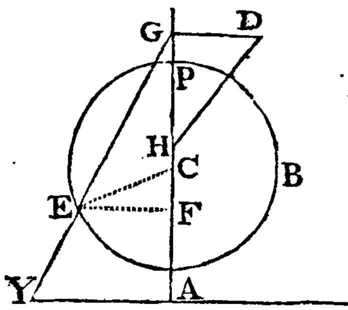
 Et

Et in hiſce duobus caſibus ſi mutetur ſitus radicis affirmativæ KY ſumendo eam ad alteram partem puncti K, per ſimilem argumentationis ſeriem devenietur ad alteros duos caſus KY *cub.* $+\text{KB}\times\text{KY}q=-\text{KA}\times\text{CX}q$, ſeu $x^3+pxx+r=0$, & KY *cub.* $-\text{KB}\times\text{KY}q=-\text{KA}\times\text{CX}q$, ſeu $x^3-pxx+r=0$. Qui omnes caſus erant demonſtrandi.

Proponatur jam æquatio cubica $x^3+pxx+qx+r=0$, *nullo (niſi forte tertio) termino carens.* Ea conſtruetur ad hunc modum.

Cape ab arbitrium longitudinem n. Ejus dimidio æqualem duc rectam quamvis GC, & ad punctum G erige perpendiculum GD æquale $\sqrt{\frac{r}{p}}$. Deinde ſi termini p & r habent contraria ſigna, centro C intervallo CD deſcribe circulum PBE. Sin eadem ſunt eorum ſigna, centro D intervallo GC deſcribe circulum occultum ſecantem rectam GA in H; dein centro C intervallo GH deſcribe circulum PBE. Tum fac $\text{GA}=-\frac{q}{n}-\frac{r}{np}$, eamque duc in linea GC ad partes puncti G verſus C ſi modo quantitas $-\frac{q}{n}-\frac{r}{np}$ (ſignis terminorum p, q, r, in æquatione conſtruendâ probe obſervatis) affirmativa obvenerit: ſecus age GA ad alteras partes puncti G, & ad punctum A erecto perpendiculo AY, inter hoc & circulum PBE ſuperius deſcriptum inſcribe lineam EY æqualem termino p, ea lege ut hæc inſcripta convergat ad punctum G. Quo facto & productâ illâ EY ad G, erit linea EG una ex radicibus æquationis conſtruendæ. Quæ quidem radices affirmativæ ſunt ubi punctum E cadit inter puncta G & Y, & negativæ ubi E cadit extra, ſi modo habeatur $+p$; & contra ſi $-p$.





Demonſtrationi hujus conſtructionis præmittimus *Lemmata* ſequentia.

LEM. I. *Demiſſo ad* AG *perpendiculo* EF *&* *acta recta* EC, *eſt* $\text{EG}q+\text{GC}q=\text{EC}q+2\text{CGF}$.

Nam per *Prop.* 12. *lib.* II. *Elem.* eſt $\text{EG}q=\text{EC}q+\text{GC}q+2\text{GCF}$. Addatur utrinque $\text{GC}q$ & fiet $\text{EG}q+\text{GC}q=\text{EC}q+2\text{GC}q+2\text{GCF}$. Sed

2GCq+2GCF eſt 2GC in GC + CF id eſt 2CGF. Ergo EGq + GCq = ECq + 2CGF. Q. E. D. *Æquationum Conſtructiones.*

LEM. II. *In conſtructionis caſu primo ubi circulus* PBE *tranſit per punctum* D, *eſt* EGq−GDq = 2CGF.

Nam per *Lemma primum* eſt EGq+GCq = ECq+ 2CGF, & ablato utrinque GCq, fit EGq = ECq−GCq+ 2CGF. Sed ECq−GCq idem eſt quod CDq−GCq, hoc eſt idem quod GDq. Ergo EGq = GDq+ 2CGF, & ſubducto utrobique GDq, fit EGq−GDq = 2CGF. Q. E. D.

LEM. III. *In conſtructionis caſu ſecundo, ubi circulus* PBE *non tranſit per punctum* D, *eſt* EGq+GDq = 2CGF.

Namque in *Lemmate primo* erat EGq+GCq = ECq+ 2CGF. Aufer utrinque ECq & fiet EGq+GCq−ECq = 2CGF. Sed GC = DH & EC = CP = GH: ergo GCq−ECq = DHq − GHq = GDq, atque adeo EGq+GDq = 2CGF. Q. E. D.

LEM. IV. *Eſt* 2CGF *in* GY = 2CG *in* AGE.

Namque ob ſimilia triangula GEF, GYA eſt GF ad GE ut AG ad GY; hoc eſt (per *Prop.* I. *lib.* VI. *Elem.*) ut 2CG × AG ad 2CG × GY. Ducantur extrema & media in ſe, & fiet 2CG × GY × GF = 2CG × AG × GE. Q. E. D.

Tandem ope horum Lemmatum conſtructio Problematis ſic demonſtratur.

In caſu *primo* eſt (per *Lem.* 2.) EGq−GDq = 2CGF, & ductis omnibus in GY fit EGq × GY − GDq × GY = 2CGF × GY (hoc eſt per *Lem.* 4.) = 2CG × AGE. Pro GY ſcribe EG + EY, & fiet EG *cub.* + EY × EGq − GDq × EG − GDq × EY = 2CGA × EG, ſeu EG *cub.* + EY × EGq $\begin{smallmatrix}-\text{GD}q\\ -2\text{CGA}\end{smallmatrix}$ × EG − GDq × EY = 0.

In caſu *ſecundo* eſt (per *Lem.* 3.) EFq+GEq = 2CGF, & ductis omnibus in GY fit EGq × GY + GDq × GY = 2CGF × GY (hoc eſt per *Lem.* 4.) = 2CG × AGE. Pro GY ſcribe EG+EY, & fiet EG *cub.* + EY × EGq + GDq × EG + GDq × EY = 2CGA × EG, ſeu EG *cub.* + EY × EGq $\begin{smallmatrix}+\text{GD}q\\ -2\text{CGA}\end{smallmatrix}$ × EG + GDq × EY = 0.

Jam vero erat EG radix æquationis conſtructæ dicta x; item GD = $\sqrt{\frac{r}{p}}$, EY = p, 2CG = n, & GA = $-\frac{q}{n}-\frac{r}{np}$, id eſt in caſu primo ubi terminorum p & r diverſa ſunt ſigna: at in caſu ſecundo ubi alterutrius p vel r mutatur ſignum fiet $-\frac{q}{n}+\frac{r}{np}$ = GA. Scribantur igitur

APPENDIX. tur pro EG, GD, EY, 2CG, & GA quantitates x, $\sqrt{\frac{r}{p}}$, p, n, & $-{}_{n} \mp \frac{\;}{np}$,

& caſu *primo* fiet $x^3 + px^2 \begin{matrix} -\frac{r}{p} \\ +q+\frac{r}{p} \end{matrix} x - r = 0$, id eſt $x^3 + pxx + qx - r = 0$,

caſu autem *ſecundo* $x^3 + px^2 \begin{matrix} +\frac{r}{p} \\ +q-\frac{r}{p} \end{matrix} x + r = 0$, id eſt $x^3 + pxx + qx + r = 0$,

Eſt igitur in utroque caſu EG vera longitudo radicis x. Q. E. D.

Subdiſtinguitur autem caſus uterque in caſus plures particulares: Nimirum prior in hoſce $x^3+px^2+qx-r=0$, $x^3+pxx-qx-r=0$, $x^3-pxx+qx+r=0$, $x^3-pxx-qx+r=0$, $x^3+px^2-r=0$, & $x^3-pxx+r=0$; poſterior in hoſce $x^3+pxx+qx+r=0$, $x^3+pxx-qx+r=0$, $x^3-pxx+qx-r=0$, $x^3-pxx-qx-r=0$, $x^3+pxx+r=0$, & $x^3-pxx-r=0$. Quorum omnium demonſtrationes verbis iiſdem ac duorum jam demonſtratorum, mutato tantum linearum ſitu, compinguntur.

Hæ ſunt Problematum conſtructiones præcipuæ per inſcriptionem rectæ longitudine datæ inter circulum, & rectam lineam poſitione datam, eâ lege, ut inſcripta ad datum punctum convergat. Inſcribitur autem talis recta, ducendo *Conchoidem* veterum, cujus Polus ſit punctum illud, ad quod recta inſcribenda debet convergere, Regula ſeu Aſymptotos recta altera poſitione data, & intervallum longitudo rectæ inſcribendæ. Secabit enim hæc Conchoides circulum præfatum in puncto E, per quod recta inſcribenda duci debet. Suffecerit vero in rebus practicis, rectam illam, inter circulum & alteram poſitione datam rectam, ratione quâcunque mechanicâ interponere.

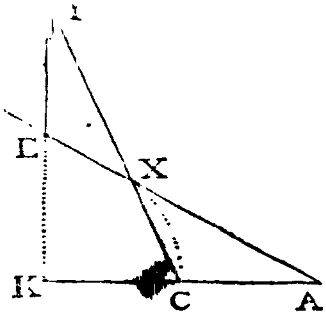
In hiſce autem conſtructionibus notandum eſt quod quantias n, ubique indeterminata & ad arbitrium aſſumenda relinquitur; id adeo ut ſingulis problematis conſtructiones commodius aptentur. Hujus rei exempla in inventione duarum medie proportionalium, & anguli triſectione dabimus.

Inveniendæ ſint inter a & b duæ medie proportionales x & y. Quoniam ſunt $a.\ x.\ y.\ b$ continue proportionales erit aa ad xx ut x ad b, adeoque $x^3 = aab$, ſeu $x^3 - aab = 0$. Hic deſunt æquationis termini p & q, & loco termini r habetur $-aab$. Igitur in conſtructionum formulâ primâ, ubi recta EY ad datum punctum K

convergens

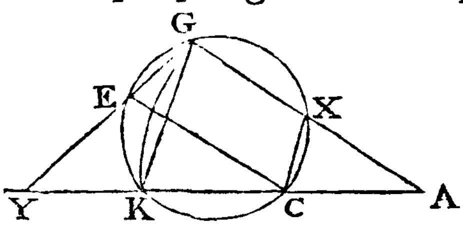
convergens inſeritur inter alias duas poſitione datas rectas EX & YC, & recta CX ponitur æqualis $\frac{r}{nn}$ id eſt æqualis $\frac{-aab}{nn}$, aſſumo n æqualem a, & ſic fit CX æqualis $-b$. Unde talis emerget conſtructio. *Æquationum Constructiones.*

Duco quamvis KA æqualem a, eamque biſeco in C, centroque K intervallo KC deſcribo circulum CX ad quem apto rectam CX



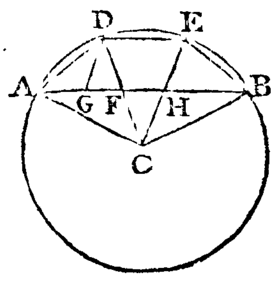
æqualem b & inter rectas AX, CX infinite productas pono EY æqualem CA, & convergentem ad punctum K. Sic erunt KA, XY, KE, CX, continue proportionales, id eſt XY & KE duæ medie proportionales inter a & b. Conſtructio nota eſt.

Inter altera autem conſtructionum formula ubi recta EY ad datum punctum G convergens ponitur inter circulum GECX & rectam AK, eſtque CX $= \frac{r}{nn}$ id eſt (in hoc Problemate $= \frac{-aab}{nn}$, pono ut prius $n = a$, & ſic fit CX $= b$, cæteraque peraguntur ut ſequitur.



Duco rectam quamvis KA æqualem a, eamque biſeco in C & centro A intervallo AK deſcribo circulum KG ad quem apto rectam KG æqualem $2b$ conſtituendo triangulum æquicrurum AKG. Dein per puncta C, K, G circulum deſcribo & inter hujus perimetrum & rectam productam AK inſcribo rectam EY æqualem KC, & convergentem ad punctum G. Quo facto continue proportionales erunt AK, EC, KY, $\frac{1}{2}$ KG, id eſt EC & KY duæ medie proportionales erunt inter datas a & b.

Secandus jam ſit angulus in partes tres æquales. Sitque angulus ſecandus ACB, *partes ejus inveniendæ* ACD, DCE, ECB.

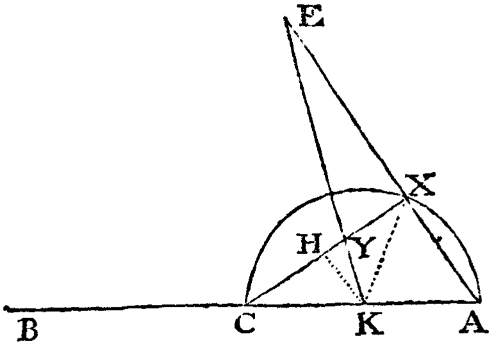


Centro C intervallo CA deſcribatur circulus ADEB ſecans rectas CA, CD, CE, CB in A, D, E, B. Jungantur AD, DE, EB ut & AB ſecans rectas CD, CE in F & H, & ipſi CE parallela agatur DG occurrens AB in G. Ob ſimilia triangula CAD, ADF, DFG, continue proportionales ſunt CA, AD, DF, FG. Ergo ſi dicatur AC $= a$,

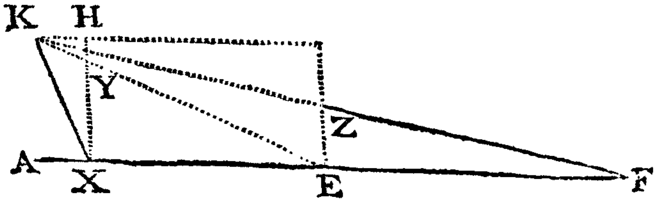
& AD

& AD=x, ſiet DF = $\frac{xx}{a}$, & FG = $\frac{x^3}{aa}$. Eſt autem AB = BH + HG + FA − GF = 3AD − GF = $3x - \frac{x^3}{aa}$. Dic AB=b, & fiet $b = 3x - \frac{x^3}{aa}$, ſeu $x^3 - 3aax + aab = 0$. Hic deeſt æquationis terminus ſecundus p, & loco q & r habentur $-3aa$ & aab. Ergo in conſtructionum formula prima ubi erat $p = 0$, KA = n, KB = $\frac{q}{n}$, & CX = $\frac{r}{nn}$, id eſt in problemate jam conſtruendo KB = $-\frac{3aa}{n}$, & CX = $\frac{aab}{nn}$, ut hæ quantitates evadant quam ſimpliciſſimæ pono $n = a$, & ſic fit KB = $-3a$, & CX = b. Unde talis emergit Problematis *conſtructio*.

Ago quamvis KA = a, & ad contrarias partes KB = $3a$. Biſeco BA in C, centroque K intervallo KC deſcribo circulum, cui inſcribo rectam CX = b. Et actâ rectâ AX, inter ipſam infinite productam & rectam CX pono rectam EY æqualem AC, & convergentem ad punctum K. Sic fit XY = x. Quinetiam ob æquales circulos ADEB, CXA, & æquales ſubtenſas AB, CX, nec non æquales ſubtenſarum partes BH, XY, æquales erunt anguli ACB, CKX, ut & anguli BCH, XKY, atque adeo anguli CKX tertia pars erit angulus XKY. Dati igitur cujuſvis anguli CKX pars tertia XKY invenietur ponendo inter chordas CX, AX infinitè productas rectam EY æqualem diametro AC, & convergentem ad circuli centrum K.



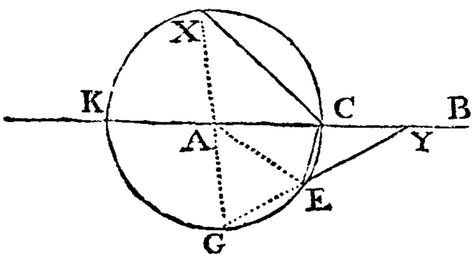
Hinc ſi à circuli centro K ad ſubtenſam CX demittas perpendiculum KH, erit angulus HKY tertia pars anguli HKX, adeo ut ſi detur quilibet angulus HKX inveniri poſſit ejus pars tertia HKY demittendo à quolibet lateris utriuſvis KX puncto X ad latus alterum KH perpendiculum XH, & lateri KH ducendo parallelam XE, dein rectam YE duplam ipſius KX, & convergentem ad punctum K ponendo inter rectas XH & XE. *Vel ſic.* Detur angulus quilibet AXK. Ad latus alterutrum AX erigatur perpendiculum XH, & à lateris alterius XK puncto quovis K agatur



gatur recta KE cujus pars YE interjacens lateri AX producto, & ejus perpendiculo XH ſit dupla lateris XK, & erit angulus KEA tertia pars anguli dati AXK. Tum rurſus erecto perpendiculo EZ, & acta KF cujus pars ZF inter EF & EZ ſit dupla ipſius KE, fiet angulus KFA tertia pars anguli KEA, & ſic pergitur per continuam anguli triſectionem in infinitum. Exſtat autem hæc triſectio apud *Pappum*, *lib.* 4. *Prop.* 32. *Æquationum Conſtructiones.*

Quod ſi angulum per alteram conſtructionum formulam ubi recta inter aliam rectam & circulum ponenda eſt, trifariam dividere malueris: hic etiam erunt $KB = \frac{q}{n}$, & $CX = \frac{r}{nn}$, id eſt in problemate de quo nunc agimus $KB = \frac{-3aa}{n}$, & $CX = \frac{aab}{nn}$, adeoque ponendo $n = a$ fiet $KB = -3a$, & $CX = b$. Et inde talis emerget *conſtructio.*

A puncto quovis K ducantur ad eaſdem partes rectæ duæ $KA = a$, & $KB = 3a$. Biſeca AB in C, centroque A intervallo AC deſcribe circulum. In eo pone rectam $CX = b$. Junge AX, & junctam produc donec ea iterum ſecet circulum jam deſcriptum in G. Tum inter hunc circulum & rectam KC infinite productam pone rectam EY æqualem rectæ AC, & convergentem ad punctum G, & acta recta EC erit longitudo quæſita x, qua tertia pars anguli dati ſubtenditur.

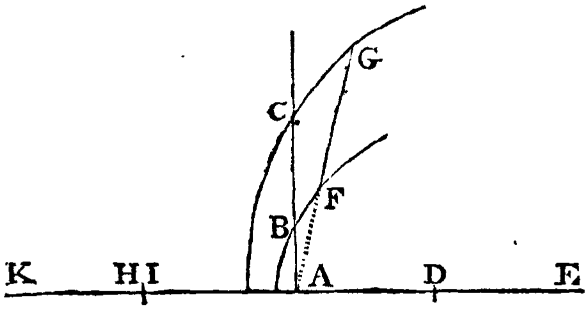


Talis conſtructio conſequitur formulam ſuperius allatam: quæ tamen ſic evadet concinnior. Ob æquales circulos ADEB & KXG, & æquales ſubtenſas CX & AB, æquales ſunt anguli CAX ſive KAG & ACB, adeoque CE ſubtenſa eſt tertiæ partis anguli KAG. Quare dato quovis angulo KAG, ut ejus inveniatur pars tertia CAE, pone inter circulum KGC, & anguli latus KA infinité productum rectam EY æqualem circuli ſemidiametro AG, & convergentem ad punctum G. Sic docuit *Archimedes* angulum trifariam ſecare. Eædem conſtructiones facilius explicari poſſint quam hic factum eſt; ſed in his volui oſtendere quomodo ex generalibus Problematum conſtructionibus ſuperius

Lemma Archim 8.

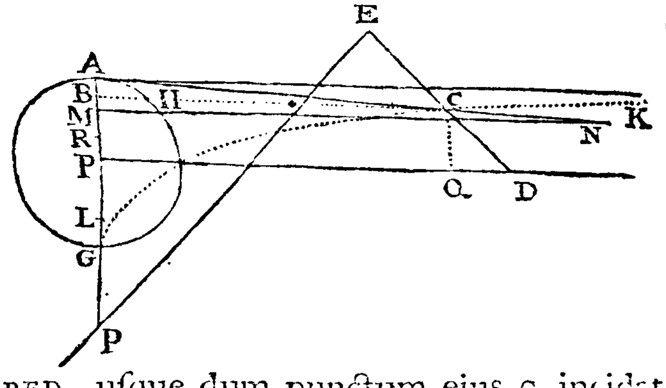
Appendix. expoſitis conſtructiones ſimpliciſſimas particularium Problematum derivare liceat.

Præter conſtructiones hic expoſitas adjungere liceret quamplurimas. *Ut ſi inter a & b inveniendæ eſſent duæ medié proportionales.* Age quamvis AK $=b$, & huic perpendiculare AB $=a$. Biſeca AK in I, & in eâdem AK, ſubtenſæ BI æqualem pone AH; ut & in lineâ AB productâ ſubtenſæ BH æqualem AC.



Tum in lineâ AK ad alteras partes puncti A cape AD cujuſvis longitudinis, & huic æqualem DE, centriſque D & E, intervallis DB, EC deſcribe circulos duos BF, CG, & inter eos pone rectam FG æqualem rectæ AI, & convergentem ad punctum A, & erit AF, prima duarum medié proportionalium quas invenire oportuit.

Docuerunt Veteres inventionem duarum medie proportionalium per *Ciſſoidem*; ſed lineæ hujus deſcriptionem commodam manualem nemo, quod ſcio, appoſuit. Sit AG diameter & F centrum circuli ad quem *Ciſſois* pertinet. Ad punctum F erigatur normalis FD, eaque producatur in infinitum. Et producatur FG ad P, ut FP æqualis ſit circuli Diametro. Moveatur norma rectangula PED eâ lege ut crus ejus EP perpetuó tranſeat per punctum P, & crus alterum ED circuli Diametro AG ſeu FP æquale, termino ſuo D tangat ſemper lineam FD, & cruris hujus medium punctum C deſcribet *Ciſſoidem* deſideratam GCK ut ſupra expoſui. Quare ſi inter duas quaſvis a & b inveniendæ ſint duæ mediæ proportionales: cape AM $=a$, erige perpendiculum MN $= b$. Junge AN; & lege præfatâ moveatur norma



PED, uſque dum punctum ejus C incidat in rectam AN. Tum demiſſo ad AP perpendiculo CB, cape t ad BH, & u ad BC, ut eſt

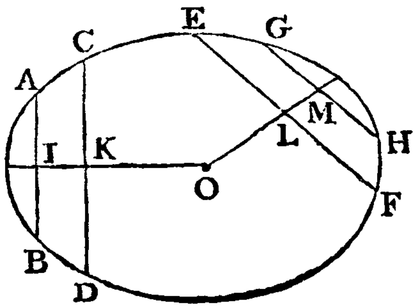
MN ad BC, & ob continue proportionales AB, BH, BG, BC erunt etiam continue proportionales a, t, v, b. *Æquationum Conſtructiones.*

Simili normæ applicatione conſtrui poſſunt etiam alia Problemata ſolida. Verbi gratiâ proponatur æquatio cubica $x^3 \mp pxx + qx - r = 0$: ubi q ſemper affirmativum ſit, r negativum, & p ſigni utriuſvis. Fac AG $= \frac{r}{q}$, eamque biſeca in F, & cape FR & GL$=\frac{1}{2}p$, idque verſus A ſi habeatur $+p$ aliter verſus P. Erige inſuper normalem FD, inque eà cape FQ$=\sqrt{q}$ huic etiam erige normalem QC. In normæ autem crure ED, cape ED & EC ipſis AG & AR æquales reſpectivè, & applicetur deinceps norma ad Schema ſic ut punctum ejus D tangat rectam FD, & punctum C rectam QC, tum ſi compleatur parallelogrammum BQ; erit LB æquationis radix quæſita x.

Hactenus conſtructionem ſolidorum Problematum, per operationes, quarum praxis manualis maxime ſimplex eſt & expedita, exponere viſum fuit. Sic Veteres, poſtquam confectionem horum problematum per compoſitionem locorum ſolidorum aſſecuti fuerant, ſentientes ejuſmodi conſtructiones, ob difficilem Conicarum ſectionum deſcriptionem, inutiles eſſe, quærebant conſtructiones faciliores per Conchoidem, Ciſſoidem, extenſionem filorum, & figurarum adaptiones quaſcunque mechanicas: prælatâ mechanicâ utilitate inutili ſpeculationi Geometricæ, ut ex *Pappo* diſcimus. Sic magnus ille *Archimedes* triſectionem anguli, per coni ſectiones, à ſuperioribus Geometris expoſitam, neglexit, & in Lemmatis ſuis angulum, modo à nobis ſuperius expoſito, trifariam ſecarè docuit. Si veteres problemata, per figuras, eâ tempeſtate in Geometriam non receptas, conſtruere maluerint, quanto magis præferendæ nunc ſunt illæ figuræ, in Geometriam æque ac ipſæ coni ſectiones à pleriſque receptæ.

Verumtamèn novo huic Geometrarum generi haud aſſentior, qui figuras haſce omnes in Geometriam recipiunt. Eorum regula admittendi lineas omnes ad conſtructionem Problematum, eo ordine, quo æquationes, quibus lineæ illæ definiuntur, numero dimenſionum aſcendunt, arbitraria eſt, & in Geometriâ fundamentum non habet. Imo falſa eſt, propterea quod circulus hâc lege, cum coni ſectionibus conjungendus eſſet, quem tamen Geometræ omnes cum lineâ rectâ conjungunt. Vacil-

Appendix. lante autem hâc regulâ, tollitur fundamentum admittendi certo ordine lineas omnes Analyticas in Geometriam. In Geometriam planam, meo quidem judicio, lineæ nullæ præter rectam & circulum admitti debent; niſi forte linearum diſtinctio aliqua prius excogitetur, quâ linea circularis conjungatur cum rectâ, & à reliquis omnibus ſegregetur. Quinimo ne tum quidem augenda eſt Geometria plana numero linearum. Nam figuræ omnes ſunt planæ, quæ admittuntur in Geometriam planam, id eſt quas Geometræ poſtulent in plano deſcribere. Et problema omne planum eſt, quod per figuras planas conſtrui poteſt. Sic igitur admiſſis in Geometriam planam conicis ſectionibus, aliiſque magis compoſitis figuris, problemata omnia ſolida, & plus quam ſolida, quæ per has figuras conſtrui poſſunt, evadent plana. Sunt autem problemata omnia plana ejuſdem ordinis. Linea recta Analyticè ſimplicior eſt quam circulus; hoc non obſtante problemata ejuſdem ſunt ordinis, quæ per rectas ſolas, & quæ per circulos conſtruuntur. Solis poſtulatis reducitur circulus ad eundem ordinem cum rectâ. Et multo magis Ellipſis, quæ minus differt à circulo, quam circulus à rectâ, poſtulando conſimiliter deſcriptionem ejus in plano, reduceretur ad eundem ordinem cum circulo. Siquis, ſpeculando Ellipſin, incideret in problema aliquod ſolidum, et ipſum beneficio ejuſdem Ellipſeos & circuli conſtrueret; hoc problema jam pro plano habendum eſſet: eo, quod Ellipſis jam ante in plano deſcripta haberi ſupponitur, & conſtructio omnis, quæ ſupereſt, abſolvitur per circuli ſolius deſcriptionem. Eâdem de cauſâ problemata quævis plana per datam Ellipſin conſtruere licitum eſt. Verbi gratiâ ſi datæ Ellipſeos ADFG requireretur centrum O, ducerem parallelas duas AB, CD Ellipſi occurrentes in A, B, C, D, aliaſque duas EF, GH Ellipſi occurrentes in E, F, G, H. Has biſecarem in I, K, L, M, & junctas IK, LM producerem, uſque ad concurſum ſuum in O. Legitima eſt hæc conſtructio plani problematis per Ellipſin. Nil refert quod Ellipſis Analyticè definiatur per æquationem duarum dimenſionum. Nil quod Ellipſis Geometricè generetur ſectione figuræ ſolidæ. Hypotheſis ſola, quod Ellipſis jam deſcripta habetur in plano, problemata omnia ſolida, per ipſam conſtructa, reducit ad ordinem planorum, efficitque, ut plana omnia



omnia per ipſam legitimè conſtruantur. Et eadem eſt ratio Poſtulati. Quod vi poſtulatorum fieri poteſt, ut jam factum, & datum aſſumere conceſſum eſt. Poſtuletur igitur Ellipſin in plano deſcribere, & ad ordinem planorum problematum reducentur, ea omnia, quæ per Ellipſin conſtrui poſſunt, planaque omnia per Ellipſin licebit conſtruere. Æquationum Conſtructiones.

Neceſſe eſt igitur aut Problemata plana & ſolida inter ſe confundi, aut lineas omnes rejici è Geometriâ planâ præter rectam & circulum, & ſiqua forſan alia detur aliquando, in ſtatu conſtruendi alicujus Problematis. Verum genera problematum confundi, nemo certe permiſerit. Rejiciantur igitur è Geometriâ planâ ſectiones Conicæ, aliæque figuræ omnes præter rectam & circulum, & quas contigerit in ſtatu problematum dari. Alienæ ſunt igitur à Geometriâ deſcriptiones illæ omnes conicarum ſectionum in plano, quibus hodierni Geometræ tantoperè indulgent. Nec tamen ideo Coni ſectiones è Geometriâ rejiciendæ erunt. Hæ in plano non deſcribuntur Geometricè, generantur vero in ſolidi Geometrici ſuperficie plane. Conus conſtituitur Geometrice, & plano Geometrico ſecatur. Tale Coni ſegmentum figura Geometrica eſt, eundemque habet locum in Geometriâ ſolidâ ac ſegmentum circuli in planâ, & hâc ratione baſis ejus, quam Coni ſectionem vocant, figura Geometrica eſt. Locum igitur habet Coni ſectio in Geometriâ, quatenus ea ſuperficies eſt ſolidi Geometrici. Aliâ autem nullâ ratione Geometricâ, quam ſolidi ſectione, generatur, & ideo non niſi in Geometriam ſolidam antiquitùs admiſſa fuit. Talis autem Conicarum ſectionum generatio difficilis eſt, & in rebus practicis, quibus Geometria potiſſimum inſervire debet, prorſus inutilis. Ideo veteres ſe ad varias figurarum in plano deſcriptiones mechanicas receperunt, & nos ad eorum exemplar conſtructiones præcedentes concinnavimus. Sunto conſtructiones illæ Mechanicæ: ſic & conſtructiones per Coni ſectiones, in plano ut jam moris eſt deſcriptas, Mechanicæ ſunt. Sunto conſtructiones per datas Coni ſectiones Geometricæ: ſic & conſtructiones per alias quaſcunque

APPENDIX. cunque figuras datas Geometricæ ſunt, & ejuſdem ordinis cum conſtructionibus planorum Problematum. Nullâ ratione preferendæ ſunt in Geometriâ Sectiones conicæ figuris aliis, niſi quatenus illæ à ſectione Coni, praxi ad ſolutionem problematum prorſus inutili, derivantur. Verumtamen ne conſtructiones per Conicas ſectiones omnino prætercam, viſum fuit aliqua de his ſubjungere, in quibus etiam praxi manuali non incommodæ conſulatur.

Conicarum ſectionum ſimpliciſſima eſt Ellipſis. Hæc notior eſt, & circulo magis affinis, & praxi manuali facilius deſcribitur in plano. Parabolam præferunt plerique, ob ſimplicitatem æquationis, per quam ea exprimitur. Verum hâc ratione Parabola ipſi etiam circulo præferenda eſſet, contra quam ſit. Falſa eſt igitur argumentatio à ſimplicitate æquationum. Æquationum ſpeculationi nimium indulgent hodierni Geometræ. Harum ſimplicitas eſt conſiderationis Analyticæ. Nos in compoſitione verſamur, & compoſitioni leges dandæ non ſunt ex Analyſi. Manuducit Analyſis ad Compoſitionem: ſed Compoſitio non prius verè confit, quam liberatur ab omni Analyſi. Inſit compóſitioni vel minimum Analyſeos, & compoſitionem veram nondum aſſecutus es. Compoſitio in ſe perfecta eſt, & à miſturâ ſpeculationum Analyticarum abhorret. Pendet Figurarum ſimplicitas à ſimplicitate geneſeos & Idearum, & æquatio non eſt, ſed deſcriptio, ſive Geometrica ſive Mechanica, quâ figura generatur, & redditur conceptu facilis. Ellipſi igitur primum locum tribuentes, docebimus jam, quomodo æquationes per ipſam conſtruere licet.

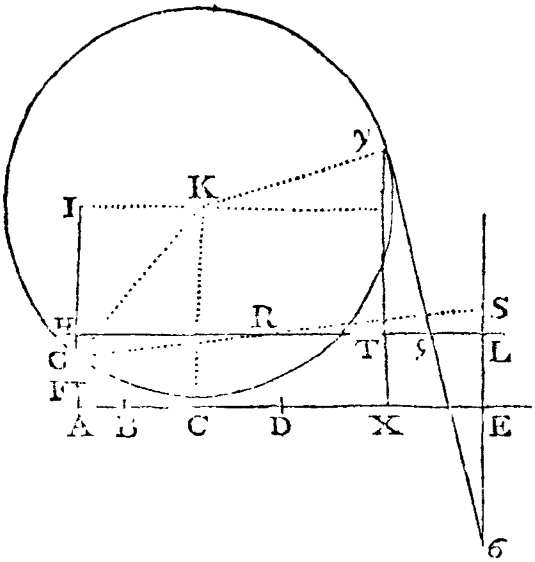
Proponatur æquatio quævis cubica $x^3 = pxx + qx + r$, *ubi* p, q *&* r *datas terminorum æquationis coefficientes cum ſignis ſuis* $+$ *&* $-$ *ſignificant, & alteruter terminorum* p *&* q, *vel etiam uterque deeſſe poteſt.* Sic enim æquationum omnium cubicarum conſtructiones unâ illâ operatione quæ ſequitur exhibebimus.

A puncto D in rectâ quavis datâ cape duas quaſcunque rectas BC, BE ad eaſdem partes; ut & inter ipſas mediam proportionalem BD. Et BC dicta n, cape etiam in eâdem rectâ $BA = \frac{q}{n}$, idque verſus punctum C ſi habeatur $-q$, aliter ad partes contrarias.

Ad

Ad punctum A erige perpendiculum AI, inque eo cape AF æqualem p, FG æqualem AF, FI æqualem $\frac{r}{nn}$, & FH in ratione ad FI ut eſt BC ad BE. FH vero & FI capiendæ ſunt ad partes puncti F verſus G ſi termini p & r habent eadem ſigna, aliter ad partes verſus A. Compleantur parallelogramma IACK & HAEL, centroque K, & intervallo KG deſcribatur circulus. Tum in lineâ HL capiatur ad utramvis partem puncti H longitudo HR, quæ ſit ad HL ut BD ad BE: agatur GR ſecans EL in S, & moveatur lineâ GRS puncto ejus R ſuper lineâ HL, & puncto S ſuper lineâ EL incedente, donec tertium ejus punctum G deſcribendo Ellipſin, occurrat circulo, quemadmodum videre eſt in poſitione $\gamma\varrho\sigma$. Nam dimidium perpendiculi γX ab occurſus illius puncto γ in rectam AE demiſſi erit radix æquationis. Poteſt autem Regulæ GRS vel $\gamma\varrho\sigma$ terminus G vel γ, circulo in tot punctis occurrere quot ſunt poſſibiles radices. Et è radicibus hæ ſunt affirmativæ quæ cadunt ad eas partes rectæ AE ad quas recta FI ducitur à puncto F, & illæ negativæ quæ cadunt ad contrarias partes lineæ AE, ſi modo habeatur $+r$: & contra ſi habeatur $-r$.

Æquationum Constructiones.



Demonſtratur autem hæc conſtructio ſubſidio *Lemmatum* ſequentium.

LEM. I. *Poſitis quæ in ſuperiore conſtructione, eſt* $2\text{CAX} - \text{AX}q = \gamma\text{X}q - 2\text{AI} \times \gamma\text{X} + 2\text{AG} \times \text{FI}$.

Namque ex natura circuli eſt $\text{K}\gamma q - \text{CX}q$, æquale quadrato ex $\gamma\text{X} - \text{AI}$. Sed eſt $\text{K}\gamma q$ æquale $\text{GI}q + \text{AC}q$, & $\text{CX}q$ æquale quadrato ex $\text{AX} - \text{AC}$ hoc eſt æquale $\text{AX}q - 2\text{CAX} + \text{AC}q$, atque adeo horum differentia $\text{GI}q + 2\text{CAX} - \text{AX}q$, æquatur quadrato ex $\gamma\text{X} - \text{AI}$, id eſt ipſi $\gamma\text{X}q - 2\text{AI} \times \gamma\text{X} + \text{AI}q$. Auferatur utrinque $\text{GI}q$, & manebunt æqualia $2\text{CAX} - \text{AX}q$, & $\gamma\text{X}q - 2\text{AI} \times \gamma\text{X} + \text{AI}q - \text{GI}q$. Verum $\text{AI}q$ (per *Prop.* 4. *Lib.* II. *Elem.*) æquale eſt $\text{AG}q + 2\text{AGI} + \text{GI}q$, atque

APPENDIX. adeo $AIq - GIq$ æquale eſt $AGq + 2AGI$, hoc eſt æquale $2AG$ in $\frac{1}{2}AG + GI$, ſeu æquale $2AG \times FI$, & proinde $2CAX - AXq$, æquale eſt $\gamma Xq - 2AI \times \gamma X + 2AG \times FI$. Q. E. D.

LEM. II. *Poſitis quæ in ſuperiore conſtructione, eſt* $2EAX - AXq$ *æquale* $\frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$.

Notum eſt enim quod punctum γ motu regulæ $\gamma\varrho\sigma$ ſuperius aſſignato deſcribit Ellipſin cujus centrum eſt L, & axes duo cum rectis LE & LH coincidunt, quorum qui in LE æquatur $2\gamma\varrho$ ſive $2GR$, & alter in LH æquatur $2\gamma\sigma$ ſive $2GS$. Et horum ratio ad invicem ea eſt quæ lineæ HR ad lineam HL, ſive lineæ BD ad lineam BE. Unde latus tranſverſum eſt ad latus rectum principale ut BE ad BC ſive ut FI ad FH. Quare cum γT ordinatim applicetur ad HL, erit ex natura Ellipſeos $GSq - LTq$ æquale $\frac{FI}{FH} T\gamma q$. Eſt autem LT æquale $AE - AX$, & $T\gamma$ æquale $X\gamma - AH$. Scribantur horum quadrata pro LTq & $T\gamma q$, & fiet $GSq - AEq + 2EAX - AXq = \frac{FI}{FH}$ in $X\gamma q - 2AH \times X\gamma + AHq$. Eſt autem $GSq - AEq$ æquale quadrato ex $GH + LS$, propterea quod GS hypotenuſa eſt trianguli rectanguli cujus latera ſunt ipſis AE & $GH + LS$ æqualia. Eſt & (ob ſimilia triangula RGH, RSL) LS ad GH ut LR ad HR, & componendo $GH + LS$ ad GH ut HL ad HR, & duplicando rationes, quadratum ex $GH + LS$, eſt ad GHq ut HLq ad HRq, hoc eſt (per conſtructionem) ut BEq ad BDq, id eſt ut BE ad BC, ſeu FI ad FH, adeoque quadratum ex $GH + LS$ æquale eſt $\frac{FI}{FH} GHq$. Eſt itaque $GSq - AEq$ æquale $\frac{FI}{FH} GHq$, atque adeo $\frac{FI}{FH} GHq + 2EAX - AXq = \frac{FI}{FH}$ in $X\gamma q - 2AH \times X\gamma + AHq$. Auferatur utrinque $\frac{FI}{FH} GHq$, & reſtabit $2EAX - AXq = \frac{FI}{FH}$ in $X\gamma q - 2AH \times X\gamma + AHq - GHq$. Eſt autem $AH = AG + GH$, adeoque $AHq = AGq + 2AGH + GHq$ & ſubducto utrinque GHq reſtat $AHq - GHq = AGq + 2AGH$, hoc eſt $= 2AG$ in $\frac{1}{2}AG + GH$, ſeu $= 2AG \times FH$, atque adeo eſt $2EAX - AXq = \frac{FI}{FH}$ in $X\gamma q - 2AH \times X\gamma + 2AG \times FH$, i. e. $= \frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$. Q. E. D.

LEM. III. *Iiſdem poſitis eſt* AX *ad* $X\gamma - AG$ *ut* $X\gamma$ *ad* $2BC$.

Nam ſi de æqualibus in *Lemmate ſecundo* ſubducantur æqualia in *Lemmate primo*, reſtabunt æqualia $2CE \times AX$ & $\frac{HI}{FH} X\gamma q - \frac{2FI}{FH} AH$

×

$\times X\gamma + 2AI \times X\gamma$. Ducatur pars utraque in FH, & fiet $2FH \times CE \times AX$ æquale $HI \times X\gamma q - 2FI \times AH \times X\gamma + 2AI \times FH \times X\gamma$. Eſt autem $AI = AH + HI$, adeoque $2FI \times AH - 2FH \times AI = 2FI \times AH - 2FHA - 2FHI$. Sed $2FI \times AH - 2FHA = 2AHI$, & $2AHI - 2FHI = 2HI \times AF$. Ergo $2FI \times AH - 2FH \times AI = 2HI \times AF$, adeoque $2FH \times CE \times AX = HI \times X\gamma q - 2HI \times AF \times X\gamma$. Et inde HI ad FH ut $2CE \times AX$ ad $X\gamma q - 2AF \times X\gamma$. Sed per conſtructionem HI eſt ad FH ut CE ad BC, atque adeo ut $2CE \times AX$ ad $2BC \times AX$ & proinde $2BC \times AX$ & $X\gamma q - 2AF \times X\gamma$ (per *Prop.* 9. *lib.* V. *Elem.*) erunt æqualia. Æqualium vero rectangulorum proportionalia ſunt latera, AX ad $X\gamma - 2AF$, id eſt ad $X\gamma - AG$ ut $X\gamma$ ad 2BC. Q. E. D. Æquationum Conſtructiones.

LEM. IV. *Iiſdem poſitis, eſt* 2FI *ad* $AX - 2AB$ *ut* $X\gamma$ *ad* 2BC.

Nam de æqualibus in Lemmate tertio, nimirum $2BC \times AX = X\gamma q - 2AF \times X\gamma$, ſubducantur æqualia in *Lemmate primo*, & reſtabunt æqualia $-2AB \times AX + AXq = 2FI \times X\gamma - 2AG \times FI$, hoc eſt AX in $AX - 2AB = 2FI$ in $X\gamma - AG$. Æqualium vero rectangulorum proportionalia ſunt latera 2FI ad $AX - 2AB$ ut AX ad $X\gamma - AG$, hoc eſt (per *Lemma tertium*) ut $X\gamma$ ad 2BC. Q. E. D.

Præſtratis his Lemmatibus, Conſtructio Problematis ſic tandem demonſtratur.

Per *Lemma quartum* eſt $X\gamma$ ad 2BC ut 2FI ad $\overline{AX - 2AB}$, hoc eſt (per *Prop.* 1. *lib.* VI. *Elem.*) ut $2BC \times 2FI$ ad $2BC \times \overline{AX - 2AB}$, ſeu ad $2BC \times AX - 2BC \times 2AB$. Sed per *Lemma tertium* eſt AX ad $X\gamma - 2AF$ ut $X\gamma$ ad 2BC, ſeu $2BC \times AX = X\gamma q - 2AF \times X\gamma$, adeoque $X\gamma$ eſt ad 2BC ut $2BC \times 2FI$ ad $X\gamma q - 2AF \times X\gamma - 2BC \times 2AB$. Et ductis extremis & mediis in ſe, fit $X\gamma$ *cub.* $- 2AF \times X\gamma q - 4BC \times AB \times X\gamma = 8BCq \times FI$. Addantur utrinque $2AF \times X\gamma q + 4BC \times AB \times X\gamma$, & fiet $X\gamma$ *cub.* $= 2AF \times X\gamma q + 4BC \times AB \times X\gamma + 8BCq \times FI$. Erat utem in conſtructione demonſtrandâ, $\frac{1}{2}X\gamma$ radix æquationis dicta x, nec non $AF = p$, $BC = n$, $AB = \frac{q}{n}$, & $FI = \frac{r}{nn}$, adeoque $BC \times AB = q$. Et $BCq \times FI = r$. Quibus ſubſtitutis fiet $x^3 = px^2 + qx + r$. Q. E. D.

Corol. Hinc ſi AF & AB ponantur nulla, per *Lemma* tertium & quartum fiet 2FI ad AX ut AX ad $X\gamma$ & $X\gamma$ ad 2BC. Unde conſtat inventio duarum mediè proportionalium inter datas quaſlibet FI & BC.

Scholium.

Appendix. *Scholium.* Hactenus æquationis cubicæ conſtructionem per Ellipſin ſolummodo expoſui: ſed regula ſuâ naturâ generalior eſt, ſeſe ad omnes coni ſectiones indifferenter extendens. Nam ſi loco Ellipſeos velis Hyperbolam adhiberi, cape lineas BC, BE ad contrarias partes puncti B, dein puncta A, F, G, I, H, K, L & R determinentur ut antè; excepto tantùm quòd FH debet ſumi ad partes ipſius F contra I, & quòd HR non in lineâ HL, ſed in lineâ AI ad utramque partem puncti H capi debet, & vice rectæ GRS duæ aliæ rectæ à puncto L ad puncta duo R & R hinc inde duci pro aſymptotis Hyperbolæ. Cum iſtis itaque aſymptotis LR, LR deſcribe Hyperbolam per punctum G, ut & circulum centro K intervallo KG: & dimidia perpendiculorum ab eorum interſectionibus ad rectam AE demiſſorum erunt radices æquationis propoſitæ. Quæ omnia, ſignis + & − probè mutatis, demonſtrantur ut priùs.

Quòd ſi Parabolam velis adhiberi, abibit punctum E in infinitum, atque adeo nullibi capendum erit, & punctum H cum puncto F coincidet, eritque Parabola circa axem HL cum latere recto principali BC per puncta G & A deſcribenda, ſito vertice ad partes puncti F ad quas punctum B ſitum eſt reſpectu puncti C.

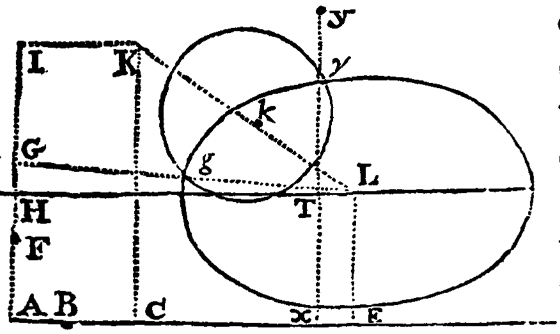
Sic ſunt conſtructiones per Parabolam, ſi ſimplicitatem analyticam ſpectes, ſimpliciſſimæ omnium. Eæ per Hyperbolam proximum locum obtinent, & ultimum locum tenent quæ per Ellipſin abſolvuntur. Quòd ſi praxeos manualis in deſcribendis figuris ſpectetur ſimplicitas, mutandus eſt ordo.

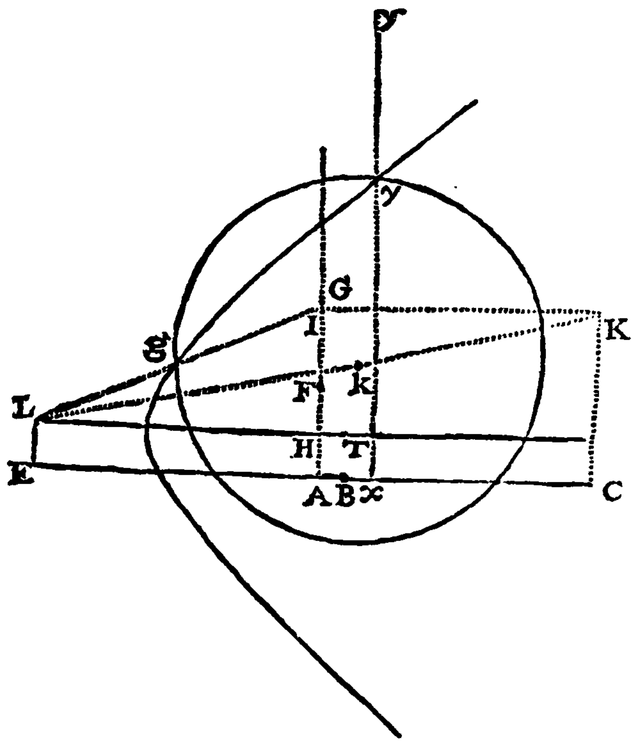
In hiſce autem conſtructionibus obſervandum venit, quòd proportione lateris recti principalis ad latus tranſverſum determinatur ſpecies Ellipſeos & Hyperbolæ; & proportio illa eadem eſt quæ linearum BC & BE, atque adeo aſſumi poteſt: Parabolæ verò ſpecies eſt unica quam artifex ponendo BE infinitè longam aſſequitur. Sic igitur penes artificem eſt, æquationem quamcunque cubicam per conicam ſectionem imperatæ ſpeciei conſtruere. A figuris autem ſpecie datis, ad figuras magnitudine datas devenietur, augendo vel diminuendo in ratione datâ lineas omnes, quibus figuræ ſpecie dabantur, atque ita æquationes omnes cubicas, per datam quamvis Conicam ſectionem, conſtruere licebit. Id quod ſic pleniùs explico.

Proponatur

Proponatur æquationem quamcunque cubicam $x^3 = pxx.\ qx.\ r.$ *ope datæ cujuſcunque ſectionis conicæ conſtruere.* Æquationum Conſtructiones.

A puncto quovis B in rectâ quâvis infinitâ BCE, cape duas quaſcunque longitudines BC, BE ad eaſdem partes ſi data Coni ſectio ſit Ellipſis, ad contrarias ſi ea ſit Hyperbola. Sit autem BC ad BE ut datæ ſectionis latus rectum principale ad latus tranſverſum; & BC nominatâ n, cape $BA = \frac{q}{n}$, idque verſus C ſi habeatur $-q$, aliter ad partes contrarias. Ad punctum A erige perpendiculum AI, inque eo cape AF æqualem p & FG æqualem AF; item FI æqualem $\frac{r}{nn}$. Capiatur verò FI verſus G ſi termini p & r habent eadem ſigna, aliter verſus A. Dein fac ut ſit FH ad FI ut BC ad BE, & hanc FH cape à puncto F verſus I ſi ſectio ſit Ellipſis, aut ad partes contrarias ſi ea ſit Hyperbola. Porro compleantur parallelogramma IACK & HAEL, & hæ omnes jam





deſcriptæ lineæ transferantur ad datam ſectionem Conicam, aut quod perinde eſt, his ſuperponatur curva, ita ut axis ejus ſive tranſverſa diameter principalis conveniat cum rectâ LH & centrum cum puncto L. His ita conſtitutis agatur recta KL ut & recta GL ſecans conicam ſectionem in g. In LK cape Lk quæ ſit ad LK ut Lg ad LG, centroque k & intervallo kg deſcribe circulum. A punctis ubi hic ſecuerit curvam impoſitam demitte perpendicula ad lineam LH, cujuſmodi ſit γT. Denique verſus γ, cape TY

APPENDIX. quæ ſit ad Tγ ut LG ad Lg, & hæc TY producta ſecet rectam AB in X, eritque recta $\frac{1}{2}$XY una ex radicibus æquationis. Sunt autem radices affirmativæ quæ jacent ad partes rectæ AB ad quas recta FI jacet à puncto F, & negativæ quæ jacent ad contrarias partes, ſi modò habeatur $+r$, & contrà ſi $-r$ obvenerit.

Hoc modo conſtruuntur æquationes cubicæ per Ellipſes & Hyperbolas datas: Quòd ſi detur Parabola, capienda eſt BC æqualis lateri recto ipſius. Dein punctis A, F, G, I & K inventis ut antè, centro K intervallo KG deſcribendus eſt circulus, & Parabola ita applicanda ad Schema jam deſcriptum (aut Schema ad Parabolam) ut ipſa tranſeat per puncta A & G, & axis ejus ipſi AC parallelus per punctum F, cadente vertice ad partes puncti illius F ad quas punctum B cadit à puncto C. His ita conſtitutis, ſi perpendicula ab ejus occurſibus cum circulo demittantur ad lineam BC, eorum dimidia erunt radices æquationis conſtruendæ.

Et notes quòd ubi ſecundus æquationis terminus deeſt, & latus rectum Parabolæ ponitur numerus binarius, hæc conſtructio evadet eadem cum illâ quam Carteſius attulit in Geometriâ ſuâ, præterquam quòd lineamenta hîc ſunt illorum duplicia.

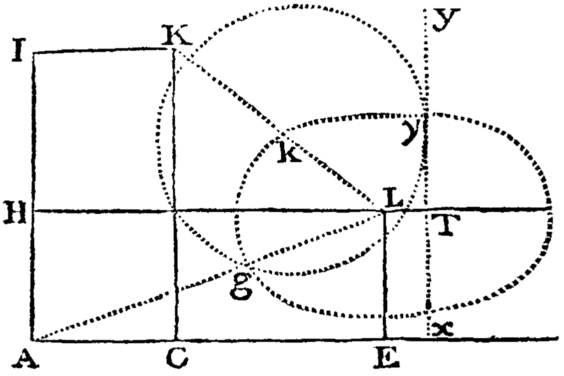
Hæc eſt conſtructionum regula generalis. Verùm ubi problemata particularia proponuntur, conſulendum eſt conſtructionum formulis ſimpliciſſimis. Libera enim manet quantitas n, cujus aſſumptione conſtructio plerumque ſimplicior reddi poteſt. Ejus rei exemplum unum ſubjungo.

Detur Ellipſis, & inter datas lineas a & b inveniendæ ſint duæ mediæ proportionales. Sit earum prima x, & $a. x. \frac{xx}{a}. b$ erunt continuè proportionales, adeoque $ab = \frac{x^3}{a}$, ſeu $x^3=aab$, æquatio eſt quam conſtruere oportet. Hîc deſunt termini p, & q, & terminus r eſt aab, adeoque BA & AF nullæ ſunt, FI & eſt $\frac{aab}{nn}$. Ut terminus noviſſimus evadat ſimplicior aſſumatur $n=a$, & fiet $\text{FI}=b$. Deinde conſtructio ita ſe habebit.

A puncto quovis A in rectâ quâvis infinitâ AE cape $\text{AC}=a$, & ad eaſdem partes puncti A cape AC ad AE ut eſt Ellipſeos latus rectum principale ad latus tranſverſum. Tum in perpendiculo

AI

AI cape AI $= b$, & AH ad AI ut eſt AC ad AE. Compleantur parallelogramma IACK, HAEL. Jungantur LA, LK. Huic ſchemati imponatur Ellipſis data. Secet ea rectam AL in puncto g, Fiat *Æquationum Conſtructiones.*



Lk ad LK ut Lg ad LA. Centro k intervallo kg deſcribatur circulus ſecans Ellipſin in γ. Ad AE demittatur perpendiculum γX ſecans HL in T, & producatur id ad Y ut ſit TY ad Tγ ſicut LA ad Lg. Sic fiet $\frac{1}{2}$XY prima duarum mediè proportionalium x. Q. E. I.

FINIS ARITHMETICÆ UNIVERSALIS.

TOMI PRIMI

PARS SECUNDA,

IN QUA CONTINENTUR

TRACTATUS OMNES

AD

FUNDAMENTA GEOMETRIÆ SUBLIMIORIS

PERTINENTES.

MONITUM EDITORIS.

CUM in eâ diu fuerim opinione, fore ut ſublimioris Geometriæ ſcientia faciliùs atque meliùs ex ipſius Newtoni libris quàm aliunde ferè haurienda eſſet, modò ea ſuppleta eſſent quæ Newtonus reliquit imperfecta, et pauca quædam accuratiùs expoſita, quæ ille breviùs dixit quàm ut diſcentibus perſpicua eſſent; nonnulla Newtonianis adjungere decrevimus, quæ tironibus viam complanare poſſint, modò ipſi à Matheſi elementariâ ſatis inſtructi ad hæc ſublimiora accedant. Exigit igitur inſtituti ratio, ut quo ordine hæc omnia ſint legenda, ſi quis tempore breviſſimo minimoque labore fructus ſtudiorum maximos percipere cupiat, breviter exponam.

CUI *igitur nova ſit quæ in hiſce traditur diſciplina, is omnium primum Tractatum ipſius Newtoni de Rationibus Primis Ultimiſque diligenter velim perlegat.*

2. *Dein noſtram Fluxionum Geometriam.*

3. *Newtoni Introductionem ad Quadraturam Curvarum, et libri ipſius de Quadraturis initium, abſolutam uſque ſecundi Problematis enodationem.*

4. *Analyſin per æquationes numero terminorum infinitas, collatis Geometriæ Analyticæ locis cognatis, ad quæ notationes noſtræ legentes ablegant.*

5. *Noſtram Infinitorum Logiſticam.*

6. *Excerpta ex Epiſtolis Newtoni ad Series Infinitas Fluxioneſque pertinentia.*

7. *Newtoni Geometriam Analyticam, ab initio doctrinæ fluxionalis abſolutum uſque nonum Problema.*

8. *Librum de Quadraturâ Curvarum, ab initio reſumptum, finem uſque*; *conferendo Geometriæ Analyticæ loca cognata.*

9. *Methodum Differentialem.*

10. *Enumerationem linearum tertii ordinis. Nos verò in condendis demonſtrationibus noſtris, eam nobis legem tulimus, ut non aliis unquam tanquam cognitis uteremur, præter ea quæ non cognoviſſe lectori non licebit, qui ordinem legendi à nobis præſcriptum obſervaverit.*

DE

RATIONIBUS

PRIMIS ULTIMISQUE

E

LIBRO PRIMO PRINCIPIORUM.

D E

R A T I O N I B U S

P R I M I S U L T I M I S Q U E

E

L I B R O P R I M O P R I N C I P I O R U M.

L E M M A I.

Quantitates, ut & quantitatum rationes, quæ ad æqualitatem tempore quovis finito conſtanter tendunt, & ante finem temporis illius propiùs ad invicem accedunt, quàm pro datâ quâvis differentiâ, fiunt ultimò æquales.

SI negas; fiant ultimò inæquales, & ſit earum ultima differentia D. Ergo nequeunt propiùs ad æqualitatem accedere quàm pro datâ differentiâ D: contra hypotheſin ([a]).

L E M M A

([a]) Siquis ea quæ ſub Lemmatis Primi titulo hîc proferuntur, pro definitione potiùs haberi velit, quâ nimirum conſtituit Newtonus quidnam ſit ultima illa, quam ſæpè adeo locuturus eſt, ſive magnitudinum ſive rationum æqualitas, is ſententiæ ſuæ me certè haud magnoperè refragantem habiturus eſt. Quid enim? Si duarum magnitudinum, A, B, quarum illam A minorem ponamus, harum ſi major B fixa maneat, minor A ſenſim increſcat, eâ quidem lege, ut licèt nunquam ea major evadat quàm B, intra certum tamen tempus minùs abiit à B quàm pro differentiâ ullâ quam minimam quis poſuerit, quidni habeatur illa A alteri B ultimò æqualis? Cùm dari nequeat magnitudo ulla illâ B vel tantillo minor, quam A ultimò non exuperabit; cùm tamen ipſam B nunquam ei exuperare liceat. Vel ſi, manente magnitudine minore A, major B ſenſim minuatur, eâ quidem lege, ut licet nunquam ea minor fiat quàm A, intra certum tamen tempus minùs exuperet A, quàm pro differentiâ ullâ quam minimam quis poſuerit, quidni habeatur illa B alteri A ultimò æqualis? Cùm dari nequeat magnitudo ulla illâ A vel tantillo major, quâ altera B ultimò minor non fiat; cùm tamen ipſa A nunquam ea fiat minor.

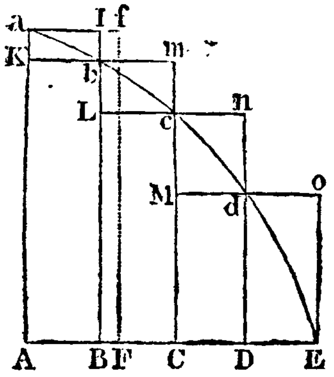


Rurſum duæ magnitudines, A, B, increſcendo vel decreſcendo quovis modo mutationes ſubeant. Sint datæ duæ, C, D; ſitque E magnitudo cui mutabilis B ultimò æqualis fiat. Sit etiam alia F, quæ habeat ad E rationem eam quam C ad D. Jam ſi magnitudini mutabili, A, is ſit mutationum

De Rationibus

LEMMA II.

Si in figurâ quâvis AacE, *rectis* Aa, AE *&* *curvâ* acE *comprehenſâ, inſcribantur parallelogramma quotcunque,* Ab, Bc, Cd, *&c. ſub baſibus* AB, BC, CD, *&c. æqualibus, & lateribus* Bb, Cc, Dd, &c. *figuræ lateri* Aa *parallelis contenta; & compleantur parallelogramma* aKbl, bLcm, cMdn, &c. *Dein horum parallelogrammorum latitudo minuatur, & numerus augeatur in infinitum: dico quòd ultimæ rationes, quas habent ad ſe invicem figura inſcripta* AKbLcMdD, *circumſcripta* AalbmcndoE, *& curvilinea* AabcdE, *ſunt rationes æqualitatis.*

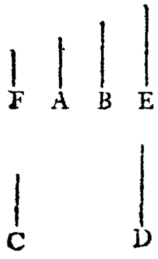


Nam figuræ inſcriptæ & circumſcriptæ differentia eſt ſumma parallelogrammorum Kl, Lm, Mn, Do, hoc eſt (ob æquales omnium baſes) rectangulum ſub unius baſi Kb & altitudinum ſummâ Aa, id eſt, rectangulum ABla. Sed hoc rectangulum, eò quòd latitudo ejus AB in infinitum minuitur, fit minus quovis dato. Ergo (per Lemma I) figura inſcripta & circumſcripta, & multo magis figura curvilinea intermedia, fiunt ultimò æquales. Q. E. D.

LEMMA III.

Eædem rationes ultimæ ſunt etiam rationes æqualitatis, ubi parallelogrammorum latitudines AB, BC, CD, &c. *ſunt inæquales, & omnes minuuntur in infinitum.*

Sit enim AF æqualis latitudini maximæ, & compleatur parallelogrammum FAaf. Hoc erit majus quàm differentia figuræ inſcriptæ



tionum ſuarum omnium terminus atque finis, ut illi F ultimò æqualis fiat, his poſitis, quidni ratio illius C ad D, magnitudinis A ad F ultima dicatur ratio? Vel quidni dicantur rationes illæ, C ad D, A ad B, ultimò æquales? Siquidem illa A cuivis magnitudini ultimò inæqualis fiat, quæ ad ultimam ipſius B magnitudinem rationem habeat aliam atque C habet ad D. Si neges, inquit Newtonus, magnitudines, A, B, in caſu primo habendas eſſe ultimò æquales, neceſſe eſt contrarium velis; eas habendas eſſe ultimò inæquales. Si hoc velis, neceſſe erit aliquam inter eas ponas differentiam ultimam: nam inæqualium eſt ſemper aliquid quo alterum ab altero abſit: neque propiùs quam pro differentiâ illâ, quam ultimam habituræ ſunt, magnitudines ultimò inæquales accedere poterunt. Hoc autem contrarium eſt legi mutationum à nobis conſtitutæ. Haud igitur inæquales ultimò habendæ ſunt magnitudines, quarum mutationes legi, quam tulimus, obtemperant. Si non inæquales,

ſcriptæ & figuræ circumſcriptæ; at latitudine ſuâ AF in infinitum diminutâ, minus fiet dato quovis rectangulo. Q. E. D. PRIMIS ULTIMISQUE.

Corol. 1. Hinc ſumma ultima parallelogrammorum evaneſcentium coincidit omni ex parte cum figurâ curvilineâ.

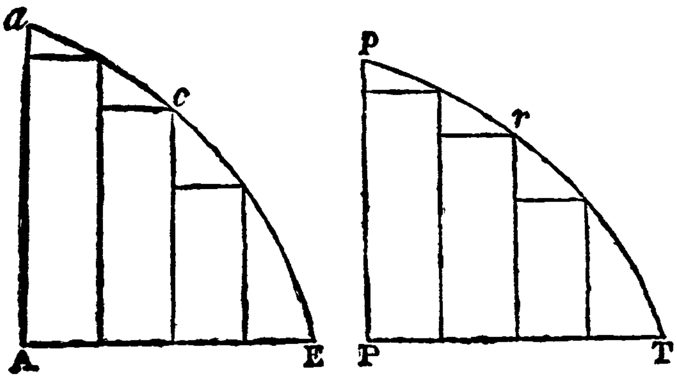
Corol. 2. Et multo magis figura rectilinea, quæ chordis evaneſcentium arcuum *ab*, *bc*, *cd*, &c. comprehenditur, coincidit ultimò cum figurâ curvilineâ.

Corol. 3. Ut & figura rectilinea circumſcripta, quæ tangentibus eorundem arcuum comprehenditur.

Corol. 4. Et propterea hæ figuræ ultimæ (quoad perimetros *ac*E,) non ſunt rectilineæ, ſed rectilinearum limites curvilinei.

LEMMA IV.

Si in duabus figuris A*ac*E, P*pr*T, *inſcribantur (ut ſuprà) duæ parallelogrammorum ſeries, ſitque idem amborum numerus, & ubi latitudines in infinitum diminuuntur, rationes ultimæ parallelogrammorum in unâ figurâ ad parallelogramma in alterâ, ſingulorum ad ſingula, ſint eædem; dico quòd figuræ duæ* A*ac*E, P*pr*T, *ſunt ad invicem in eâdem illâ ratione.*



Etenim ut ſunt parallelogramma ſingula ad ſingula, ita (componendo) fit ſumma omnium ad ſummam omnium, & ita figura ad figuram; exiſtente nimirum figurâ priore (per lem-

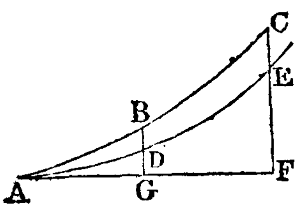
quales, æquales certè. Namque Æqualis & Inæqualis haud ſanè novimus intermedium quid, cui nec æquali nec inæquali eſſe liceat. Jam verò ſi magnitudines illæ A, B jure optimo cenſeantur ultimò æquales, rationes etiam, ſecundum eandem legem noſtram mutabiles, nihilo ineptiùs ultimò æquales dici, id etiam invictis argumentis obtinebimus. Quid enim? Cùm in caſu ſecundo magnitudinum mutabilium, A, B, datæ, F, E, ſint ultimæ magnitudines, hoc eſt cùm A fiat ultimò æqualis illi F, necnon B ultimò æqualis illi E, nonne dicendum A eſſe ultimò ad F ut B ultimò ad E. Id verò ſi rectè dicitur, nonne Euclides ipſe dicere jubebit, eſſe permutando A ultimò ad B ſicut F ad E. Quòd ſi illud etiam rectè dicitur, cùm ſit præterea F ad E ut C ad D (id enim poſuimus) nonne iterum ex Euclidis mente dicendum erit, eſſe A ad B ultimò ſicut C ad D, ſive rationes, A ad B, C ad D, eſſe ultimò æquales. Evicimus igitur, inquit Newtonus, magnitudines rationeſque mutabiles, quarum mutationes legem à nobis conſtitutam ſervant, habendas eſſe ultimò æquales.

De Rationibus

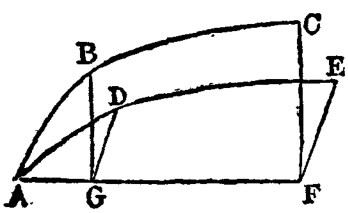
ma III) ad ſummam priorem, & figurâ poſteriore ad ſummam poſteriorem in ratione æqualitatis. Q. E. D ([b]).

Corol. Hinc ſi duæ cujuſcunque generis quantitates in eundem partium numerum utcunque dividantur; & partes illæ, ubi numerus earum augetur & magnitudo diminuitur in infinitum, datam obtineant rationem ad invicem, prima ad primam, ſecunda ad ſecundam, cæteræque ſuo ordine ad cæteras: erunt tota ad invicem in eâdem illâ datâ ratione. Nam ſi in lemmatis hujus figuris ſumantur parallelogramma inter ſe ut partes, ſummæ partium

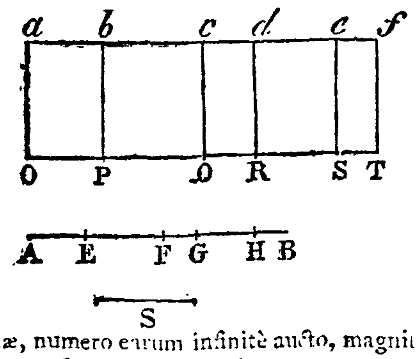
([b]) *Cor. H.* 1. Si duæ curvæ (ABC, ADE) axem habeant communem (AF) hujus autem curvæ ordinatæ ad ordinatas alterius datam aliquam rationem gerant, binas utique ſemper conferendo, quæ ex eodem axis puncto eductæ ſint, areæ curvarum AFC, AFE, datam ordinatarum inter ſe rationem habebunt. Nempe hoc dico; à puncto G in axe curvarum, AF, pro lubitu aſſumpto eductâ ad perpendiculum rectâ GB, quæ curvis duabus, ABC, ADE, illi in puncto B, huic in puncto D, occurrat, ſi rectis BG, DG, data aliqua ratio intercedat, quæ eadem obtineat à quocunque demum axis puncto ordinata educatur, eadem arearum etiam, AFC, AFE, inter ipſas ratio erit.

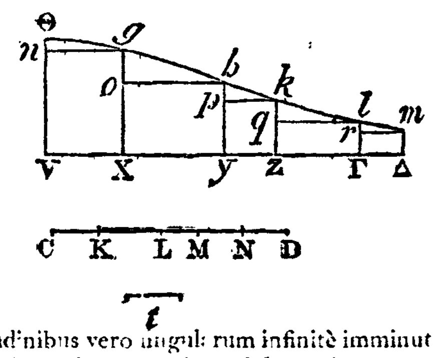


Cor. H. 2. Quòd ſi recta GB non ad perpendiculum, ſed cum datâ quâvis ad axem communem AF inclinatione, educatur, idem obtinebit.



Cor. H. 3. Si verò curvæ duæ, ABC, ADE, eo modo ad ſe mutuò ſint affectæ, ut rectæ, GB, GD, à puncto aliquo G in axe communi, AF, ordinatim eductæ, datis quidem ſed diverſis ad axem illam inclinationibus, datam inter ſe rationem gerant; quæ eadem obtineat, à quocunque demum axis puncto ordinatæ illæ educantur; vel ſic etiam areis ABCF, ADEF, data quædam ratio intercedet; ea nimirum, quam rectæ à punctis B, D in axem AF ad perpendiculum demiſſæ inter ſe habent.





([c]) Sint duæ quantitates AB, CD cujuſcunque generis quæ in partes quotcunque dividantur, AE, EF, FG, GH, HB; CK, KL, LM, MN, ND; quæ partes totidem in hàc et in illâ ſint. Si partes illæ, numero earum infinitè aucto, magnitudinibus vero ſingulorum infinitè imminutis, prima ad primam, ſecunda ad ſecundam, cæteræque ordine ad cæteras, hoc eſt ſi AE ad CK, EF ad KL, FG ad LM, GH ad MN, HB ad ND, datam inter ſe rationem habeant; hoc ſi fiat, dicit Newtonus inter quantitates ipſas, AB, CD, eandem rationem datam obtinere, quæ partium evaneſcentium communis eſt.

DEMONSTRATIO AD MENTEM NEWTONI.

Exponatur recta OP datæ cujuſlibet longitudinis; et in eâdem rectâ, infinitè productâ, capiatur PQ, ad quam OP rationem habeat eandem ac AE ad EF; et QR, ad quam PQ, rationem habeat quam

EF

partium ſemper erunt ut ſummæ parallelogrammorum; atque ideo, ubi partium & parallelogrammorum numerus augetur & magnitudo diminuitur in infinitum, in ultimâ ratione parallelogrammi ad parallelogrammum, id eſt (per hypotheſin) in ultimâ ratione partis ad partem (c). Primis Ultimisque.

LEMMA V.

Similium figurarum latera omnia, quæ ſibi mutuò reſpondent, ſunt proportionalia, tam curvilinea quàm rectilinea; & areæ ſunt in duplicatâ ratione laterum.

EF ad FG; et RS, ad quam QR rationem habeat quam FG ad GH; et ST, ad quam RS rationem habeat quam GH ad HB. Rectam OT ad perpendiculum inſiſtat *oa* datæ cujuſlibet longitudinis, et rectangulum *oaf*T compleatur; cujus cum lateribus parallelæ ducantur P*b*, Q*c*, R*d*, S*e*. Exponatur alia recta VΔ rectæ OT æqualis, in quâ capiantur partes, VX, X*y*, *y*Z, ZΓ, ΓΔ, partibus OP, PQ, QR, RS, ST, ordine æquales. Rectam VΔ ad puncta V, X, *y*, Z, Γ, Δ, rectæ V*n*, X*g*, *yh*, Z*k*, Γ*l*, Δ*m*, ad perpendiculum inſiſtant.

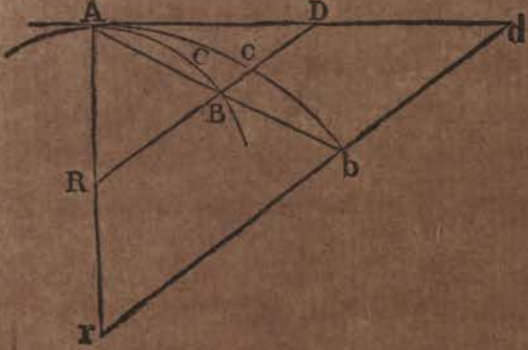
Capiantur X*g*, *yh*, Z*k*, Γ*l*, Δ*m*, ad quas *oa* rationes habeat quas AE ad CK, EF ad KL, FG ad LM, GH ad MN, HB ad ND; et rectangula V*g*, X*h*, *yk*, Z*l*, Γ*m* compleantur. Jam cùm OP ſit ad PQ ut AE ad EF, componendo erit OQ : PQ = AF : FE. Sed PQ : QR = EF : FG. Id enim factum eſt. Igitur ex æquo OQ : QR = AF : FG, et componendo, OR : QR = AG : GF. Sed, QR : RS = FG : GH. Igitur ex æquo, OR : RS = AG : GH, et componendo OS : SR = AH : HG. Sed RS : ST = GH : HB. Igitur ex æquo OS : ST = AH : HB, et componendo OT : TS = AB : BH. Atque hinc colligitur rectam OT eſſe ad partes ſuas, OP, PQ, QR, RS, ST, ſicut quantitas AB ad partes ſuas, AE, EF, FG, GH, HB, ordine ſumendas. Et cùm parallelogramma ejuſdem altitudinis ſint inter ſe ut baſes, rectangula *ob*, P*c*, Q*d*, R*e*, S*f* erunt inter ſe et ad rectangulum totum *of*, ut AE, EF, FG, GH, HB, partes quantitatis AB, inter ſe et ad totam. Atqui propter rectas VX, OP æquales, rectangula V*g*, *ob*, ſunt inter ſe ut rectæ X*g*, P*b*; et X*g*, P*b*, ſunt ut CK, AE; ita enim factæ ſunt. Rectangulum igitur V*g* ad *ob* ut CK ad AE. Et ex modò oſtenſis, *ob* : *of* = AE : AB. Quare ex æquo erit V*g*, *of* ut CK ad AB. Rurſum propter æquales X*y*, PQ, erit rectangulum *yo* ad Q*b* ut *yh* ad P*b*, id eſt ut KL ad EF. Et ex modò oſtenſis, Q*b* : *of* = EF : AB. Quare ex æquo, *yo* : *of* = KL : AB. Sed jam oſtenſum eſt X*n* eſſe ad P*a* ut CK ad AE. Ergo duo rectangula *yo*, X*n* ſimul ſumpta erunt ad rectangulum *of*, ut CK, KL ſimul ſumptæ ad AB (per 24. 5. Elem.) Rurſum propter æquales *y*Z, QR, rectangulum Z*p* : R*c* = Z*k* : Q*c* = LM : FG. Et ex modò oſtenſis, RC : *of* = FG : AB. Quare ex æquo Z*p* : *of* = LM : AB. Tria igitur rectangula ſimul ſumpta Z*p*, *yo*, X*n* erunt ad *of*, ut tres ML, LK, CK ſimul ſumptæ, hoc eſt ut CL ad AB (per 24. 5. Elem.) Atque eandem eundo argumentationis viam eò tandem pervenietur, eſſe omnia rectangula Δ*r*, Γ*q*, Z*p*, *yo*, X*n* ſimul ſumpta ad rectangulum *of* ſicut quantitas CD ad quantitatem AB. Atque hæc ita erunt quotcunque fuerint rectangula illa, et quantacunque fuerit cujuſque magnitudo. Quare et figura Δ*v*Θ*m*, cui rectangulorum illorum omnium ſumma, numero eorum infinitè aucto, magnitudinibus ſingulorum infinitè imminutis, ultimò fit æqualis, hæc etiam erit ad rectangulum *of* ſicut CD ad AB.

Sit jam ratio illa data, quæ ultima eſt evaneſcentium partium quantitatis AB ad partes quantitatis CD ſimul evaneſcentes, ea quam recta *s* habet ad rectam *t*.

Jam numero partium quantitatum, AB, CD, infinitè creſcente, magnitudinibus ſingularum infinitè decreſcentibus, rectangulorum *ob*, P*c*, Q*d*, R*e*, S*f* necnon V*g*, X*h*, *yk*, Z*l*, Γ*m* numerum infinitè pariter augeri, magnitudines ſingularum infinitè pariter diminui, ex conſtructione et è demonſtratis ſatis patet. Cùm vero recta *oa* ſemper ſit ad rectam X*g* ut AE ad CK (id enim poſuimus) et cùm AE fiat ultimò ad CK, ut *s* ad *t*, erit *oa* ad X*g* ultimò ut *s* ad *t*. Sed ut *oa* ad X*g* ita ſemper erit rectangulum *ob* ad rectangulum V*g*, propter OP, VX ſemper æquales. Ergo *ob* ad V*g* ultimò ut *s* ad *t*. Ac ſimili ratione rectangula reliqua P*c*, Q*d*, R*e*, S*f* ad reliqua X*h*, *yk*, Z*l*, Γ*m*, ordine ſumenda, fient ultimò ut *s* ad *t*. Ergo ut *s* ad *t* ita erit rectangulum *of* ad figuram Θ*v*Δ*m* (per Lemma). Sed ex ſuprà oſtenſis rectangulum *of* ad figuram Θ*v*Δ*m* ut AB ad CD. Habet igitur AB ad CD rationem eam quam *s* ad *t*; hoc eſt, datam illam, quæ partium evaneſcentium ultimò ſit communis. Q. E. D.

De Rationibus

LEMMA VI.



Si arcus quilibet poſitione datus, ACB, *ſubtendatur chordâ* AB, *& in puncto aliquo* A, *in medio curvaturæ continuæ* (d), *tangatur à rectâ utrinque productâ* AD; *dein puncta* A, B *ad invicem accedant, & coeant; dico, quòd angulus* BAD, *ſub chordâ & tangente contentus, minuetur in infinitum, & ultimò evaneſcet.*

Nam ſi angulus ille non evaneſcit, continebit arcus ACB cum tangente AD angulum rectilineo æqualem; & propterea curvatura ad punctum A non erit continua, contra hypotheſin.

LEMMA VII.

Iiſdem poſitis; dico quòd ultima ratio arcûs, chordæ, & tangentis ad invicem eſt ratio æqualitatis.

Nam dum punctum B ad punctum A accedit, intelligantur ſemper AB & AD ad puncta longinqua *b* ac *d* produci, & ſecanti BD parallela agatur *bd*. Sitque arcus A*cb* ſemper ſimilis arcui ACB. Et punctis A, B coeuntibus, angulus *d*A*b*, per lemma ſuperius, evaneſcet; ideoque rectæ ſemper finitæ A*b*, A*d*, & arcus intermedius A*cb* coincident, & propterea æquales erunt. Unde & hiſce ſemper proportionales rectæ AB, AD, & arcus intermedius ACB evaneſcent, & rationem ultimam habebunt æqualitatis. Q. E. D.

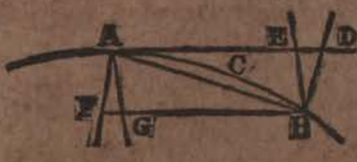
Corol. 1. Unde ſi per B ducatur tangenti parallela BF, rectam quamvis AF per A tranſeuntem perpetuò ſecans in F, hæc BF ultimò

DEFINITIO.



(d) Curvæ cujuſpiam, EAC, curvatura in puncto quovis, A, *continua* dicitur, ſi recta quævis, FA, quæ curvæ in puncto illo occurrat, angulos hinc inde faciat, FAE, FAC, quorum ſimul ſumptorum ſumma duos rectos propiùs quàm pro datâ quâvis differentiâ accedat.

(*) Si



ultimò ad arcum evaneſcentem ACB rationem habebit æqualitatis; eo quòd completo parallelogrammo AFDB rationem ſemper habet æqualitatis ad AD.

PRIMIS ULTIMISQUE.

Corol. 2. Et ſi per B & A ducantur plures rectæ BE, BD, AF, AG, ſecantes tangentem AD & ipſius paralellam BF; ratio ultima abſciſſarum omnium AD, AE, BF, BG, chordæque & arcûs AB ad invicem erit ratio æqualitatis.

Corol. 3. Et propterea hæ omnes lineæ, in omni de rationibus ultimis argumentatione, pro ſe invicem uſurpari poſſunt.

LEMMA VIII.

Si rectæ datæ AR, BR (e) *cum arcu* ACB, *chordâ* AB, *& tangente* AD, *triangula tria* RAB, RACB, RAD *conſtituunt, dein puncta* A, B *accedunt ad invicem: dico, quòd ultima forma triangulorum evaneſcentium eſt ſimilitudinis, & ultima ratio æqualitatis. (Vide fig. Lemma* VI.)

Nam dum punctum B ad punctum A accedit, intelligantur ſemper AB, AD, AR ad puncta longinqua *b*, *d* & *r* produci, ipſique RD parallela agi *rbd*, & arcui ACB ſimilis ſemper ſit arcus A*cb*. Et coeuntibus punctis A, B, angulus *b*A*d* evaneſcet, & propterea triangula tria ſemper finita *r*A*b*, *r*A*cb*, *r*A*d* coincident, ſuntque eo nomine ſimilia & æqualia. Unde & hiſce ſemper ſimilia & propoıtionalia RAB, RACB, RAD fient ultimò ſibi invicem ſimilia & æqualia. Q. E. D.

Corol. Et hinc triangula illa, in omni de rationibus ultimis argumentatione, pro ſe invicem uſurpari poſſunt.

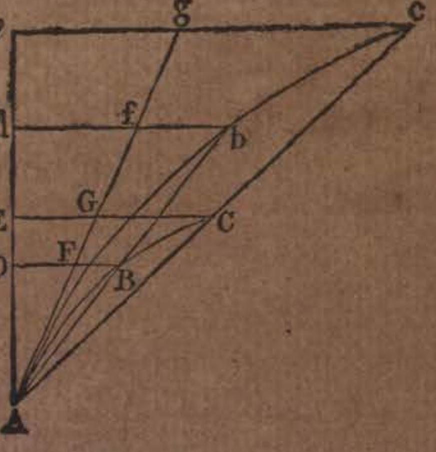
LEMMA IX.

Si recta AE *& curva* ABC *poſitione datæ ſe mutuò ſecent in angulo dato,* A, *& ad rectam illam in alio dato angulo ordinatim applicentur* BD, CE, *curvæ occurrentes in* B, C, *dein puncta* B, C *ſimul*

(e) Intellige rectam AR poſitione datam; rectam verò BR eâ lege mobilem, ut cum aliâ aliquâ, poſitione datâ, ſemper ea parallela maneat.

DE RATIONIBUS *accedant ad punctum* A: *dico, quòd areæ triangulorum* ABD, ACE *erunt ultimò ad invicem in duplicatâ ratione laterum.*

Etenim dum puncta B, C accedunt ad punctum A, intelligatur ſemper AD produci ad puncta longinqua *d* & *e*; ut ſint A*d*, A*e* ipſis AD, AE proportionales, & erigantur ordinatæ *db*, *ec* ordinatis DB, EC parallelæ, quæ occurrant ipſis AB, AC productis in *b* & *c*. Duci intelligatur, tum curva A*bc* ipſi ABC ſimilis, tum recta A*g*, quæ tangat curvam utramque in A, & ſecet ordinatam applicatas DB, EC, *db*, *ec* in F, G, *f*, *g*. Tum, manente longitudine A*e*, coeant puncta B, C cum puncta A; &, angulo *c*A*g* evaneſcente, coincident areæ curvilineæ A*bd*, A*ce*

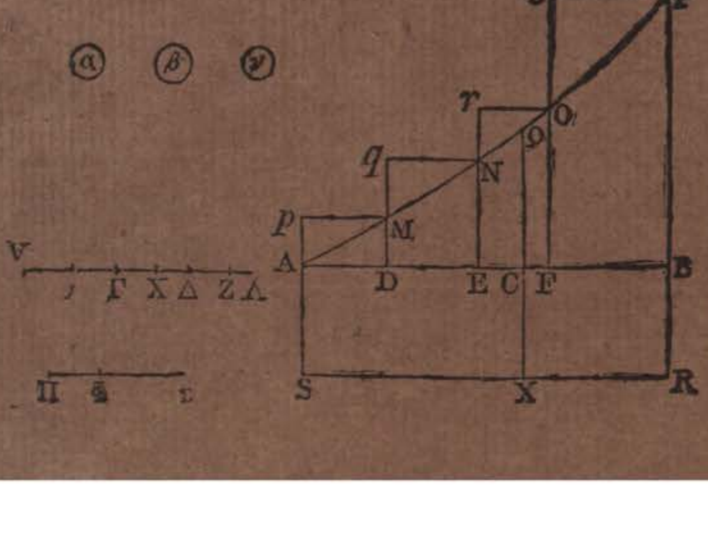


(f) Ideoque (per Lemma 5.) erunt, &c.

Hæc ſic emendarem, " ideòque erunt in duplicatâ ratione laterum A*d*, A*e*. Sed his areis proportionales ſemper ſunt areæ ABD, ACE, (per Lemma 5.) et his lateribus latera AD, AE. Ergo &c."

Nimirum areas curvilineares A*bd*, A*ce*, cum rectilinearibus A*fd*, A*ge* ultimò congruentes, habere inter ſe ultimò rationem laterum A*d*, A*e* duplicatam ſatis patet ex ſimilitudine triangulorum A*fd*, A*ge*, et ex propoſitione 19 Libri 6 Elementorum Euclidis. Et arearum curvilinearium A*bd*, A*ce*, ABD, ACE conſtans analogia è lemmate quinto ad hunc modum colligenda eſt. Areæ curvilineæ ABD, A*bd* inter ſe ſunt ſimiles (id enim ponitur). Ergo $ABD : Abd = AD^2 : Ad^2$ (per Lemma 5.) Simili ratione $ACE : Ace = AE^2 : Ae^2$, (per Lemma 5^m.) Sed $AD^2 : Ad^2 = AE^2 : Ae^2$; nimirum cùm rectæ quatuor AD, A*d*, AE, A*e* proportione inter ſe conveniant. Ergo $ABD : Abd = ACE : Ace$ (per V. 11. El.) Permutando $ABD : ACE = Abd : Ace$.

T H E O R E M A.



(g) Exponatur recta AB datæ cujuſvis longitudinis, quæ in puncto C ita dividatur, ut pars AC ſit ad totam AB, ut tempus quoddam *t* ad tempus majus T. Jam dividatur recta AB in partes quotvis, AD, DE, EF, FB; quæ ſint inter ſe et ad totam AB ut temporis T portiones totidem, ordine ſumendæ, inter ſe & ad totum tempus T. Rectam AB in punctis D, E, C, F, B, rectæ, DM, EN, FO, BP, ad perpendiculum inſiſtant; capianturque DM, EN, FO, BP, quæ ſint inter ſe ſicut velocitates, quas corpus quodpiam

ACe cum rectilineis Afd, Age; ideoque (per Lemma v.) erunt in duplicatâ ratione laterum Ad, Ae: ſed his areis proportionales ſemper ſunt areæ ABD, ACE, & his lateribus latera AD, AE (f). Ergo & areæ ABD, ACE ſunt ultimò in duplicatâ ratione laterum AD, AE. Q. E. D. PRIMIS ULTIMISQUE.

LEMMA X.

Spatia, quæ corpus urgente quâcunque vi finitâ deſcribit, ſive vis illa determinata & immutabilis ſit, ſive eadem continuò augeatur vel continuò diminuatur, ſunt ipſo motûs initio in duplicatâ ratione temporum.

Exponantur tempora per lineas AD, AE, & velocitates genitæ per ordinatas DB, EC; & ſpatia his velocitatibus deſcripta, erunt ut areæ ABD, ACE his ordinatis deſcriptæ (g), hoc eſt, ipſo motûs initio

quodpiam α, urgente vi quâpiam finitâ utcunque mutabili, temporibus AD, AE, AF, AB adeptum fuerit. Idque ſemper fieri intelligatur, dum partium AD, DE, EF, FB numerus infinitè augeatur magnitudinibus ſingularum infinitè decreſcentibus: ductâque curvâ, quæ per omnium perpendiculorum puncta extrema A, M, N, O, P, tranſeat, et a puncto C eductâ ad perpendiculum rectâ CQ, quæ curvæ in Q occurrat, ſpatia quæ corpus α urgente vi illâ temporibus t, T tranſiverit, arearum ACQ, ABP inter ſe proportionem gerent.

Sint enim VX, VZ ſpatia, quæ corpus α urgente vi illâ temporibus t, T tranſivit. Rectam AB in puncto A recta datæ longitudinis, AS, ad perpendiculum inſiſtat. Puta corpus quoddam, β, toto tempore T velocitate æquabili latum, quæ ad velocitatem rectâ BP expoſitam datæ rectæ AS ad rectam BP rationem habeat; puta, inquam, corpus β, hâc velocitate latum, tempore T ſpatium ΠΣ tranſiiſſe. Capiantur rectæ, υy, yΓ, ΓΔ, ΔΛ, æquales ſpatiis quæ corpus quodlibet tertium γ temporibus AD, DE, EF, FB, velocitatibus DM, EN, FO, BP latum, deſcriberet. Hoc eſt corpori γ, in loco υ quieſcenti, puta ictu quodam ſubito velocitatem omnem DM ſimul imprimi; quâ corpus illud γ totum tempus AD curſu recto υΛ perſeverans ſpatium υy abſolvat. Sub fine autem temporis AD, ſimul ac corpus in locum y pervenerit, puta velocitatem primùm impreſſam, DM, novo ictu accedente, in velocitatem EN ſubito mutari. Et corpus novâ hâc velocitate totum tempus DE curſu recto perſeverans ſpatium yΓ conficiat. Et ſub fine temporis DE, ſimul ac corpus in locum Γ pervenerit, puta velocitatem EN novo rurſum ictu in aliam, FO, mutari. Atque idem ſemper fieri intelligatur; nimirum ut ſub fine temporis cujuſque ex illis AD, DE, EF, FB, quæ tempus totum AB, ſive T, conſtituunt, corporis γ velocitas in illam ſubito mutetur, quam corpus α ſub fine temporis proximè ſequentis acquiſierit: et velocitatibus hiſce diverſis, temporibus illis AD, DE, EF, FB ordine ſuccedentibus, corpus γ ſpatia υy, yΓ, ΓΔ, ΔΛ ordine conficiat; donec, conſumpto tandem toto tempore AB, in locum Λ pervenerit. Compleantur rectangula AR, Bs, Fr, Eq, Dp. Jam ſpatium υy habet ad ſpatium ΠΣ rationem compoſitam è rationibus temporis ad tempus et velocitatis ad velocitatem; hoc eſt rectæ AD ad AB, et DM ad AS, ſive rationem rectanguli AM ad rectangulum BS. Ac ſimili ratione ſpatium yΓ erit ad ſpatium ΠΣ ut rectangulum DN ad rectangulum BS. Ergo ſpatia υy, yΓ ſimul ſumpta erunt ad ſpatium ΠΣ ut rectangula AM, DN ſimul ſumpta ad rectangulum BS. (per 24. 5. Elem.) Rurſum ſpatium ΓΔ ad ſpatium ΠΣ ut rectangulum EO ad rectangulum BS. Ergo tria ſpatia υy, yΓ, ΓΔ, ſive ſpatium υΔ, ad ΠΣ, ut tria rectangula AM, DN, EO ſimul ſumpta ad rectangulum BS. Atque eâdem argumentationis viâ eò tandem perventum erit, eſſe omnia rectangula AM, DN, EO, FP, ſimul ſumpta ad rectangulum

De Rationibus initio (per Lemma IX) in duplicatâ ratione temporum AD, AE. Q. E. D.

Corol. 1. Et hinc facilè colligitur, quòd corporum, ſimiles ſimilium figurarum partes temporibus proportionalibus deſcribentium, errores, qui viribus quibuſvis æqualibus ad corpora ſimiliter applicatis generantur, & menſurantur per diſtantias corporum à figurarum ſimilium locis illis, ad quæ corpora eadem temporibus iiſdem proportionalibus, ſine viribus iſtis, pervenirent, ſunt ut quadrata temporum in quibus generantur quam proximè.

Corol. 2. Errores autem qui viribus proportionalibus, datam aliquam inter ſe ipſas proportionem ſervantibus, ad ſimiles figurarum ſimilium partes ſimiliter applicatis, generantur, ſunt ut vires & quadrata temporum conjunctim.

Corol. 3. Idem intelligendum eſt de ſpatiis quibuſvis quæ corpora urgentibus diverſis viribus deſcribunt. Hæc ſunt, ipſo motûs initio, ut vires & quadrata temporum conjunctim.

Corol. 4. Ideoque vires ſunt ut ſpatia, ipſo motûs initio, deſcripta directè, & quadrata temporum inverſè.

Corol. 5. Et quadrata temporum ſunt ut deſcripta ſpatia directè, & vires inverſè.

SCHOLIUM.

Si quantitates indeterminatæ diverſorum generum conferantur inter ſe, & earum aliqua dicatur eſſe ut eſt alia quævis directè vel inverſè: ſenſus eſt, quòd prior augetur vel diminuitur in eâdem ratione cum poſteriore, vel cum ejus reciprocâ. Et ſi earum aliqua dicatur eſſe ut ſunt aliæ duæ vel plures directè vel inverſè:

gulum BS ut ſpatia omnia $V\gamma$, $\gamma\Gamma$, $\Gamma\Delta$, $\Delta\Lambda$ ſimul ſumpta ad ſpatium $\Pi\Sigma$. Hinc, ſi partium AD, DE, EF, FB numerus infinitè augeatur, magnitudinibus ſingularum infinitè decreſcentibus, (ex quo fiet ut ſpatiorum etiam, $V\gamma$, $\gamma\Gamma$, $\Gamma\Delta$, $\Delta\Lambda$ numerus pariter infinitè creſcet, ſingulorum verò magnitudines infinitè diminuentur) ſumma rectangulorum omnium, AM, DN, EO, FP, quæ ultimò ſcilicet evaneſcentium fit, hoc eſt (per Lemma 3.) area ABP, erit ad rectangulum BS, ut ultima ſpatiorum omnium $V\gamma$, $\gamma\Gamma$, $\Gamma\Delta$, $\Delta\Lambda$ evaneſcentium ſumma ad ſpatium $\Pi\Sigma$.

Sed ſumma ultima ſpatiorum omnium $V\gamma$, $\gamma\Gamma$, $\Gamma\Delta$, $\Delta\Lambda$ evaneſcentium, ſive longitudo rectæ $V\Lambda$ quæ ultimò fit, rectæ VZ æqualis eſt. Numero enim partium, ſive temporum, AD, DE, EF, FB, infinitè creſcente, æqualis erit ultimò velocitas, quæ, tempore quovis finito, corpori γ acceſſerit, ei quæ eodem tempore corpori α. Si igitur recta $V\Lambda$ non ſit ultimò æqualis rectæ VZ, ſint illæ rectæ ultimò inæquales. Corpus igitur α, quod urgente vi finitâ ſecundum legem quandam mutabili, motu à puncto V incepto, curſu recto tempore T perſeveravit, ſub fine temporis T in locum Z devenerit. Et corpus aliud γ, quod urgente eâdem vi finitâ juxta eandem legem mutabili, motu

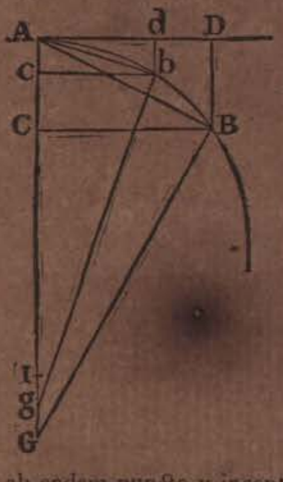
ab

inverſè: ſenſus eſt, quòd prima augetur vel diminuitur in ratione, quæ componitur ex rationibus, in quibus aliæ, vel aliarum reciprocæ, augentur vel diminuuntur. Ut ſi A dicatur eſſe ut B directè & C directè & D inverſè: ſenſus eſt, quòd A augetur, vel diminuitur, in eâdem ratione cum $B \times C \times \frac{1}{D}$; hoc eſt, quòd A & $\frac{BC}{D}$ ſunt ad invicem in ratione datâ. PRIMIS ULTIMISQUE.

LEMMA XI.

Subtenſa evaneſcens anguli contactûs, in curvis omnibus curvaturam finitam ad punctum contactûs habentibus, eſt ultimò in ratione duplicatâ ſubtenſæ arcûs contermini.

Caſ. 1. Sit arcus ille AB, tangens ejus AD, ſubtenſa anguli contactûs ad tangentem perpendicularis BD, ſubtenſa arcûs AB. Huic ſubtenſæ AB & tangenti AD perpendiculares erigantur AG, BG, occurrentes in G; dein accedant puncta D, B, G, ad puncta *d*, *b*, *g*, ſitque J interſectio linearum BG, AG ultimò facta, ubi puncta D, B accedunt uſque ad A. Manifeſtum eſt, quòd diſtantia GJ



minor eſſe poteſt quàm aſſignata quævis. Eſt autem (ex naturâ circulorum per puncta ABG, A*bg* tranſeuntium) AB *quad.* æquale AG × BD, & A*b* *quad.* æquale A*g* × *bd*; ideoque ratio AB *quad.* ad A*b* *quad.* componitur ex rationibus AG ad A*g* & BD ad *bd*. Sed quoniam GJ aſſumi poteſt minor longitudine quâvis aſſignatâ, fieri poteſt, ut ratio AG ad A*g* minùs differat à ratione æqualitatis, quàm pro differentiâ quâvis aſſignatâ; ideoque ut ratio AB *quad.* ad A*b* *quad.* minùs

ab eodem puncto V incepto, eodem curſu recto eodem tempore perſeveravit, ſub ejuſdem temporis fine in alium locum Λ devenerit. Quod eſt abſurdum. Non igitur ſunt inæquales recta VZ et ſpatiorum V*y*, *y*Γ, ΓΔ, ΛΛ ſumma quæ ultima evaneſcentium ſit. Æquales igitur. Ergo area ABP erit ad rectangulum BS ut VZ ad ΠΣ.

Jam verò recta CQ producta rectæ SR in X occurrat, et capiatur ΠΦ æqualis ſpatio, quod corpus velocitate æquabili AS latum, tempore *t*, ſeu AC, abſolveret. Atque eodem argumento quo hactenus uſi ſumus, oſtendere licet aream ACQ eſſe ad rectangulum CS ut ſpatium VX ad ſpatium ΠΦ. Invertendo CS : ACQ = ΠΦ : VX.

Cùm igitur ſit ABP : BS = VZ : ΠΣ
et BS : CS = BA : AC = T : *t* = ΠΣ : ΠΦ
et CS : ACQ = ΠΦ : VX.
Erit, ex æquo, ABP : ACQ = VZ : VX. Q. E. D.

differat à ratione BD ad *bd* quàm pro differentiâ quâvis aſſignatâ. Eſt ergo, per Lemma 1, ratio ultima AB *quad.* ad A*b* *quad.* eadem cum ratione ultimâ BD ad *bd*. Q. E. D.

Caſ. 2. Inclinetur jam BD ad AD in angulo quovis dato, & eadem ſemper erit ratio ultima BD ad *bd* quæ priùs, ideoque eadem ac AB *quad.* ad A*b* *quad.* Q. E. D.

Caſ. 3. Et quamvis angulus D non detur, ſed recta BD ad datum punctum convergat, vel aliâ quâcunque lege conſtituatur; tamen anguli D, *d*, communi lege conſtituti, ad æqualitatem ſemper vergent; & propiùs accedent ad invicem quàm pro differentiâ quâvis aſſignatâ, ideoque ultimò æquales erunt, per Lem. 1; & propterea lineæ BD, *bd* ſunt in eâdem ratione ad invicem ac priùs. Q. E. D.

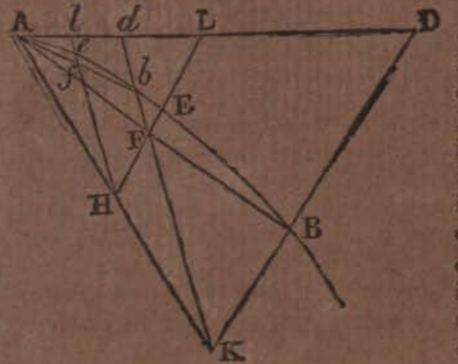
Corol. 1. Unde cùm tangentes AD, A*d*, arcus AB, A*b*, & eorum ſinus BC, *bc* fiant ultimò chordis AB, A*b* æquales; erunt etiam illorum quadrata ultimò ut ſubtenſæ BD, *bd*.

Corol. 2. Eorundem quadrata ſunt etiam ultimò ut ſunt arcuum ſagittæ, quæ chordas biſecant, & ad datum punctum convergunt. Nam ſagittæ illæ ſunt ut ſubtenſæ BD, *bd* (h).

Corol. 3. Ideoque ſagitta eſt in duplicatâ ratione temporis quo corpus datâ velocitate deſcribit arcum.

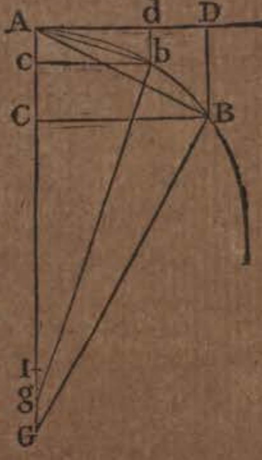
Corol. 4. Triangula rectilinea ADB, A*db* ſunt ultimò in triplicatâ ratione laterum AD, A*d*, inque ſeſquiplicatâ laterum DB, *db*; utpote in compoſitâ ratione laterum AD & DB, A*d* & *db* exiſtentia.

Sic



(h) Sagittæ, EF, *ef*, chordas, AB, A*b*, medias dividentes, in dato puncto H concurrant. Jungatur AH, et in AH productâ capiatur HK = AH; junctæque K*b*, KB et productæ contingenti AD in punctis *d*, D occurrant. Eidem contingenti occurrant ſagittæ, *fe*, FE, productæ in punctis *l*, L. Ex conſtructione AH : AK = AF : AB. Sunt igitur HF, KB, ſive FL, BD inter ſe parallelæ. Coeuntibus igitur punctis B, A ratio ultima evaneſcentis EL ad BD evaneſcentem, quadrati ex evaneſcente AE ad quadratum ex evaneſcente AB ultima erit (per Lemma). Eadem igitur quæ eſt quadrati ex AF ad quadratum ex AB, ſive unitatis ad quaternarium: fiunt enim AF, AE ultimo æquales. Sed propter rectas BD, FL parallelas, erit BD ad

Sic & triangula ABC, A*bc* ſunt ultimò in triplicatâ ratione laterum BC, *bc*. Rationem verò ſeſquiplicatam voco triplicatæ ſubduplicatam, quæ nempe ex ſimplici & ſubduplicatâ componitur. PRIMIS ULTIMISQUE.



Corol. 5. Et quoniam DB, *db* ſunt ultimò parallelæ & in duplicatâ ratione ipſarum AD, A*d*: erunt areæ ultimæ curvilineæ ADB, A*db* (ex naturâ parabolæ) ([i]) duæ tertiæ partes triangulorum rectilineorum ADB, A*db*; & ſegmenta AB, A*b* partes tertiæ eorundem triangulorum. Et inde hæ areæ et hæc ſegmenta erunt in duplicatâ ratione tum tangentium AD, A*d*, tum chordarum & arcuum AB, A*b*.

SCHOLIUM.

Cæterùm in his omnibus ſupponimus angulum contactûs nec infinitè majorem eſſe angulis contactuum, quos circuli continent cum tangentibus ſuis, nec iiſdem infinitè minorem; hoc eſt, curvaturam ad punctum A, nec infinitè parvam eſſe, nec infinitè magnam; ſeu intervallum AJ finitæ eſſe magnitudinis. Capi enim poteſt DB ut AD^3: quo in caſu circulus nullus, per punctum A, inter tangentem AD & curvam AB, duci poteſt, proindeque angulus contactûs erit infinitè minor circularibus. Et ſimili argumento ſi fiat DB ſucceſſivè ut AD^4, AD^5, AD^6, AD^7, &c. habebitur ſeries angulorum contactûs pergens in infinitum, quorum quilibet poſterior eſt infinitè minor priore. Et ſi fiat DB ſucceſſivè ut AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. habebitur alia ſeries infinita angulorum contactûs, quorum primus eſt ejuſdem generis cum circularibus, ſecundus infinitè major, & quilibet poſterior infi-

ad FL, ut AE ad AF hoc eſt ut quaternarius ad binarium. Quare ex æquo ratio ultima rectæ EL ad FL unitatis erit ad binarium. Ergo FE, EL ſunt ultimò æquales, ac proinde EF erit ultimò ad BD ſicut unitas ad quaternarium. Simili ratione oſtendetur eſſe *ef* ad *bd* ultimò ſicut unitas ad quaternarium. Quare ultimò EF : BD = *ef* : *bd*; et permutando, ultimò, EF : *ef* = BD : *bd*. Sed BD, *bd* ad datum punctum K vergunt. Quare (per Lemma II.) punctis, B, *b*, in A coeuntibus, quadrata ex ſubtenſis evaneſcentibus AB, A*b* ſunt ultimò inter ſe ſicut evaneſcentes BD, *bd*: hoc eſt, ſicut ſagittæ EF, *ef* evaneſcentes. Q. E. D.

([i]) *Ex natura parabolæ.* Etenim cùm BD, *bd* evaneſcentes rationem inter ſe ultimam habeant evaneſcentium AD, A*d* duplicatam, arcus evaneſcens AB, cujuſcunque demum generis curva AB fuerit, ab evaneſcente arcu parabolico ultimò non differt.

De Rationibus

nitè major priore. Sed & inter duos quoſvis ex his angulis poteſt ſeries, utrinque in infinitum pergens, angulorum intermediorum inſeri, quorum quilibet poſterior erit infinitè major minorve priore. Ut ſi inter terminos AD^2 & AD^3 inſeratur ſeries $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. Et rurſus inter binos quoſvis angulos hujus ſeriei inſeri poteſt ſeries nova angulorum intermediorum, ab invicem infinitis intervallis differentium. Neque novit natura limitem (k).

Quæ de curvis lineis, deque ſuperficiebus comprehenſis, demonſtrata ſunt, facilè applicantur ad ſolidorum ſuperficies curvas, & contenta. Præmiſi verò hæc lemmata, ut effugerem tædium deducendi longas demonſtrationes, more veterum geometrarum, ad abſurdum. Contractiores enim redduntur demonſtrationes per methodum indiviſibilium. Sed quoniam durior eſt indiviſibilium hypotheſis, & propterea methodus illa minùs geometrica cenſetur; malui demonſtrationes rerum ſequentium ad ultimas quantitatum evaneſcentium ſummas & rationes, primaſque naſcentium, id eſt, ad limites ſummarum & rationum deducere; & propterea limitum illorum demonſtrationes, quâ potui brevitate, præmittere. His enim idem præſtatur quod per methodum indiviſibilium; & principiis demonſtratis jam tutiùs utemur. Proinde in ſequentibus, ſiquando quantitates tanquam ex particulis conſtantes conſideravero, vel ſi pro rectis uſurpavero lineolas curvas; nolim indiviſibilia, ſed evaneſcentia diviſibilia, non ſummas & rationes partium determinatarum, ſed ſummarum & rationum limites ſemper intelligi; vimque talium demonſtrationum ad methodum præcedentium lemmatum ſemper revocari.

Objectio eſt, quòd quantitatum evaneſcentium nulla ſit ultima proportio; quippe quæ, antequam evanuerunt, non eſt ultima, ubi evanuerunt, nulla eſt. Sed & eodem argumento æquè contendi poſſet, nullam eſſe corporis ad certum locum, ubi motus finiatur, pervenientis velocitatem ultimam: hanc enim, antequam corpus attingit locum, non eſſe ultimam, ubi attingit, nullam eſſe. Et reſponſio facilis eſt: per velocitatem ultimam intelligi eam, quâ corpus movetur, neque antequam attingit lo-

(k) Vide Keil. Introduct. ad Veram Phyſicam. Lect. 4.

cum ultimum & motus ceſſat, neque poſtea, ſed tunc cùm attingit; id eſt, illam ipſam velocitatem quâcum corpus attingit locum ultimum, & quâcum motus ceſſat. Et ſimiliter per ultimam rationem quantitatum evaneſcentium, intelligendam eſſe rationem quantitatum, non antequam evaneſcunt, non poſtea, ſed quâcum evaneſcunt. Pariter & ratio prima naſcentium eſt ratio quâcum naſcuntur. Et ſumma prima & ultima eſt quâcum eſſe (vel augeri aut minui) incipiunt & ceſſant. Extat limes, quem velocitas in fine motûs attingere poteſt, non autem tranſgredi. Hæc eſt velocitas ultima. Et par eſt ratio limitis quantitatum & proportionum omnium incipientium & ceſſantium. Cúmque hic limes ſit certus & definitus, problema eſt verè geometricum, eundem determinare. Geometrica verò omnia, in aliis geometricis determinandis ac demonſtrandis, legitimè uſurpantur. PRIMIS ULTIMISQUE.

Contendi etiam poteſt, quòd ſi dentur ultimæ quantitatum evaneſcentium rationes, dabuntur & ultimæ magnitudines: & ſic quantitas omnis conſtabit ex indiviſibilibus, contrà quàm *Euclides* de incommenſurabilibus, in libro decimo elementorum, demonſtravit. Verùm hæc objectio falſæ innititur hypotheſi. Ultimæ rationes illæ quibuſcum quantitates evaneſcunt, reverâ non ſunt rationes quantitatum ultimarum, ſed limites, ad quos quantitatum, ſine limite decreſcentium, rationes ſemper appropinquant; & quas propiùs aſſequi poſſunt, quàm pro datâ quâvis differentiâ, nunquam verò tranſgredi; neque priùs attingere quàm quantitates diminuuntur in infinitum. Res clariùs intelligetur in infinitè magnis. Si quantitates duæ, quarum data eſt differentia, augeantur in infinitum, dabitur harum ultima ratio, nimirum ratio æqualitatis; nec tamen ideò dabuntur quantitates ultimæ, ſeu maximæ, quarum iſta eſt ratio. In ſequentibus igitur, ſiquando, facili rerum conceptui conſulens, dixero quantitates quàm minimas, vel evaneſcentes, vel ultimas; cave intelligas quantitates magnitudine determinatas, ſed cogita ſemper diminuendas ſine limite.

DE ANALYSI

PER ÆQUATIONES NUMERO TERMINORUM

INFINITAS.

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CAP. X. Sect. 1. *Demonſtratio quadraturæ curvarum ſimplicium in regulâ prima.* p. 280. Sect. 2. *Demonſtratio reſolutionis æquationum affectarum.* p. 281

DE

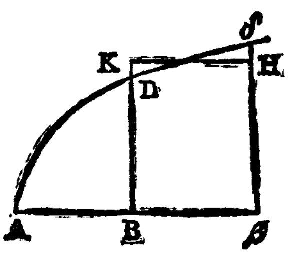
D E A N A L Y S I

PER ÆQUATIONES NUMERO TERMINORUM INFINITAS.

METHODUM generalem, quam, de Curvarum quantitate per Infinitam terminorum Seriem menſurandâ, olim excogitaveram, in ſequentibus breviter explicatam, potiùs quàm accuratè demonſtratam, habes.

C A P U T P R I M U M.

CAPUT PRIMUM.



BASI AB Curvæ alicujus AD, ſit Applicata BD perpendicularis: Et vocetur AB $= x$, BD $= y$, & ſint a, b, c, &c. Quantitates datæ, & m, n, Numeri Integri. Deinde,

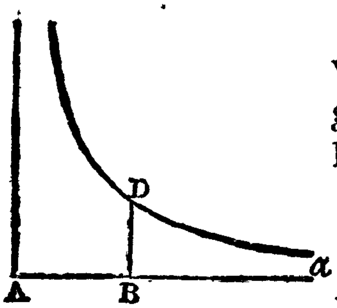
Curvarum Simplicium Quadratura.

R E G U L A I.

$$Si\ ax^{\frac{m}{n}} = y;\ Erit\ \frac{an}{m+n}x^{\frac{m+n}{n}} = Areæ\ \text{ABD}.$$

Res Exemplo patebit.

1. Si x^2 $(=1x^{\frac{2}{1}})=y$, hoc eſt, $a=1=n$, & $m=2$; Erit $\frac{1}{3}x^3=$ ABD.
2. Si $4\sqrt{x}$ $(=4x^{\frac{1}{2}})=y$; Erit $\frac{8}{3}x^{\frac{3}{2}}$ $(=\frac{8}{3}\sqrt[2]{x^3}=$ ABD.
3. Si $\sqrt[3]{x^5}$ $(=x^{\frac{5}{3}})=y$; Erit $\frac{3}{8}x^{\frac{8}{3}}$ $(=\frac{3}{8}\sqrt[3]{x^8})=$ ABD.
4. Si $\frac{1}{x^2}$ $(=x^{-2})=y$; id eſt, ſi $a=1=n$, & $m=-2$;



Erit $(-\frac{1}{1}x^{-\frac{1}{1}}=)\ -x^{-1}$ $(=-\frac{1}{x})=\alpha$BD, infinitè verſus α protenſæ; quam Calculus ponit negativam, propterea quòd jacet ex alterâ parte lineæ BD.

5. Si $\frac{1}{\sqrt{x^3}}(=x^{-\frac{3}{2}})=y$; Erit $(-\frac{2}{1}x^{-\frac{1}{2}}=\frac{2}{-\sqrt{x}}=$ BDα.
6. Si $\frac{1}{x}$ $(=x^{-1})=y$; Erit $\frac{1}{0}x^{\frac{0}{1}}=\frac{1}{0}x^0=\frac{1}{0}\times 1=\frac{1}{0}=$ Infinitæ; qualis eſt Area Hyperbolæ ex utrâque parte lineæ BD.

CAPUT SECUNDUM.

Compoſitarum Curvarum Quadratura ex Simplicibus.

REGULA II.

Si valor ipſius y *ex pluribus iſtiuſmodi Terminis componitur, Area etiam componetur ex Areis, quæ à ſingulis Terminis emanant.* ([a])

Exempla

REGULÆ PRIMÆ

([a]) Duabus hiſce regulis, quæ ex Problemate Secundo Quadraturæ Curvarum facilè ſunt deducendæ, omnis ferè Arithmetica Infinitorum Walliſii ſummatim comprehenſa eſt. Ac Primam quidem Walliſius ipſe allegavit; Arithmet. inf. prop. 64 & 102; Secundam verò, ſi minùs verbis, ſaltem exemplis patefecit. Demonſtrationem prioris ſyntheticam, ſimplicitate et elegantiâ planè admirabilem, haud ingratum certè Geometris fecero, ſi è Fermatio apponam.

“ Unico, quod notiſſimum eſt, Proportionis Geometricæ attributo tota hæc methodus innititur.

“ Theorema hoc eſt. *Datâ quâvis proportione geometricâ, cujus termini decreſcant in infinitum, eſt ut differentia terminorum progreſſionem conſtituentium ad minorem terminum, ita maximus progreſſionis terminus ad reliquos omnes in infinitum ſumptos.*

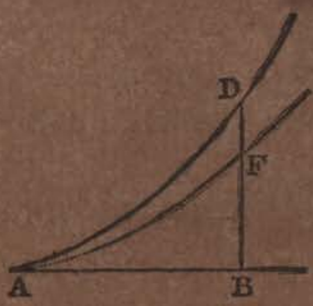
“ Hoc poſito, proponantur primò HYPERBOLÆ quadrandæ. Exponatur ſi placet hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ HA ad quadratum rectæ AG ita recta GE ad rectam HI; et ut quadratum OA ad quadratum AH ita recta HI ad rectam ON. Aio ſpatium infinitum, cujus baſis GE, & curva ES ex uno latere, ex alio vero aſymptotos infinita GOR, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis Geometricæ in infinitum extendendæ, quorum primus ſit AG, ſecundus AH, tertius AO, in infinitum: et ad ſe per approximationem tantum accedant, quantum ſatis ſit, ut juxta methodum Archimedeam parallelogrammum rectilineum GE × GH, quadrilineo mixto GHIE adæquetur. Item ut priora ex intervallis rectis proportionalium, GH, HO, OM, & ſimilia, ſint ferè inter ſe æqualia; ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones et inſcriptiones, Archimedea demonſtrandi ratio inſtitui poſſit: quod ſemel monuiſſe ſufficiat.

“ His poſitis, cùm ſit AG : AH = AH : AO = AO : AM, erit pariter ut AG ad AH ita intervallum GH ad HO, et ita intervallum HO ad OM, &c. Parallelogrammum autem EG × GH erit ad parallelogrammum HI × HO, ut parallelogrammum HI × HO ad parallelogrammum NO × OM. Cùm enim ratio parallelogrammi GE × GH ad HI × HO componatur ex ratione rectæ GE ad rectam HI, et ex ratione rectæ GH ad HO; ſit autem GH : HO = AG : AH, ut præmonuimus: ergo, ratio parallelogrammi GE × GH ad parallelogrammum HI × HO componitur ex ratione GE ad HI, et ex ratione AG ad AH. Sed GE : HI = $AH^2 : AG^2$ (ex hyp.) et $AH^2 : AG^2$ = AO : AG, propter proportionales. Ergo ratio parallelogrammi GE × GH ad HI × HO componitur ex ratione AO ad AG et AG ad AH. Sed ratio AO ad AH componitur ex illis duabus. Ergo parallelogrammum GE × GH ad parallelogrammum HI × HO ut AO ad AH, ſive ut AH ad AG. Similiter probabitur parallelogrammum HI × HO eſſe ad parallelogrammum NO × OM ut AM ad AO, hoc eſt ut AH ad AG. Ergo parallelogramma EG × GH, HI × HO, NO × OM, &c in infinitum ſumpta erunt continuè proportionalia, in ratione rectæ HA ad GA. Eſt igitur, ex theoremate hujus methodi conſtitutivo, ut GH, differentia terminorum, ad minorem terminum GA, ita primum parallelogrammum EG × GH ad reliqua in infinitum parallelogramma; hoc eſt, ex adæquatione Archimedeâ, ad figuram ſub HI aſymptoto HR, et curvâ IND, in infinitum extendendâ, contentam. Sed ut HG ad GA ita, ſumptâ communi latitudine rectâ GE, parallelogrammum GE × GH ad parallelogrammum GE × GA. Ergo parallelogrammum GE × GA, quod eſt ſpatium rectilineum datum, adæquatur figuræ prædictæ. Cui ſi addas parallelogrammum GE × GH, quod propter minutiſſimos τεμαχισμοὺς evaneſcit, et abit in nihilum, ſupereſt veriſſimum et Archimedeâ licet prolixiore demonſtratione facillimè firmandum, parallelogrammum AE, in hâc hyperbolæ ſpecie, æquari figuræ ſub baſe GE, aſymptoto GR, curvâ ED, in infinitum producendâ, contentæ. Nec operoſum ad omnes omninò hujuſmodi hyperbolas, unâ demptâ, inventionem extendere.

“ Sit

Exempla Prima.

QUADRATURA COMPOSITARUM.



1. Si $x^2+x^{\frac{3}{2}}=y$; Erit $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=$ ABD.

Etenim ſi ſemper ſit $x^2=$ BF, et $x^{\frac{3}{2}}=$ FD, erit, ex præcedente Regulâ, $\frac{1}{3}x^3=$ ſuperficiei AFB deſcriptæ per lineam BF, et $\frac{2}{5}x^{\frac{5}{2}}=$ AFD deſcriptæ per DF; Quare $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=$ toti ABD.

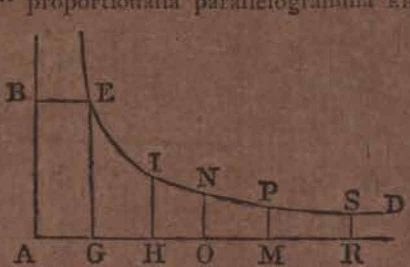
2. Sic ſi $x^2-x^{\frac{3}{2}}=y$; Erit $\frac{1}{3}x^3-\frac{2}{5}x^{\frac{5}{2}}=$ ABD.

3. Et ſi $3x-2x^2+x^3-5x^4=y$; Erit $\frac{3}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4-x^5=$ ABD.

Exempla

DEMONSTRATIO FERMATIANA.

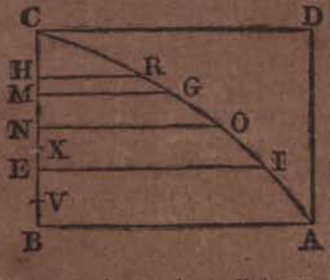
“ Sit ea alterius, ſi placet, hyperbolæ proprietas, ut ſit GE ad HI ut cubus rectæ HA ad cubum “ rectæ AG, & ſic de reliquis. Expoſitâ ex more infinitâ proportionalium, ut ſuprà, ſerie, fient “ proportionalia parallelogramma EH, IO, NM ut ſuprà in infinitum. In hoc verò caſu paral-



“ lelogrammum primum erit ad ſecundum, et ſecundum “ ad tertium, &c. ut recta AO ad AG, quod ſtatim com- “ poſitio proportionum manifeſtabit. Erit igitur ut pa- “ rallelogrammum EH ad figuram, ita recta OG ad AG; “ & ſumptâ communi latitudine GE, ita parallelogram- “ mum OG × GE ad parallelogrammum AG × GE. Eſt “ igitur ut parallelogrammum OG × GE ad parallelogram- “ mum AG × GE, ita parallelogrammum EG × GH ad figu- “ ram. Et viciſſim ut parallelogrammum OG × GE, ad “ parallelogrammum EG × GH, ita AG × GE ad figuram. Ut autem parallelogrammum OG × GE “ ad EG × GH ita OG ad GH, ſive 2 ad 1 ex adæquatione: intervalla enim baſi proxima facta “ ſunt ex conſtructione ferè (*ſive ultimò*) æqualia inter ſe. Ergo, in hâc hyperbolâ, parallelogram- “ mum EGA, quod eſt æquale ſpatio rectilineo dato, eſt duplum figuræ ſub baſe GR, aſymptoto “ GR, et curvâ ESD in infinitum protenſâ.

“ Similis in quibuſlibet aliis caſibus habebit locum demonſtratio; niſi quòd in primariâ, ſive “ Apollonianâ et ſimplici hyperbolâ, deficit eâ ſolâ ratione methodus, quia in hâc parallelo- “ gramma, EH, IO, NM ſunt ſemper inter ſe æqualia; atque ideo cùm termini progreſſionis con- “ ſtitutivi ſint inter ſe æquales, nulla inter eos eſt differentia, quæ totum in hoc negotio conficit “ myſterium.

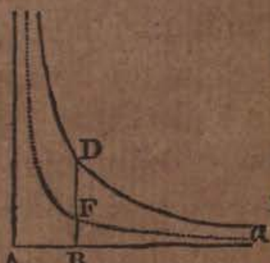
“ Eâdem ratione PARABOLÆ omnes omninò quadrantur; nec eſt ulla, quæ ab artificio noſtræ “ methodi, ut ſit in hyperbolis, poſſit eſſe immunis.



“ Sit ſemiparabola primaria AGRE cujus diameter CB, ſemibaſis “ AB: ſumptis autem applicatis IE, ON, GM, &c. ſit ſemper ut “ quadratum AB ad quadratum IE, ita recta BC, ad CE; et ut “ quadratum IE ad quadratum ON ita recta EC ad CN: et ſic in “ infinitum. Intelligantur, ex more methodi, rectæ BC, EC, NC, “ MC, HC, in infinitum, continuè proportionales. Erunt etiam, ut “ ſuperiùs probatum eſt, proportionalia parallelogramma, AE, IN, “ OM, GH in infinitum. Ut cognoſcatur ratio parallelogrammi “ AE ad parallelogrammum IN, recurrendum ex methodo ad com- “ poſitionem proportionum. Componitur autem ratio parallelo- “ grammi AE ad parallelogrammum IN ex ratione AB ad IE et ex ratione BE ad EN. Cùm autem “ ſit ut AB quadratum ad IE quadratum, ita BC ad CE; ſi inter BC & CE ſûmatur media propor- “ tionalis CV, item inter EC, & CN, media proportionalis XC, erunt continuè proportionales BC, “ VC, EC, XC, NC: et ut BC ad EC ita quadratum BC ad quadratum CV. Sed BC : EC $=$ AB^2 : IE^2. “ Ergo BC^2 : CV^2 $=$ AB^2 : IE^2, et BC : CV $=$ AB : EI. Ratio igitur parallelogrammi AE ad parallelo- “ grammum IN componetur ex ratione BC ad VC, ſive VC ad CE, ſive EC ad CX, et ex ratione “ BE ad EN, ſive BC ad CE. Ratio autem quæ componitur ex his duabus rationibus, BC nempe “ ad CE, et CE ad CX, eſt eadem quæ ratio BC ad CX. Igitur parallelogrammum AE eſt ad

 “ parallelo-

Exempla Secunda.

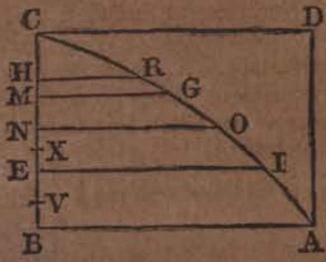


4. Si $x^{-2}+x^{-\frac{3}{2}}=y$; Erit $-x^{-1}-2x^{-\frac{1}{2}}=\alpha$BD. Vel ſi $x^{-2}-x^{-\frac{3}{2}}=y$; Erit $-x^{-1}+2x^{-\frac{1}{2}}=\alpha$BD.

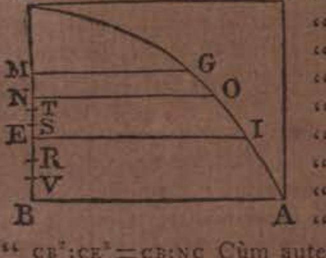
Quarum ſigna ſi mutaveris, habebis Affirmativum valorem ($x^{-1}+2x^{-\frac{1}{2}}$ vel $x^{-1}-2x^{-\frac{1}{2}}$) ſuperficiei αBD, modò tota cadat ſupra baſim ABα.

5. Sin

" parallelogrammum IN ut BC ad CX. [*Atque eadem erit ratio parallelogrammi* IN *ad parallelogrammum* OM, *et parallelogrammi* OM *ad parallelogrammum* GH, &c.] Ideoque ex theoremate methodi " conſtitutivo parallelogrammum AE erit ad figuram IRCHE ut recta EX ad rectam XC; ideoque



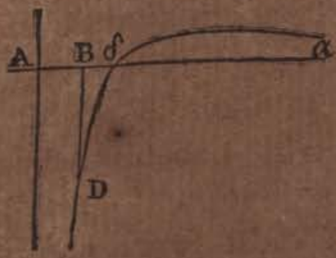
" ut idem parallelogrammum AE ad totam figuram AIGRCB ita " recta BX ad totam diametrum BC. Ut autem recta BX ad totam diametrum BC, ita, ſumptâ communi latitudine AB, parallelogrammum AB × BX ad parallelogrammum AB × BC, ſive " parallelogrammum BD, ductâ AD diametro parallelâ occurrente " tangenti CD in D. Ergo ut parallelogrammum AE ad totam " figuram ſemiparabolicam ARCB, ita parallelogrammum AB × BX " ad parallelogrammum ED. Et viciſſim ut parallelogrammum " AE ad parallelogrammum AB × BX ita, figura ad parallelogrammum BD. Ut autem parallelogrammum AE ad AB × BX, ita, " propter communem latitudinem, recta BE ad BX. Ergo ut BE ad BX ita figura ad parallelogrammum. Eſt autem BE ad BX, propter adæqualitatem, & ſectiones minutiſſimas, quæ " rectas BV, VE, EX, intervalla proportionalium repreſentantes, ferè inter ſe ſupponit æquales, " ut 2 ad 3. Ergo figura ſemiparabolica ARCB ad parallelogrammum BD, ut 2 ad 3.



" Rurſum, ſit curva AICB, cujus baſis AB diameter CB, ejus generis parabola, ut cubus applicatæ AB ſit ad cubum applicatæ IE ut " quadratum rectæ BC ad quadratum rectæ CE, et reliqua ponantur " ut ſuprà. Series nempe proportionalium rectarum BC, CE, CN, " CM, &c. item ſeries proportionalium parallelogrammorum AE, IN, " OM, &c. in infinitum. Inter BC et EC ſumantur duæ mediæ proportionales, CV, CR. Item inter EC & CN ſumantur etiam duæ " mediæ proportionales CS, CT. Conſtat ex conſtructione, cùm ratio " BC ad EC ſit eadem rationi EC ad NC, fore quoque continuè proportionales rectas BC, CV, CR, CE, CS, CT, CN. Eſt autem $AB^3:IE^3=$ " $CB^2:CE^2=CB:NC$ Çùm autem ſint, ut ſuprà probavimus, ſeptem continuè proportionales, BC, " CV, CR, CE, CS, CT, CN, ergo prima, tertia, quinta, ſeptima erunt etiam continuè proportionales: Ideoque erit BC ad RC ut RC ad CS, et ut CS ad CN. Ut igitur prima BC ad quartam " NC ita cubus primæ BC ad cubum ſecundæ RC. Sed ut BC ad NC ita probavimus eſſe cubum AB " ad cubum IE. Ergo $AB^3:IE^3=BC^3:RC^3$, et AB:IE$=$BC;RC. Cùm igitur ratio parallelogrammi " AE ad parallelogrammum IN, componatur ex ratione AE ad IE, et ex ratione BE ad EN, ſive " BC ad EC; eadem parallelogrammorum ratio componetur ex ratione BC ad RC et BC ad EC. " Ut autem BC prima ſeptem proportionalium ad EC quartam, ita RC tertia, ad CT ſextam. " Ergo parallelogrammi AE ad parallelogrammum IN ratio componitur ex ratione BC ad RC et " RC ad TC: hoc eſt parallelogrammum AE eſt ad parallelogrammum IN ut BC ad CT. Parallelogrammum igitur AE, ex prædemonſtratis, eſt ad figuram IGCE ut BT ad CT; ideoque ut " parallelogrammum AE ad totam figuram AICB, ita recta BT ad rectam BC, ſive AB×BT, ad " AB × BC, vel BD. Et viciſſim parallelogrammum AE ad AB × BT, ſive recta BE ad rectam BT, ut " figura AICB ad parallelogrammum BD. Recta autem BT continet quinque intervalla TS, SE, " ER, RV, VB, quæ inter ſe, propter noſtram methodum, cenſentur æqualia. Recta autem BE " continet tria ex iis intervallis, nempe ER, RV, VB. Ergo BE eſt ad BT, ac propterea figura " AICB ad parallelogrammum BD, ut 3 ad 5.

" CANON verò univerſalis inde nullo negotio elicietur. Patet nempe ſemper fore parallelogrammum BD ad figuram AICB ut aggregatum exponentium poteſtatum applicatæ et diametri

" ad



5. Sin aliqua pars cadat infra (quod fit cùm Curva decuſſat ſuam Baſin inter B et α, ut hìc vides in δ,) iſtâ parte à parte ſuperiori ſubductâ, habebis valorem Differentiæ:[b] Earum verò Summam ſi cupis, quære utramque Superficiem ſeorſim, et adde. Quod idem in reliquis hujus Regulæ exemplis notandum volo. QUADRATURA COMPOSITARUM.

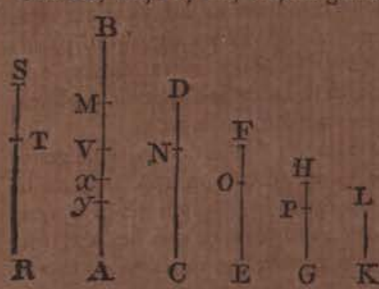
Exempla

" ad exponentem poteſtatis applicatæ. Ut in hoc exemplo videre eſt, in quo poteſtas applicatæ " AB eſt cubus, cujus exponens 3; poteſtas autem diametri eſt quadratum, cujus exponens 2. " Ergo debet eſſe BD ad AICB, ut jam demonſtravimus, & perpetuò conſtabit, ut ſumma 3 & 2, " hoc eſt 5, ad 3, exponentem applicatæ. DEMONSTRATIO FERMATIANA.

" In hyperbolis autem Canon non minori facilitate invenietur univerſalis. Erit enim in quâ-" cunque hyperbolâ, parallelogrammum BG (vid. fig. p. 259.) ad figuram in infinitum protenſam, " RGED, ut differentia poteſtatum applicatæ & diametri ad exponentem poteſtatis applicatæ." Nempe verbis hìc dicit Fermatius quæ Newtonus, Regulâ ſuâ Primâ, ſymbolis.

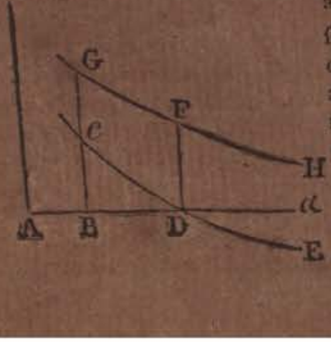
Cæterùm Theorema, ex quo deductæ ſunt Fermatii demonſtrationes, ad hunc ferè modum oſtendi poteſt.

Sint AB, CD, EF, GH, KL, magnitudines quarum ſit continua, ſecundum datam duarum RS, RT rationem, proportionis convenientia.



Si earum numerus infinitè augeatur, dico maximæ AB ad reliquas omnes ſimul ſumptas proportionem ultimam eam eſſe quam ST habet ad TR. Magnitudini enim AB auferatur AM magnitudini CD æqualis. Et huic CD auferatur CN magnitudini EF æqualis. Et huic EF auferatur EO magnitudini GH æqualis. Et huic GH auferatur GP magnitudini KL æqualis. Tum ſumatur MV = DN. Vx = FO. xy = HP. Jam cùm AB ſit ad CD, vel AM, ut SR ad RT, dividendo erit BM ad CD ut ST ad TR. Simili ratione erit DN ſive MV ad EF ut ST ad TR. Necnon FO, ſive Vx ad GH, et HP, ſive xy, ad KL, ut ST ad TR. Erunt igitur omnes BM, MV, Vx, xy, ſimul ſumptæ, ſive By, ad omnes CD, EF, GH, KL, ſimul ſumptas ut ST ad TR. (per 12. 5. Elem.) Et quotcunque fuerint magnitudines illæ AB, CD, EF, GH, KL proportione inter ſe continuè convenientes, hæc ratio ſummæ differentiarum By ad ſummam omnium à maximâ manet. Quare major erit AB quam ut habeat ad omnium reliquarum ſummam rationem eam quam ST ad TR. Sed eo minùs quidem major quo minor fuerit Ay. Et Ay ultimæ reliquarum KL ſemper eſt æqualis, ut ex conſtructione ſatis patet. Et numero omnium infinitè creſcente, ultima illarum infinitè minuetur, & minor fiet ultimò quâvis datâ. Quare magnitudo Ay minor fit ultimò quâvis datâ. Quare magnitudo By ipſi BA fit ultimò æqualis; & rationes magnitudinis AB ad ſummam omnium CD, EF, GH, KL, ſimul ſumptas, & alterius By ad eandem ſummam fiunt ultimò æquales. (Newton De Rat. prim. & ult. 1.) Sed, By ad ſummam illam rationem eam quam ST ad TR ſemper habet, ex ſuprà oſtenſis. Eadem igitur ratio ST ad TR magnitudinis AB ad omnes CD, EF, GH, KL ultima erit. Q. E. D.

([b]) Sit Curva CDE cujus abſciſſa AB, x, ordinatim applicata BC, y. Naturam vero curvæ hæc

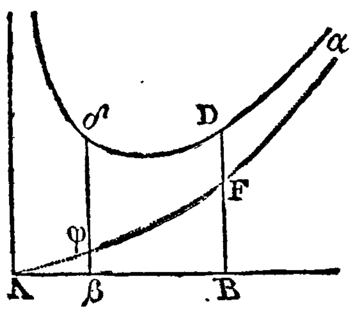


æquatio definiat, $x^{-2} - x^{-\frac{1}{2}} = y$. Talis quidem curva baſin AB ſecabit. Secet in D. Dico quantitatem $2x^{-\frac{1}{2}} - x^{-1}$, quæ ex calculo ſecundum regulam Newtonianam peracto pro areâ provenerit, arearum CDB, αDE differentiam eſſe. Per D ducatur DF cum ordinatis (BC) curvæ CDE parallela. Et capiatur DF = 1 = AD. In BC, ordinatâ quâvis curvæ CDE, capiatur BG = x^{-2}. Et per GF ſcribatur curva GFH quam punctum G perpetuò tangit; ſive cujus ea ſit natura, ut $x^{-2} = z$, deſignante x abſciſſam AB, z ordinatam BG. Jam per regulam primam $-x^{-1}$ = areæ HGBα. Sed cùm GB = x^{-2}, et BC = $x^{-2} - x^{-\frac{1}{2}}$, erit GC = $x^{-\frac{1}{2}}$. Et $-2x^{\frac{1}{2}}$ = areæ HFGCE

curvis

Caput Secundum.

Exempla Tertia.



6. Si $x^2+x^{-2}=y$; Erit $\frac{1}{3}x^3-x^{-1}=$ Superficiei deſcriptæ. Sed hîc notandum eſt, quòd dictæ Superficiei partes ſic inventæ jacent ex diverſo latere lineæ BD.

Nempe, poſito $x^2=$ BF, & $x^{-2}=$ FD; Erit $\frac{1}{3}x^3=$ ABF Superficiei per BF deſcriptæ, & $-x^{-1}=$ DFα Superficiei deſcriptæ per DF.

7. Et hoc ſemper accidit, cùm indices $(\frac{m+n}{n})$ rationum Baſis x in valore Superficiei quæſitæ, ſint variis Signis affecti. In hujuſmodi Caſibus, pars aliqua BDδβ Superficiei media (quæ ſola dari poterit, cùm Superficies ſit utrinque infinita) ſic invenitur.

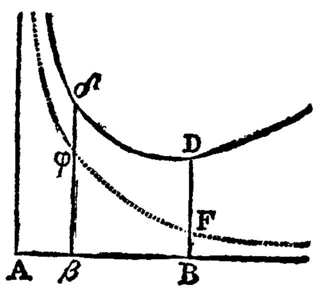
8. Subtrahe Superficiem ad minorem Baſin Aβ pertinentem, à Superficie ad majorem Baſin AB pertinente, & habebis βBDδ Superficiem differentiæ Baſium inſiſtentem. Sic in hoc Exemplo *(Vide Fig. præcedentem.)* Si AB $=2$, & Aβ $=1$; Erit βBDδ$=\frac{17}{6}$.

Etenim Superficies ad AB pertinens (viz. ABF$-$DFα) erit $\frac{8}{3}-\frac{1}{2}$, ſive $\frac{13}{6}$; et ſuperficies ad Aβ pertinens (viz. Aφβ$-$δφα) erit $\frac{1}{3}-1$, ſive $-\frac{2}{3}$: et earum differentia (viz. ABF$-$DFα$-$Aφβ$+$δφα$=$βBDδ) erit $\frac{13}{6}+\frac{2}{3}$, ſive $\frac{17}{6}$.

9. Eodem modo, ſi Aβ$=1$, AB$=x$; erit βBDδ$=\frac{2}{3}+\frac{1}{3}x^3-x^{-1}$.

10. Sic ſi $2x^3-3x^5-\frac{2}{3}x^{-4}+x^{-\frac{3}{5}}=y$, & Aβ$=1$; Erit βBDδ$=\frac{1}{2}x^4-\frac{1}{2}x^6+\frac{2}{9}x^{-3}+\frac{5}{2}x^{\frac{2}{5}}-\frac{49}{18}$.

11. Denique notari poterit, quòd ſi quantitas x^{-1} in valore ipſius y reperiatur, iſte terminus (cùm hyperbolicam ſuperficiem generat) ſeorſim à reliquis conſiderandus eſt.



Ut ſi $x^2+x^{-3}+x^{-1}=y$: ſit $x^{-1}=$ BF, & $x^2+x^{-3}=$ FD, ac Aβ$=1$; et erit δφFD$=\frac{1}{6}+\frac{1}{3}x^3-\frac{1}{2}x^{-2}$, utpote quæ ex terminis x^2+x^{-3} generatur.

Quare

curvis duabus HFG, CDE interceptæ. Hinc $2x^{-\frac{1}{2}}-x^{-1}$ arearum HGBα, HGCDE differentia erit. Sed duarum HGbα, HGCLE, eadem quæ duarum CDE, αDE differentia erit; nempe cùm pars αDCGH utrique

Quare, ſi reliqua ſuperficies βςFB, quæ hyperbolica eſt, ex calculo aliquo ſit data, dabitur tota βBDδ. QUADRATURA COMPOSITARUM.

CAPUT TERTIUM.

Aliarum Omnium Quadratura.

REGULA III.

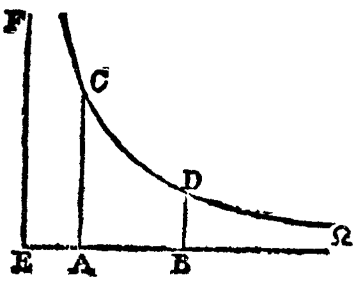
Sin valor ipſius y, vel aliquis ejus Terminus ſit præcedentibus magis compoſitus, in Terminos Simpliciores reducendus eſt; operando in Literis, ad eundem Modum quo Arithmetici, in Numeris Decimalibus, dividunt, Radices extrahunt, vel affectas Æquationes ſolvunt; & ex iſtis Terminis quæſitam Curvæ Superficiem, per præcedentes Regulas, deinceps elicies.

Exempla Dividendo.

1. SIT $\frac{aa}{b+x} = y$, curvâ nempe exiſtente Hyperbolâ.

Jam ut Æquatio iſta à Denominatore ſuo liberetur, Diviſionem ſic inſtituo.

$$
\begin{array}{l}
b+x)\; aa + 0 \left(\frac{aa}{b} - \frac{aax}{b} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4} \;\&c. \right. \\
\quad aa + \frac{aax}{b} \\
\quad\quad 0 - \frac{aax}{b} + 0 \\
\quad\quad - \frac{aax}{b} - \frac{aax^2}{b} \\
\quad\quad\quad 0 + \frac{aax^2}{b^2} + 0 \\
\quad\quad\quad + \frac{aax^2}{b} + \frac{aax^3}{b^3} \\
\quad\quad\quad\quad 0 - \frac{aax^3}{b^3} + 0 \\
\quad\quad\quad\quad - \frac{aax^3}{b^3} - \frac{aax^4}{b^4} \\
\quad\quad\quad\quad\quad 0 + \frac{aax^4}{b^4}
\end{array}
$$



utrique ſit communis. Ergo $2x^{-\frac{1}{2}} - x^{-1}$ = differentiæ arearum CBD, αDE. Q. E. D. [*E Commentariis Joannis Stewart Abredonenſis.*]

4 Et

CAPUT TERTIUM.

Et ſic vice hujus, $y=\frac{aa}{b+x}$, nova prodit, $y=\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}$, &c. ſerie iſtâc infinitè continuatâ; adeoque (per Regulam Secundam) area quæſita ABDC æqualis erit ipſi $\frac{a^2x}{b}-\frac{a^2x^2}{2b^2}+\frac{a^2x^3}{3b^3}-\frac{a^2x^4}{4b^4}$ &c. (c) infinitæ etiam ſeriei, cujus tamen Termini pauci initiales ſunt in uſum quemvis ſatis exacti, ſi modò x ſit aliquoties minor quàm b.

2. Eodem modo, ſi ſit $\frac{1}{1+xx}=y$, dividendo prodibit $y=1-x^2+x^4-x^6+x^8$ &c. Unde (per Regulam Secundam) erit ABDC $=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9$ &c.

3. Vel ſi Terminus xx ponatur in diviſore primus, hoc modo, $xx+1$, prodibit $x^{-2}-x^{-4}+x^{-6}-x^{-8}$ &c. pro valore ipſius y; unde (per Regulam Secundam) erit BDΩ $=-x^{-1}+\frac{1}{3}x^{-3}-\frac{1}{5}x^{-5}+\frac{1}{7}x^{-7}$ &c. Priori modo procede cùm x eſt ſatis parva, poſteriori cùm ſatis magna ſupponitur.

4. Denique ſi $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}=y$; dividendo prodit $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$ &c. unde erit ABDC $=\frac{4}{3}x^{\frac{3}{2}}-x^2+\frac{14}{5}x^{\frac{5}{2}}-\frac{13}{3}x^3$ &c.

Exempla Radicem Extrahendo.

5. Si ſit $\sqrt{aa+xx}=y$, radicem ſic extraho,

$$aa+xx\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\right. \text{ \&c.}$$

$$\underline{aa}$$

$$0+xx$$

$$\underline{xx+\frac{x^4}{4a^2}}$$

$$0-\frac{x^4}{4a^2}$$

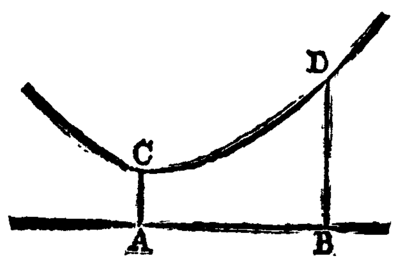
$$\underline{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}$$

$$0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\underline{+\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}}}$$

$$0-\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}}$$

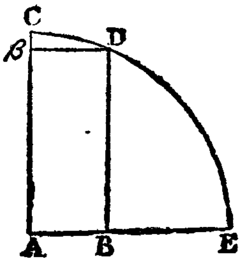
&c.



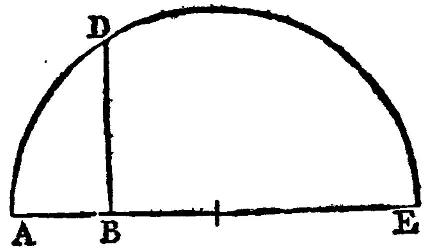
Unde,

Unde, pro æquatione $\sqrt{aa+xx}=y$, nova producitur, viz. $y=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$ &c. Et (per Reg. 2.) area quæſita ABDC erit $=ax+\frac{x^3}{6a}-\frac{x^5}{40a^3}+\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ &c. Et hæc eſt quadratura hyperbolæ ([d]). QUADRATURA COMPOSITARUM.

6. Eodem modo, ſi ſit $\sqrt{aa-xx}=y$, ejus radix erit $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$ &c. Adeoque area quæſita ABDC erit æqualis $ax-\frac{x^3}{6a}-\frac{x^5}{40a^3}-\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ &c. Et hæc eſt Quadratura Circuli ([e]).

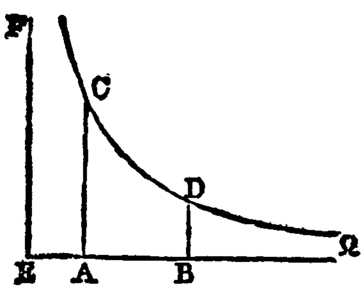


7. Vel ſi ponas $\sqrt{x-xx}=y$, erit radix æqualis infinitæ ſeriei $x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}-\frac{5}{128}x^{\frac{9}{2}}$ &c. Et area quæſita ABD æqualis erit $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}-\frac{5}{704}x^{\frac{11}{2}}$ &c. ſive $x^{\frac{1}{2}}$ in $\frac{2}{3}x-\frac{1}{5}x^2-\frac{1}{28}x^3-\frac{1}{72}x^4-\frac{5}{704}x^5$ &c. Et hæc eſt Areæ Circuli Quadratura ([f]).



8. Si $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}=y$, (Cujus Quadratura dat Longitudinem curvæ Ellipticæ ([g]), extrahendo radicem utramque prodit

1 +



([c]) Sit hyperbola ΩDC quæ centrum habeat E, aſymptotas ad perpendiculum compoſitas EF, EΩ. In aſymptotâ alterutrâ, puta EΩ, datum intelligatur punctum A. Sitque BD cum alterâ aſymptotâ parallela. Jam è naturâ curvæ rectangulum EB × BD dabitur. Si igitur rectangulum illud datum ſymbolum aa deſignet; rectam datam EA, literâ b; indefinitas AB, BD, literæ x, y; erit $by+xy=aa$, et $\frac{aa}{b+x}=y$. Et ſeries Newtoniana infinitè producta areæ hyperbolicæ ABDC ultimò æqualis erit.

([d]) Sit hyperbola æquilatera CD, cujus centrum A, ſemiaxis tranſverſus AC $=a$, abſciſſa AB axis ſecundi $=x$, ordinata BD $=y$. (Vid. fig. pag. 264.) Erit ACBD area illa hyperbolica cui ſeries $ax+\frac{x^3}{6a}-\frac{40a^3}{x^5}+$ ultimò æqualis erit.

([e]) Circuli centrum ſit A, ſemidiameter AE $=a$. Quadrans AEDC. Abſciſſa ſemidiametri à parte centri, AB $=x$; ordinata BD $=y$. Erit ABDC area circularis, cui ſeries $ax-\frac{x^3}{6a}-\frac{x^5}{40a^3}$ infinitè producta ultimò æqualis erit.

([f]) Sit AE circuli diameter $=1$; ADE ſemicirculus; AB abſciſſa $=x$; BD ordinata $=y$; ſeries $x^{\frac{1}{2}}\times\frac{2}{3}x-\frac{1}{5}x^2-\frac{1}{28}x^3$ infinitè producta areæ ADB ultimò æqualis erit.

([g]) Nempe hoc modo. Sit ellipſis CDBE cujus centrum A, axes CAB, DAE. Abſciſſa axis minoris

CAPUT TERTIUM.

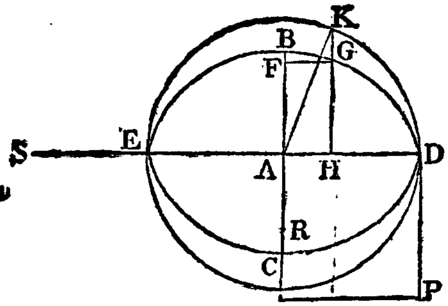
$$\frac{1+\frac{1}{2}ax^2-\frac{1}{8}a^2x^4+\frac{1}{16}a^3x^6-\frac{5}{128}a^4x^8}{1-\frac{1}{2}bx^2-\frac{1}{8}b^2x^4-\frac{1}{16}b^3x^6-\frac{5}{128}b^4x^8}\ \&c.$$

Et Dividendo, ſicut fit in Fractionibus Decimalibus, habes

$$\begin{array}{llll} 1+\frac{1}{2}b^3\,x^2 & +\frac{3}{8}b^2\,x^4 & +\frac{5}{16}b^3\,x^6 & +\frac{35}{128}b^4\,x^8\ \&c. \\ +\frac{1}{2}a & +\frac{1}{4}ab & +\frac{3}{16}ab^2 & +\frac{5}{32}ab^3 \\ & -\frac{1}{8}a^2 & -\frac{1}{16}a^2b & -\frac{3}{64}a^2b^2 \\ & & +\frac{1}{16}a^3 & +\frac{1}{32}a^3b \\ & & & -\frac{5}{128}a^4 \end{array}$$

Adeoque

LONGITUDO



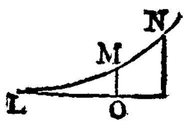
noris à parte centri, AF, deſignetur literâ x; ordinata GF literâ y; arcus DG literâ z. Sit AR ſemiparameter Ellipſeos, et ſumatur AS, ad quam ſemiaxis tranſverſus AD rationem habeat eam quam AR ad ipſum AD: et recta illa AS deſignetur literâ b: ponaturque ſemiaxis tranſverſus $AD = 1$. Jam, propter angulum AFG rectum, fluxiones rectarum AF, FG ſunt ad fluxionem arcûs DG, ſicut latera trianguli cujuſdam rectilinearis rectanguli ad hypotenuſam (Newton Introduct. ad Quad. Curv.) Erunt igitur $\dot{y}\dot{y}+\dot{x}\dot{x}=\dot{z}\dot{z}$. Et $\sqrt{\dot{y}\dot{y}+\dot{x}\dot{x}}=\dot{z}$. Centro A, ſemidiametro AD, ſcribatur circulus DKE. A puncto G in axem tranſverſum AD deducatur ad perpendiculum GH, quæ circuli circumflexum in puncto K ſecet. Jungatur AK. Jam propter circulum, $AK^2 - HK^2 = AH^2$; hoc eſt $1 - HK^2 = y^2$. Sed è naturis ellipſeos circulique quibus axis DE communis, $GH^2 : HK^2 = AB^2 : AD^2 = AR : AD = AD : AS$. Hoc eſt $x^2 : HK^2 = 1 : b$. Quare $HK^2 = bx^2$, et $1 - bx^2 = y^2$. Hinc $\dot{y} = -\frac{bx\dot{x}}{y} = -\frac{bx\dot{x}}{\sqrt{1-bx^2}}$. Ergo $\dot{y}\dot{y} = \dot{x}\dot{x} \times \frac{b^2x^2}{1-bx^2}$. Et $\dot{y}\dot{y}+\dot{x}\dot{x} = \dot{x}\dot{x} \times 1 + \frac{b^2x^2}{1-bx^2} = \frac{1+\overline{b^2-b}.x^2}{1-bx^2}\dot{x}\dot{x}$. Vel, ſi ponatur $a = b^2 - b$, $\dot{y}\dot{y}+\dot{x}\dot{x} = \frac{1+ax^2}{1-bx^2}\dot{x}\dot{x}$. Et $\sqrt{\frac{1+ax^2}{1-bx^2}}\,\dot{x} = \sqrt{\dot{y}\dot{y}+\dot{x}\dot{x}} = \dot{z}$.

$$\text{Ergo } \dot{x}\,\frac{1+\frac{ax^2}{2}-\frac{a^2x^4}{8}+\frac{a^3x^6}{16}-\frac{5}{128}a^4x^8+}{1-\frac{bx^2}{2}-\frac{1}{8}b^2x^4-\frac{1}{16}b^3x^6-\frac{5}{128}b^4x^8-} = \dot{z}.$$

Hinc, Diviſionis operâ, invenietur

$$\dot{z} = \dot{x} \times 1 \begin{array}{l} +\frac{1}{2}b \\ +\frac{1}{2}a \end{array} x^2 \begin{array}{l} +\frac{3b^2}{2^2\times 2} \\ +\frac{ab}{2^2} \\ -\frac{a^2}{2^2\times 2} \end{array} x^4 \begin{array}{l} +\frac{b^3\times 3\times 5}{2^3\times 2\times 3} \\ +\frac{3ab^2}{2^3\times 2} \\ -\frac{a^2b}{2^3\times 2} \\ +\frac{3a^3}{2^3\times 2\times 3} \end{array} x^6$$

Jam ſit curva LMN, cujus ſi abſciſſa LO capiatur $= x$, ordinata MO ad perpendiculum educta ſemper æqualis ſit quantitati $\frac{\dot{z}}{\dot{x}}$, cui ſeries utique illa $1 \begin{array}{l} +\frac{1}{2}b \\ +\frac{1}{2}a \end{array} x^2 +$, infinitè producta, ultimò ſit æqualis.



Tum

Adeoque aream quæſitam $\begin{matrix} x + \frac{1}{6} b\, x^3 + \frac{3}{40} b^2\, x^5 \ \&c. \\ \quad + \frac{1}{6} a \quad + \frac{1}{20} ab \\ \qquad\qquad - \frac{1}{40} a^2 \end{matrix}$

QUADRATURA COMPOSITARUM.

9. Sed obſervandum eſt, quòd operatio non raró abbreviatur per debitam æquationis præparationem; ut in allato exemplo

Tum hæc ſeries $x \begin{matrix} + \frac{1}{6} b \\ + \frac{1}{6} a \end{matrix} x^3 + \begin{matrix} \frac{3b^2}{2^2 . 2 \times 5} \\ + \frac{ab}{2^2 \times 5} \\ - \frac{a^2}{2^2 \times 2 \times 5} \end{matrix} x^5$

infinitè producta areæ LOM ultimò æqualis erit (per Regulam Secundam). Sed eadem ſeries quantitati z, ſive $1 \times z$, ultimò æqualis erit. (Geomet. flux. Prop. II.) Nempe cùm hæc ſeries fluens ſit alterius, quæ quantitati $\dot{z}$ oſtenſa eſt æqualis, et unà cum quantitate z à nihilo generari inceperit. Rectangulum igitur $1 \times z$, ſive AD $\times$ arc. DG areæ, LOM æqualis erit. Et ſi ad rectam AD applicetur rectangulum AP $=$ areæ LOM, latus DP, ex applicatione ortum, arcui Elliptico, DG, æquale erit. Et hoc eſt quod dixit Newtonus, per quadraturam curvæ, cujus ea ſit natura, ut ordinata ſit $\sqrt{\frac{1 + ax^2}{1 - bx^2}}\, \dot{x}$, exiſtente abſciſſâ x, dari longitudinem arcûs Elliptici.

ARCUS ELLIPTICI.

Et hujus exemplo interpretanda ſunt quæcunque ſimiliter ſunt dicta, qualia multà certè occurrunt, in ſcriptis Newtoni. Arithmeticè autem æſtimatur arcus z ſummam ſeriei colligendo. Cæterùm eo ſermone Newtonus uti voluit, qui maximè ſignificaret, geometricas etiam problematum conſtructiones è calculis, quos tradit, facillimè deducendas. Nam ſi talis eſſet curva LMN, cujus area Geometricæ alicujus lineationis ope quadraturam ſubiret, Arcûs etiam Elliptici Longitudinem geometricè definire liceret. Vel ſi aut conſtructione Geometricâ, aut artificio quopiam mechanico, aream curvæ LMN appropinquare ſaltem potueris, arcûs Elliptici longitudinem geometricè vel mechanicè æquè appropinquare poteris.

Series ſimplicior paulò exſiſtet è computatione ſequente, quæ literam b exulare coget; modò litera b non AS, ut priùs, ſed ſemiaxem ſecundum AB deſignet. E naturâ Ellipſeos, $y^2 : b^2 - x^2 = 1 : b^2$. Ergo $b^2y^2 = b^2 - x^2$, et $x^2 = b^2 - b^2y^2$. Hinc $\dot{x} = \frac{-b^2y}{x}\dot{y} = \frac{-b^2y}{\sqrt{b^2 - b^2y^2}}\dot{y} = \frac{-by}{\sqrt{1 - y^2}}\dot{y}$. Hinc $\dot{x}\dot{x} = \frac{b^2y^2}{1 - y^2}\dot{y}\dot{y}$. Et $\dot{x}\dot{x} + \dot{y}\dot{y} = \dot{y}\dot{y} \times 1 + \frac{b^2y^2}{1 - y^2} = \dot{y}\dot{y} \times \frac{1 - y^2 + b^2y^2}{1 - y^2}$. Vel, ſi ponatur $b^2 - 1 = -a$, $\dot{x}\dot{x} + \dot{y}\dot{y} = \frac{1 - ay^2}{1 - y^2}\dot{y}\dot{y}$. Ergo $\dot{y} \times \sqrt{\frac{1 - ay^2}{1 - y^2}} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = -\dot{z}$. Unde calculo ut priùs peracto veniet ſeries $y \begin{matrix} + \frac{1}{6} \\ - \frac{1}{6} a \end{matrix} x^3 + \begin{matrix} \frac{3}{2^2 \times 2 \times 5} \\ - \frac{a}{2^2 \times 5} \\ - \frac{a^2}{2^2 \times 2 \times 5} \end{matrix} y^5 \ \&c. = -z$

hoc eſt arcui Elliptico BG. Poſitis autem ſemiaxe ſecundo AB $= 1$, tranſverſo AD $= b$, arcus DG ſimili computatione prodibit $x \begin{matrix} + \frac{1}{6} \\ + \frac{1}{6} a \end{matrix} x^3 + \begin{matrix} \frac{3}{2^2 \times 2 \times 5} \\ + \frac{a}{2^2 . \times 5} \\ - \frac{a^2}{2^2 \times 2 \times 5} \end{matrix} x^5 +$

quantitate ſcilicet a, quæ in priore caſu negata, eſt in hoc poſteriore affirmatâ.

 $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}=y.$ Si utramque partem fractionis per $\sqrt{1-bxx}$ multiplices, prodibit $\frac{\sqrt{1+\begin{smallmatrix}a\\-b\end{smallmatrix}x^2-abx^4}}{1+bx^2}=y$, & reliquum opus perficitur extrahendo radicem numeratoris tantùm, & dividendo per denominatorem.

10. Ex hiſce, credo, ſatis patebit modus reducendi quemlibet valorem ipſius y (quibuſcunque radicibus vel denominatoribus ſit perplexus, ut hic videre eſt; $x^3+\frac{\sqrt{x-\sqrt{1-xx}}}{\sqrt[3]{ax^2+x^3}}-\frac{\sqrt[5]{x^4+2x^5-x^{\frac{3}{2}}}}{\sqrt[3]{x+x}-\sqrt[2]{2x-x^{\frac{2}{3}}}}$ $=y$) in ſeries infinitas ſimplicium terminorum, ex quibus, per Regulam Secundam, quæſita ſuperficies cognoſcetur.

CAPUT QUARTUM.

Exempla per Reſolutionem Æquationum.

SECT. I.

Numeralis Æquationum affectarum reſolutio.

1. QUIA tota difficultas in Reſolutione latet, modum, quo ego utor, in Æquatione Numerali primùm illuſtrabo.

Sit $y^3-2y-5=0$, reſolvenda: et ſit 2 numerus qui minùs quàm decimâ ſui parte differt à Radice quæſitâ. Tum pono $2+p=y$, & ſubſtituo hunc ipſi valorem in Æquationem, & inde nova prodit $p^3+6p^2+10p-1=0$, cujus radix p exquirenda eſt, ut quotienti addatur: nempe (neglectis p^3+6p^2 ob parvitatem) $10p-1=0$, ſive $p=0,1$ prope veritatem eſt; itaque ſcribo 0,1 in quotiente, & ſuppono $0,1+q=p$, & hunc ejus valorem, ut priùs, ſubſtituo; unde prodit $q^3+6,3q^2+11,23q+0,061=0$.

Et cùm $11,23q+0,061$ veritati propè accedit, ſive ferè ſit q æqualis $-0,0054$ (dividendo nempe donec tot eliciantur figuræ, quot locis primæ figuræ hujus & principalis quotientis excluſivè diſtant) ſcribo $-0,0054$ in inferiori parte quotientis, cùm negativa ſit.

2. Et ſupponens $-0,0054+r=q$, hunc ut priùs ſubſtituo, & operationem ſic produco quo uſque placuerit. Verùm ſi ad bis tot

tot figuras tantùm quot in Quotiente jam reperiuntur unâ demptâ, operam continuare cupiam, pro q ſubſtituo $-0,0054+r$ in hanc $6,3q^2+11,23q+0,061$, ſcilicet primo ejus termino (q^3) propter exilitatem ſuam neglecto; & prodit $6,3r^2+11,16196r+0,000541708=0$ ferè, ſive (rejecto $6,3r^2$) $r=\frac{-0,000541708}{11,16196}=-0,00004853$ ferè, quam ſcribo in negativâ parte Quotientis. Denique negativam partem Quotientis ab Affirmativâ ſubducens habeo 2,09455147 Quotientem quæſitam. Æquationum Resolutiones.

$y^3-2y-5=0$		$+2,10000000$ $-0,00544853$ $+2,09455147=y$
$2+p=y$	$+y^3$ $+2y$ -5	$+8+12p+6p^2+p^3$ $-4-2p$ -5
	Summa	$-1+10p-6p^2+p^3$
$0,1+q=p$	$+p^3$ $+6p^2$ $+10p$ -1	$+0,001+0,03q+0,3q^2+q^3$ $+0,06+1,2+6,0$ $+1,\quad +10,$ $-1,$
	Summa	$+0,061+11,23q+6,3q^2+q^3$
$-0,0054+r=q$	$+6,3q^2$ $+11,23q$ $+0,061$	$+0,000183708-0,06804r+6,3r^2$ $-0,060642\quad +11,23$ $+0,061$
	Summa	$+0,000541708+11,16196r+6,3r^2$
$-0,00004854+s=r$		

3. Æquationes plurium dimenſionum nihilo ſeciùs reſolvuntur, & operam ſub fine, ut hîc factum fuit, levabis, ſi primos ejus terminos gradatim omiſeris.

4. Præterea notandum eſt, quòd in hoc exemplo, ſi dubitarem an $0,1=p$ veritati ſatis accederet, pro $10p-1=0$, finxiſſem $6p^2+10p-1=0$, & ejus radicis primam figuram in Quotiente ſcripſiſſem; & ſecundam vel tertiam Quotientis figuram ſic explorare convenit, ubi in Æquatione iſtâ ultimò reſultante quadratum coefficientis penultimi termini, non ſit decies majus quàm factus ex ultimo termino ducto in coefficientem termini antepenultimi.

5. Imo laborem plerumque minues, præſertim in Æquationibus plurimarum dimenſionum, ſi figuras omnes Quotienti addendas dicto modo (hoc eſt extrahendo minorem radicum, ex tribus ultimis terminis Æquationis noviſſimè reſultantis) exquiras: iſto enim modo figuras duplo plures quâlibet vice Quotienti lucraberis.

CAPUT QUARTUM.

6. Hæc Methodus reſolvendi Æquationes pervulgata an ſit neſcio; certè mihi videtur præ reliquis ſimplex, & uſui accommodata. Demonſtratio ejus ex ipſo modo operandi patet; unde cùm opus ſit, in memoriam facilè revocatur.

7. Æquationes, in quibus vel aliqui vel nulli Termini deſint, eâdem fere facilitate tractantur; & Æquatio ſemper relinquitur, cujus Radix unà cum acquiſitâ Quotiente adæquat Radicem Æquationis primò propoſitæ. Unde Examinatio Operis hîc æquè poterit inſtitui ac in reliquâ Arithmeticâ, auferendo nempe Quotientem à Radice primæ Æquationis (ſicut Analyſtis notum eſt) ut Æquatio ultima, vel termini ejus duo treſve ultimi, producantur inde.

8. Quicquid laboris hîc eſt, iſtud in Operatione ſubſtituendi quantitates unas pro aliis reperietur: id quod variè perficias, at ſequentem modum maximè expeditum puto, præſertim ubi Numeri coefficientes conſtant ex pluribus Figuris.

Sit $p+3$ ſubſtituenda pro y in hanc $y^4-4y^3+5y^2-12y+17=0$.

Et cùm iſta poſſit reſolvi in hanc formam $\overline{\overline{\overline{y-4}\times y+5}\times y-12}\times y$ $+17=0$ ([h]). Æquatio nova ſic generabitur $\overline{p-1}\times\overline{p+3}=p^2+2p-3$. et p^2+2p+2 in $p+3=p^3+5p^2+8p+6$. et p^3+5p^2+8p-6 in $p+3=p^4+8p^3+23p^2+18p-18$. et $p^4+8p^3+23p^2+18p-1=0$, quæ quærebatur.

SECT. II.

Literalis Æquationum affectarum reſolutio.

1. His in numeris ſic oſtenſis: ſit Æquatio literalis $y^3+a^2y-2a^3+axy-x^3=0$, reſolvenda.

Primùm inquiro valorem ipſius y, cùm x ſit nulla; hoc eſt, elicio Radicem hujus Æquationis, $y^3+a^2y-2a^3=0$, & invenio eſſe $+a$. Itaque ſcribo $+a$ in Quotiente, & ſupponens $+a+p=y$, ſubſtituo pro y valorem ejus, & Terminos inde reſultantes ($p^3+3ap^2+4a^2p$, &c) margini appono; ex quibus aſſumo $+4a^2p+a^2x$ terminos

(h) Senſus eſt, æquationem propoſitam ita generandam eſſe, ſi apotomen $y-4$ cum y multiplicaveris: factum quinario auctum cum y rurſum multiplicaveris: Novum factum duodenario imminutum

minos utique ubi p & x ſeorſim ſunt minimarum dimenſionum, & eos nihilo ferè æquales eſſe ſuppono, ſive $p=-\frac{1}{4}x$ ferè, vel $p=-\frac{1}{4}x+q$. Et ſcribens $-\frac{1}{4}x$ in Quotiente, ſubſtituo $-\frac{1}{4}x+q$ pro p; et terminos inde reſultantes iterum in margine ſcribo, ut vides in annexo ſchemate, & inde aſſumo Quantitates $+4a^2q-\frac{1}{16}ax^2$, in quibus utique q & x ſeorſim ſunt minimarum dimenſionum; & fingo $q=\frac{xx}{64a}$ ferè, ſive $q=+\frac{xx}{64a}+r$; & adnectens $+\frac{xx}{64a}$ quotienti, ſubſtituo $\frac{xx}{64a}+r$ pro q; & ſic procedo quo uſque placuerit. Æquationum Reſolutiones.

$y^3+a^2y-2a^3+axy-x^3=0$		
$y=a-\frac{x}{4}+\frac{x^2}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$ &c.		
$+a+p=y$	$+y^3$ $+a^2y$ $+axy$ $-2a^3$ $-x^3$	$+a^3+3a^2p+3ap^2+p^3$ $+a^3+a^2p$ $+a^2x+axp$ $-2a^3$ $-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$ $+3ap^2$ $+4a^2p$ $+axp$ $+a\,x$ $-x^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$ $+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$ $-a^2x+4a^2q$ $-\frac{1}{4}ax^2+axq$ $+a^2x$ $-x^3$
$+\frac{x^2}{64a}+r=q$	$+3aq^2$ $+4a^2q$ $-\frac{1}{2}axq$ $+\frac{3}{16}x^2q$ $-\frac{1}{16}ax^2$ $-\frac{65}{64}x^3$	$+\frac{3x^4}{4096a}+\frac{3}{32}x^2r+3ar^2$ $+\frac{1}{16}ax^2+4a^2r$ $-\frac{1}{128}x^3-\frac{1}{2}axr$ $+\frac{3x^4}{1024a}+\frac{3}{16}x^2r$ $-\frac{1}{16}ax^2$ $-\frac{65}{64}x^3$
$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$		

2. Sin duplo tantùm plures quotienti terminos, uno dempto, jungendos adhuc vellem: primo termino (q^3) æquationis noviſſimè reſultantis miſſo, & iſtâ etiam parte ($-\frac{3}{4}xq^2$) ſecundi, ubi x eſt tot dimenſionum quot in penultimo termino quotientis; in reliquos terminos ($3aq^2+4a^2q$, &c.) margini adſcriptos ut vides, ſubſtituo $\frac{x^2}{64a}+r$ pro q; & ex ultimis duobus terminis ($\frac{15x^4}{4096a}-\frac{131}{128}x^3+\frac{9}{32}x^2r-\frac{1}{2}axr+4a^2r$) æquationis inde reſultantis, factâ diviſione, $4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}$ (elicio $+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$ quotienti adnectendos.

imminutum in y rurſum multiplicaveris, factoque rurſum novo· deniſeptenarium tandem adjeceris.

3. De-

CAPUT QUARTUM. 3. Denique quotiens iſta ($a - \frac{x}{4} + \frac{xx}{64a}$, &c.) per Regulam Secundam, dabit $ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a} + \frac{509x^5}{81920a}$, &c. pro areâ quæſitâ; quæ ad veritatem tanto magis accedit, quanto x ſit minor.

SECT. III.

Alius modus eaſdem Reſolvendi.

1. Sin valor areæ tanto magis ad veritatem accedere debet quanto x ſit major; exemplum eſto $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Itaque hanc reſoluturus excerpo terminos $y^3 + x^2y - 2x^3$, in quibus x & y vel ſeorſim, vel ſimul multiplicatæ, ſunt & plurimarum, & æqualium ubique dimenſionum; & ex iis quaſi nihilo æqualibus radicem elicio. Hanc invenio eſſe x, & in quotiente ſcribo. Vel quod eodem recidit, ex $y^3 + y - 2$ (unitate pro x ſubſtitutâ) radicem extraho; quæ hic prodit 1, & eam per x multiplico, & factum (x) in quotiente ſcribo. Denique pono $x + p = y$; & ſic procedo ut in priori exemplo, donec habeam quotientem $x - \frac{a}{4} + \frac{aa}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$, &c. adeoque aream $\frac{x^2}{2} - \frac{ax}{4} + \boxed{\frac{aa}{64x}}$ (¹) $- \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$, de quâ vide exempla tertia Regulæ Secundæ. Lucis gratiâ dedi hoc exemplum in omnibus idem cum priori, modò x & a ſibi invicem ibi ſubſtituantur, ut non opus eſſet aliud reſolutionis exemplum hic adjungere.

2. Area autem ($\frac{xx}{2} - \frac{x}{4} + \boxed{\frac{aa}{64x}}$ &c. terminatur ad curvam, quæ juxta Aſymptoton aliquam in infinitum ſerpit; & Termini initiales ($x - \frac{1}{4}a$) valoris extracti de y, in Aſymptoton iſtam ſemper terminantur; unde portionem Aſymptoti facilè invenies. Idem ſemper notandum eſt, cùm area deſignatur terminis plus pluſque diviſis per x continuè, præterquam quòd vice Aſymptoti rectæ quandoque habeatur Parabola Conica, vel alia magis compoſita.

3. Sed

(¹) Nempe hujuſmodi ſymbolis $\boxed{\frac{a}{b.x^n}}$ curvarum hyperboliformium areas deſignari voluit, quarum abſciſſas litera x deſignat, ordinatas ſymbolum $\frac{a}{bx^n}$. Hinc pleriſque noſtratium uſu receptum

3. Sed hunc modum miſſum faciens, utpote particularem, quia non applicabilem curvis in orbem, ad inſtar Ellipſium, flexis; de altero modo per exemplum, $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, ſuprà oſtenſo (ſcilicet quo dimenſiones ipſius x in numeratoribus quotientis perpetuò augeantur) annotabo ſequentia. ÆQUATIONUM RESOLUTIONES.

I. Si quando, accidit quòd valor ipſius y, cùm x nullum eſſe fingitur, ſit quantitas ſurda vel penitùs ignota, licebit illam literâ aliquâ deſignare. Ut in exemplo, $y^3 + a^2y + axy - 2a^3 - x^3 = 0$; ſi radix hujus $y^3 + a^2y - 2a^3$ fuiſſet ſurda, vel ignota, finxiſſem quamlibet (b) pro eâ ponendam; et reſolutionem ut ſequitur perfeciſſem. Scribens b in quotiente, ſuppono $b + p = y$, & iſtum pro y ſubſtituo, ut vides; unde nova $p^3 + 3bp^2$, &c. reſultat, rejectis terminis $b^3 + a^2b - 2a^3$, qui nihilo ſunt æquales, propterea quòd b ſupponitur radix hujus $y^3 + a^2y - 2a^3 = 0$. Deinde termini $3b^2p + a^2p + abx$ dant $- \frac{abx}{3b^2 + a^2}$ quotienti apponendum, et $- \frac{abx}{3b^2 + a^2} + q$ ſubſtituendum pro p, &c.

$y^3 + aay + axy - 2a^3 - x^3 = 0$. Sit $cc = 3b^2 + a^2$.		
$y = b - \frac{abx}{c^2} + \frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^6} - \frac{a^5bx^3}{c^8} + \frac{a^5b^3x^3}{c^{10}}$ &c.		
$b + p = y$	y^3	$+ b^3 + 3b^2p + 3bp^2 + p^3$
	$+ axy$	$+ abx + axp$
	$+ aay$	$+ aab + aap$
	$- x^3$	$- x^3$
	$- 2a^3$	$- 2a^3$
$\frac{-abx}{cc} + q = p$	p^3	$- \frac{a^3b^3x^3}{c^6}$ &c.
	$+ 3bp^3$	$+ \frac{3a^2b^3x^2}{c^4} - \frac{6ab^2x}{c^2}q$ &c.
	$+ axp$	$- \frac{a^2bx^2}{c^2} + axq$
	$+ ccp$	$- abx + ccq$
	$- x^3$	$- x^3$
	$+ abx$	$+ abx$
$c^2 + ax - \frac{6ab^2x}{cc}) \frac{a^4bx^2}{c^4} + x^3 + \frac{a^3b^3x^3}{c^6} (\frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8}$ &c.		

Completo opere, ſumo numerum aliquem pro a, & hanc $y^3 + a^2y - 2a^3 = 0$, ſicut de numerali æquatione oſtenſum ſuprà, reſolvo; & radicem ejus pro b ſubſtituo.

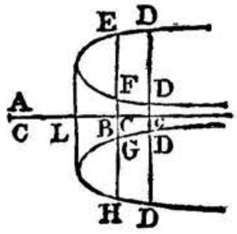
II. Si dictus valor ſit nihil, hoc eſt, ſi in æquatione reſolvendâ nullus ſit terminus, niſi qui per x vel y ſit multiplicatus, ut in hâc $y^3 - axy + x^3 = 0$; tum terminos $(-axy + x^3)$ ſeligo in quibus x ſeorſim & y etiam ſeorſim, ſi fieri poteſt, aliàs per x multiplicata, ſit minimarum dimenſionum. Et illi dant $+ \frac{xx}{a}$ pro primo termino quotientis, & $\frac{xx}{a} + p$

tum eſt, ut hoc ſymbolo, $ax^{\frac{m}{n}}\dot{x}$, fluxionem deſignante, hoc aliud $\boxed{ax^{\frac{m}{n}}\dot{x}}$ generaliter deſignet fluentem.

CAPUT QUARTUM.

pro y ſubſtituendum. In hâc $y^3-a^2y+axy-x^3=0$, licebit primum terminum quotientis vel ex $-a^2y-x^3$, vel ex y^3-a^2y elicere.

III. Si valor iſte ſit imaginarius, ut in hâc $y^4+y^2-2y+6-x^2y^2-2x+x^2+x^4=0$, augeo vel imminuo quantitatem x, donec dictus valor evadat realis.



Sic in annexo ſchemate (k), cum AC (x) nulla eſt, tum CD (y) eſt imaginaria.

Sin minuatur AC per datam AB, ut BC fiat x; tum, poſito quòd BC (x) ſit nulla, CD (y) erit valore quadruplici (CE, CF, CG, vel CH) realis; quarum radicum (CE, CF, CG, vel CH) quælibet poteſt eſſe primus terminus quotientis, prout ſuperficies BEDC, BFDC, BGDC, vel BHDC deſideratur. In aliis etiam caſibus, ſi quando hæſitas, te hoc modo extricabis.

Denique ſi index poteſtatis ipſius x vel y ſit fractio, reduco ipſum ad integrum: ut in hoc exemplo $y^3-xy^{\frac{1}{2}}+x^{\frac{4}{3}}=0$. Poſito $y^{\frac{1}{2}}=v$, & $x^{\frac{1}{3}}=z$, reſultabit $v^6-z^3v+z^4=0$, cujus radix eſt $v=z+z^3$, &c. ſive (reſtituendo valores) $y^{\frac{1}{2}}=x^{\frac{1}{3}}+x$, &c. & quadrando $y=x^{\frac{2}{3}}+2x^{\frac{4}{3}}$, &c.

Et hæc de areis curvarum inveſtigandis dicta ſufficiant (l). Imo cùm Problemata omnia de curvarum Longitudine, de quantitate & ſuperficie ſolidorum, deque Centro Gravitatis, poſſunt eò tandem reduci, ut quæratur quantitas ſuperficiei planæ lineâ curvâ terminatæ, non opus eſt quicquam de iis adjungere. In iſtis autem quo ego operor modo dicam breviſſimè.

CAPUT

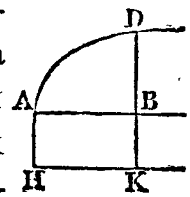
(k) Sit curva DEHD cujus abſciſſam, AC, litera x deſignet; ordinatam, CD, litera y; æquatione biquadraticâ modò expoſitâ abſciſſa AC ordinataque CD quomodo inter ſe mutuò habeant generaliter ſignificante. Hæc quidem curva talis erit, ut ſi per datum punctum A, unde abſciſſæ initium ducant, recta ducatur cum ordinatis CD parallela, ea curvæ nullibi occurrat. Ergo quando x nulla eſt, hoc eſt punctis, A, C, coeuntibus, æſtimatio literæ y imaginaria erit. Duci intelligatur recta cum ordinatis parallela quæ curvam contingat; axem verò contingens illa in puncto L fecet. Sumatur recta AB datæ cujuſvis longitudinis, modò rectâ AL non ſit minor. Si per datum punctum B ducatur recta BE, cum ordinatis parallela, quatuor illa locis cum curvâ concurret. Concurſus quatuor ſint F, E, G, H. Jam ſi litera x, pro indefinitâ AC, indefinitæ illius datæque AB differentiam BC deſignet, et in æquationem biquadraticam pro x, illud $x+AB$ ubique ſubſtituatur; quo facto illa in hanc migraverit, $y^4+y^2-2y+6-x^2y^2-2AB.x.y^2-AB^2.y^2-2x-$

2 AB.

CAPUT QUINTUM.

Curvarum Longitudines.

Applicatio prædictorum ad reliqua iſtiuſmodi Problemata.

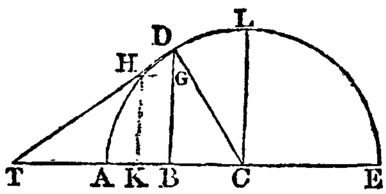


SIT ABD curva quævis, & AHKB rectangulum cujus latus AH, vel BK, eſt unitas. Et cogita rectam DBK, uniformiter ab AH motam, areas ABD & AK deſcribere; & quòd BK (1) ſit momentum quo AK (x) & BD (y) momentum quo ABD gradatim augetur; & quòd ex momento BD perpetim dato, poſſis, per prædictas regulas, aream ABD ipſo deſcriptam inveſtigare; ſive cum AK (x) momento 1 deſcriptâ conferre.

Jam quâ ratione ſuperficies ABD ex momento ſuo perpetim dato, per præcedentes regulas, elicitur, eâdem quælibet alia quantitas ex momento ſuo ſic dato elicietur. Exemplo res fiet clarior (m).

CAPUT SEXTUM.

Longitudines Curvarum invenire.



1. SIT ADLE circulus, cujus arcûs AD longitudo eſt indaganda. Ductâ tangente DHT, & completo indefinite parvo rectangulo HGBK, & poſito AE = 1 = 2AC: erit ut BK ſive GH, momentum baſis AB (x), ad HD momentum arcûs AD : : BT : DT : : BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) : : 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). Adeoque $\frac{1}{2\sqrt{x-xx}}$ ſive $\frac{\sqrt{x-xx}}{2x-2xx}$, eſt momentum arcûs AD. Quod reductum fit $\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{4}x^{\frac{1}{2}}+\frac{3}{16}x^{\frac{3}{2}}+\frac{5}{32}x^{\frac{5}{2}}+\frac{35}{256}x^{\frac{7}{2}}+\frac{63}{512}x^{\frac{9}{2}}$ &c. Quare, per Regulam

$2AB+x^2+2AB.x+AB^2+x^4+4AB.x^3+6AB^2x^2+4AB^3x+AB^4=0$; æquationis, $y^4+y^2-2y+6-2AB+AB^2+AB^4=0$, quæ ex tranſmutatâ fiet, poſito quòd x nihilo æqualis ſit; hujus inquam nulla radix impoſſibilis erit, ſed veræ quatuor exiſtent. Coeuntibus enim punctis B, C, quod fieri quidem neceſſe eſt, ſi indefinita $x=0$, hoc eſt ſi illius AC datæque AE differentia nulla ſit, ſignificantia literæ y quadruplex erit, cùm ex quatuor, BF, BE, BG, BH, quamlibet illa deſignârit. Harum autem aliam vel aliam pro membro primo ſeriei ſumere oportet, prout aream BEDC, vel BFDC, vel EGDC, vel BHDC, æſtimare velis.

(l) Quæ hîc verò breviter dixit eadem in Geometriâ Analyticâ pluribus et explanatiùs tradit. Ea quidem quæ Æquationum in ſeries infinitas reſolutionem ſpectant, capite ejus libri primo ſecundo et tertio: quæ Curvarum per ſeries Quadraturas, Problemate 9°.

(m) Vide (f) & confer Newton. de Rat. Prim. et Ult. Prop. IV. Cor.

Secundam, longitudo arcûs AD eſt $x^{\frac{1}{2}}+\frac{1}{6}x^{\frac{3}{2}}+\frac{3}{40}x^{\frac{5}{2}}+\frac{5}{112}x^{\frac{7}{2}}+\frac{35}{1152}x^{\frac{9}{2}}+\frac{63}{2816}x^{\frac{11}{2}}$ &c. ſive $x^{\frac{1}{2}}$ in $1+\frac{1}{6}x+\frac{3}{40}x^2+\frac{5}{112}x^3+\frac{35}{1152}x^4+\frac{63}{2816}x^5$ &c.

2. Non ſecus ponendo CB eſſe x, & radium CA eſſe 1, invenies arcum LD eſſe $x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c.

3. Sèd notandum eſt, quòd unitas iſta, quæ pro momento ponitur, eſt ſuperficies cùm de ſolidis, & linea cùm de ſuperficiebus, & punctum cùm de lineis (ut in hoc exemplo) agitur.

Nec vereor loqui de unitate (n) in punctis, ſive lineis infinitè parvis, ſiquidem proportiones ibi jam contemplantur Geometræ, dum utuntur methodis Indiviſibilium.

Ex his fiat conjectura de ſuperficiebus & quantitatibus ſolidorum; ac de centris gravitatum.

CAPUT SEPTIMUM.

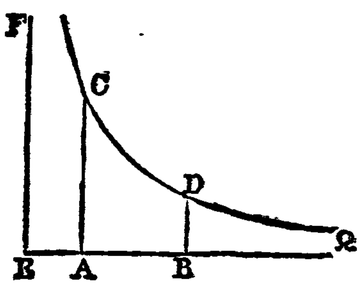
Invenire prædictorum converſum.

VERUM ſi è contrà ex areâ vel longitudine, &c. curvæ alicujus datâ longitudo baſis AB deſideratur, ex æquationibus per præcedentes regulas inventis extrahatur radix x.

SECT. Iª.

Inventio Baſis ex Areâ datâ.

Ut ſi ex areâ ABDC hyperbolæ ($\frac{1}{1+x}=y$) datâ, cupiam baſim AB inveſtigare, areâ iſtâ z nominatâ, extraho radicem hujus z (ABCD) $=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$, &c. neglectis illis terminis in quibus x eſt plurium dimenſionum quàm z, in quotiente, deſideratur.



Ut ſi vellem quòd z ad quinque tantùm dimenſiones in quotiente aſcendat, negligo omnes $-\frac{1}{6}x^6+\frac{1}{7}x^7-\frac{1}{8}x^8$, &c. et radicem hujus tantùm $\frac{1}{5}x^5-\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2+x-z=0$ extraho (o).

Analyſin

(n) Quam tamen loquendi formam ita non probavit Newtonus, ut non alium in finem, quàm ut illa ex argumentationibus Geometrarum exularet, viam ſuam Rationum Primarum Ultimarumque ex-

$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.		
$z + p = x$	$+\frac{1}{5}x^5$	$+\frac{1}{5}z^5$, &c.
	$-\frac{1}{4}x^4$	$-\frac{1}{4}z^4 - z^3p$, &c.
	$+\frac{1}{3}x^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$, &c.
	$-\frac{1}{2}x^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+ x$	$+z + p$
	$- z$	$-z$
$\frac{1}{2}z^2 + q = p$	$+zp^2$	$+\frac{1}{4}z^5$, &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$, &c.
	$-z^3p$	$-\frac{1}{2}z^5$, &c.
	$+z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$-zp$	$-\frac{1}{2}z^3 - zq$
	$+ p$	$+\frac{1}{2}z^2 - q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}x^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z\ )\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

Analyſin, ut vides, exhibui propter adnotanda duo ſequentia. BASIS EX AREA VEL ARCU.

I. Quòd inter ſubſtituendum, iſtos terminos ſemper omitto, quos nulli deinceps uſui fore prævideam. Cujus rei regula eſto, quòd poſt primum terminum, ex quâlibet quantitate ſibi collaterali reſultantem, non addo plures terminos dextrorſum, quàm iſtius primi termini index dimenſionis ab indice dimenſionis maximæ unitatibus diſtat. Ut in hoc exemplo, ubi maxima dimenſio eſt 5, omiſi omnes terminos poſt z^5, poſt z^4 poſui unicum, & duos tantùm poſt z^3. Cùm radix extrahenda (x) ſit parium ubique, vel imparium dimenſionum, hæc eſto regula; quòd poſt primum terminum, ex quâlibet quantitate ſibi collaterali reſultantem, non addo plures terminos dextrorſum, quàm iſtius primi termini index dimenſionis ab indice dimenſionis maximæ binis unitatibus diſtat; vel ternis unitatibus, ſi indices dimenſionum ipſius x unitatibus ubique ternis à ſe invicem diſtant, & ſic de reliquis.

II. Cùm video p, q, vel r, &c. in æquatione noviſſimè reſultante eſſe unius tantùm dimenſionis, ejus valorem, hoc eſt, reliquos terminos quotienti addendos, per diviſionem quæro. Ut hîc vides factum.

S E C T. II.

Inventio Baſis ex datâ Longitudine Curvæ.

Si ex dato arcu CD ſinus AB deſideratur; æquationis $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. ſuprà inventæ, (poſito nempe AB $= x$,

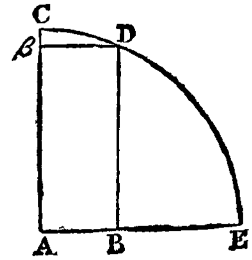
excogitaverit.

(c) Vide Geometr. Analyt. Cap. III. § 8.

CD $= z$,

CD $=z$, & AC $=1$,) radix extracta erit $x=z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{5040}z^7+\frac{1}{362880}z^9$, &c (p).

Et præterea ſi coſinum Aβ ex iſto arcu dato cupis, fac Aβ $(=\sqrt{1-xx})=1-\frac{1}{2}z^2+\frac{1}{24}z^4-\frac{1}{720}z^6+\frac{1}{40320}z^8-\frac{1}{3628800}z^{10}$, &c (q).



C A P U T O C T A V U M.

De Serie progreſſionum continuandâ.

Hic obiter notetur, quòd 5 vel 6 terminis iſtarum radicum cognitis, eas plerumque ex analogiâ obſervatâ poteris ad arbitrium producere.

Sic hanc $x=z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5$, &c. produces dividendo ultimum terminum per hos ordine numeros 2, 3, 4, 5, 6, 7, &c.

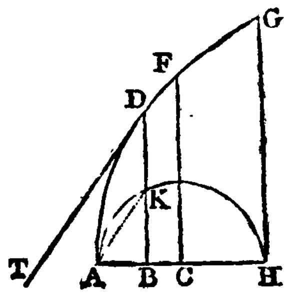
Et hanc $x=z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{5040}z^7$, &c. per hos 2×3, 4×5, 6×7, 8×9, 10×11, &c.

Et hanc $x=1-\frac{1}{2}z^2+\frac{1}{24}z^4-\frac{1}{720}z^6$, &c. per hos 1×2, 3×4, 5×6, 7×8, 9×10, &c.

Et hanc $z=x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c. multiplicando per hos $\frac{1\times 1}{2\times 3}$, $\frac{3\times 3}{4\times 5}$, $\frac{5\times 5}{6\times 7}$, $\frac{7\times 7}{8\times 9}$, &c. Et ſic in reliquis (r).

C A P U T N O N U M.

Applicatio prædictorum ad Curvas Mechanicas.



1. Et hæc de curvis Geometricis ſufficiant. Quinetiam curva etiamſi mechanica ſit, methodum tamen noſtram nequaquam reſpuit.

Exemplo ſit Trochoides, ADFG, cujus vertex A, & axis AH, & AKH rota quâ deſcribitur: et quæratur ſuperficies ABD. Jam poſito AB $=x$, BD $=y$, ut ſuprà, & AH $=1$; primò quæro

longitu-

(p) Vide Geometr. Analyt. C. III. § 9.

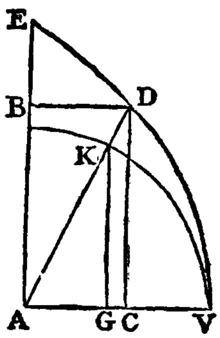
(q) Nempe ex æquatione priore, $x=z-\frac{1}{6}z^3+\frac{1}{120}z^5-$, quadraticè multiplicando, efficitur $x^2=z^2-\frac{1}{3}z^4+\frac{2}{45}z^6-\frac{1}{315}z^8+\frac{2}{14175}z^{10}-$. Ergo $\overline{1-x^2}=1-z^2+\frac{1}{3}z^4-\frac{2}{45}z^6+\frac{1}{315}z^8-\frac{2}{14175}z^{10}+$. Unde radicem extrahendo, veniet $\sqrt{1-x^2}=1-\frac{1}{2}z^2+\frac{1}{24}z^4-\frac{1}{720}z^6+\frac{1}{40320}z^8-$

1

longitudinem ipſius BD. Nempe, ex naturâ trochoidis, eſt KD = arcui AK. Quare tota BD = BK + arc. AK. Sed eſt BK ($=\sqrt{x-xx}$) $= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16} - x^{\frac{7}{2}}$, &c. & (ex prædictis) arcus AK $= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}$, &c. Ergo tota BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$, &c. Et (per Reg. 2.) area ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}}$, &c. De Curvis Mechanicis.

2. Vel breviùs ſic: cùm recta AK tangenti TD parallela ſit, erit AB ad BK ſicut momentum lineæ AB ad momentum lineæ BD, hoc eſt $x : \sqrt{x-xx} :: 1 : \frac{1}{x}\sqrt{x-xx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}}$, &c. Quare (per Reg. 2.) BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}}$, &c. Et ſuperficies ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3156}x^{\frac{11}{2}}$, &c.

3. Non diſſimili modo (poſito C centro circuli, & CB $= x$) obtinebis aream CBDF, &c.



4. Sit area ABDV Quadratricis VDE (cujus vertex eſt V, & A centrum circuli interioris VK cui aptatur) invenienda. Ductâ quâlibet AKD, demitto perpendiculares DB, DC, KG. Eritque KG : AG :: AB (x) : BD (y), ſive $\frac{x \times AG}{KG} = y$. Verùm ex naturâ Quadratricis eſt BA ($=$ DC) $=$ arcui VK, ſive VK $= x$. Quare poſito AV $= 1$, erit GK $= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c. ex ſuprà oſtenſis; & GA $= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c.

Adeoque $y \left(= \frac{x \times AG}{KG}\right) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6}$ &c. ſive, diviſione factâ, $y = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6$, &c. et (per Reg. 2.) area AVDB $= x - \frac{1}{9}x^3 - \frac{1}{225}x^5 - \frac{2}{6615}x^7$, &c.

Sic longitudo Quadratricis VD, licet calculo difficiliori, determinabilis eſt ([f]).

5. Nec quicquam hujuſmodi ſcio, ad quod hæc methodus, idque variis modis, ſeſe non extendit. Imo tangentes ad curvas mechanicas (ſi quando id non aliâs fiat) hujus ope ducantur. Et quicquid vulgaris Analyſis per æquationes ex finito terminorum numero conſtantes (quando id ſit poſſibile) perficit, hæc per

$\frac{1}{3628800}z^{10}$ +

([r]) Vide Geom. Analyt. Cap. III. § 12.

( ) Vide Geometr. Analyt. Problem ult. Exemp. ult. Cæterùm in quarto *Excerptorum ex Epiſtolis* habes Aream et Longitudinem Quadratricis Dinoſtrati.

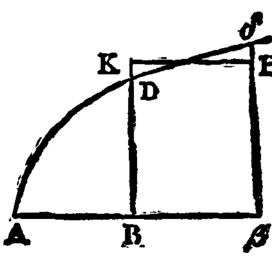
Caput Decimum. æquationes infinitas ſemper perficiat: ut nil dubitaverim nomen Analyſis etiam huic tribuere. Ratiocinia nempe in hâc non minùs certa ſunt quàm in illâ, nec æquationes minùs exactæ; licet omnes earum terminos, nos homines, & rationis finitæ, nec deſignare neque ita concipere poſſimus, ut quantitates inde deſideratas exactè cognoſcamus: ſicut radices ſurdæ finitarum æquationum nec numeris, nec quâvis arte Analyticâ, ita poſſunt exhiberi, ut alicujus quantitas, à reliquis diſtincta, exactè cognoſcatur.

6. Denique ad Analyticam meritò pertinere cenſeatur, cujus beneficio curvarum areæ & longitudines, &c. (id modò fiat) exactè & Geometricè determinentur. Sed iſta narrandi non eſt locus. Reſpicienti duo præ reliquis demonſtranda occurrunt.

CAPUT DECIMUM.

1. *Demonſtratio Quadraturæ Curvarum Simplicium in Regulâ primâ.*

Præparatio pro Regulâ primâ demonſtrandâ.



1. SIT itaque curvæ alicujus ADδ baſis AB $=x$, perpendiculariter applicata BD $=y$, & area ABD $=z$, ut priùs. Item ſit Bβ $=o$, BK $=v$, & rectangulum BβHK (ov) æquale ſpatio BβδD.

Eſt ergo Aβ $=x+o$, & Aδβ $= z+ov$.

His præmiſſis, ex relatione inter x & z, ad arbitrium aſſumptâ, quæro y iſto, quem ſequentem vides, modo.

Pro lubitu ſumatur $\frac{2}{3}x^{\frac{3}{2}}=z$, ſive $\frac{4}{9}x^3=zz$. Tum $x+o$ (Aβ) pro x, & $z+ov$ (Aδβ) pro z ſubſtitutis, prodibit $\frac{4}{9}$ in $x^3+3x^2o+3xo^2+o^3=$ (ex naturâ curvæ) $z^2+2zov+o^2v^2$. Et ſublatis ($\frac{4}{9}x^3$ & zz) æqualibus, reliquiſque per o diviſis, reſtat $\frac{4}{9}$ in $3x^2+3xo+o^2=2zv+ov^2$. Si jam ſupponamus Bβ in infinitum diminui & evaneſcere, ſive o eſſe nihil, erunt v & y æquales, & termini per o multiplicati evaneſcent: quare reſtabit $\frac{4}{9}\times 3xx=2zv$, ſive $\frac{2}{3}xx$ $(=zy)=\frac{2}{3}x^{\frac{3}{2}}y$, ſive $x^{\frac{1}{2}}\ (=\frac{x^2}{x^{\frac{3}{2}}})=y$. Quare è contra ſi $x^{\frac{1}{2}}=y$, erit $\frac{2}{3}x^{\frac{3}{2}}=z$.

Demonſtratio.

REGULARUM DEMONSTRATIONES.

Demonſtratio.

2. Vel generaliter, ſi $\frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z$; ſive, ponendo $\frac{na}{m+n} = c$, & $m+n=p$, ſi $cx^{\frac{p}{n}} = z$, ſive $c^n x^p = z^n$: tum $x+o$ pro x, & $z+ov$ (ſive, quod perinde eſt, $x+oy$) pro z, ſubſtitutis, prodit c^n in $x^p + pox^{p-1}$, &c. $= z^n + noyz^{n-1}$, &c. reliquis nempe terminis, qui tandem evaneſcerent, omiſſis. Jam ſublatis $c^n x^p$ & z^n æqualibus, reliquiſque per o diviſis, reſtat $c^n p x^{p-1} = nyz^{n-1}$ $\left(= \frac{nyz^n}{n} = \frac{nyc^n x^p}{cx^{\frac{p}{n}}}\right)$ ſive, dividendo per $c^n x^p$, erit $px^{-1} = \frac{ny}{cx^{\frac{p}{n}}}$ ſive $pcx^{\frac{p-n}{n}} = ny$; vel, reſtituendo $\frac{na}{m+n}$ pro c, & $m+n$ pro p, hoc eſt, m pro $p-n$, & na pro pc, fiet $ax^{\frac{m}{n}} = y$. Quare è contra, ſi $ax^{\frac{m}{n}} = y$, erit $\frac{n}{m+n} ax^{\frac{m+n}{n}} = z$. Q. E. D.

Inventio curvarum quæ poſſunt quadrari.

3. Hinc in tranſitu notetur modus, quo curvæ tot quot placuerit, quarum areæ ſunt cognitæ, poſſunt inveniri; ſumendo nempe quamlibet æquationem pro relatione inter aream z & baſin x, ut inde quæratur applicata y. Ut ſi ſupponas $\sqrt{aa+xx} = z$, ex calculo invenies $\frac{x}{\sqrt{aa+xx}} = y$. Et ſic de reliquis (t).

SECT. II.

2. Demonſtratio reſolutionis æquationum affectarum.

1. Alterum demonſtrandum eſt literalis æquationum affectarum reſolutio. Nempe quòd quotiens, cùm x ſit ſatis parva, quo magis producitur, eo magis ad veritatem accedit; ut defectus (p, q, vel r, &c.) quo diſtat ab exacto valore ipſius y, tandem evadat minor quâvis datâ quantitate; &, in infinitum producta, ſit ipſi y æqualis. Quod ſic patebit.

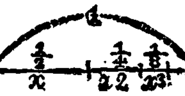
2. I. Quoniam ex ultimo termino æquationum, quarum p, q, r, &c. ſunt radices, quantitas illa in quâ x eſt minimæ dimenſi-

(t) Geometr. Analyt. Prob. VII. & Quadrat. Curv. Prop. II.

Caput Decimum.

onis (hoc eſt, pluſquam dimidium iſtius ultimi termini, ſi ſupponis x ſatis parvam eſſe) in quâlibet operatione perpetuò tollitur: iſte ultimus terminus (per 1. 10. *Elem.*) tandem evadet minor quâvis datâ quantitate; & prorſus evaneſcet, ſi opus infinitè continuatur.

Nempe ſi $x=\frac{1}{2}$, erit x dimidium omnium $x+x^2+x^3+x^4$, &c. et x^2 dimidium omnium $x^2+x^3+x^4+x^5$, &c. Itaque ſi $x \sqsupset \frac{1}{2}$, erit x pluſquam dimidium omnium $x+x^2+x^3$, &c. et x^2 pluſquam dimidium omnium $x^2+x^3+x^4$, &c. Sic ſi $\frac{x}{b} \sqsupset \frac{1}{2}$, erit x pluſquam dimidium omnium $x+\frac{x^2}{b}+\frac{x^3}{bb}$, &c (u). Et ſic de reliquis. Et numeros coefficientes quod attinet, illi plerumque decreſcunt perpetuò, vel ſi quando increſcant, tantùm opus eſt, ut x aliquoties adhuc minor ſupponatur.



II. Si ultimus terminus alicujus æquationis continuò diminuatur, donec tandem evaneſcat, una ex ejus radicibus etiam diminuetur, donec cum ultimo termino ſimul evaneſcat.

III. Quare quantitatum p, q, r, &c. unus valor continuò decreſcit, donec tandem, cùm opus in infinitum producitur, penitùs evaneſcat.

IV. Sed valores iſtarum p, q, vel r, &c. unà cum quotiente eatenus extractâ, adæquant radices æquationis propoſitæ (ſic in reſolutione æquationis $y^3+aay+axy-2a^3-x^3=0$, ſuprà oſtenſâ, percipies $y=a+p=a-\frac{1}{4}x+q=a-\frac{1}{4}x+\frac{xx}{64a}+r$, &c.) Unde ſatis liquet propoſitum, quòd quotiens, infinitè producta, eſt una ex valoribus de y (x).

3. Idem patebit ſubſtituendo quotientem pro y in æquationem propoſitam. Videbis enim terminos illos ſeſe perpetuò deſtruere, in quibus x eſt minimarum dimenſionum.

(u) Hæc ſatis patent ex theoremate illo, quod ſuprà demonſtratum dedimus, de progreſſionibus geometricis.

(x) Vide Geometr. Analyt. Cap. III. § 18 & 19.

EXCERPTA QUÆDAM

EX

EPISTOLIS NEWTONI

AD

SERIES FLUXIONESQUE PERTINENTIA.

E X C E R P T U M I.

EX EPISTOLA NEWTONI AD OLDENBURGUM PRIMA.

[Vide Commerc. Epiſtolic. N. 48.]

DE FORMULIS ALGEBRAICIS IN SERIES INFINITAS RESOLVENDIS.

FRactiones in Infinitas Series reducuntur per diviſionem; & quantitates radicales per extractionem radicum, perinde inſtituendo operationes iſtas in ſpeciebus, ac inſtitui ſolent in decimalibus numeris. Hæc ſunt fundamenta harum reductionum; ſed extractiones radicum, multùm abbreviantur per hoc *Theorema*. THEOREMA BINOMIALE.

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ +, \text{ \&c.}$$

2. Ubi $P+PQ$ ſignificat quantitatem cujus Radix, vel etiam dimenſio quævis, vel radix dimenſionis, inveſtiganda eſt; P, primum terminum quantitatis ejus; Q, reliquos terminos diviſos per primum. Et $\frac{m}{n}$, numeralem indicem dimenſionis ipſius $P+PQ$: ſive dimenſio illa integra ſit; ſive (ut ita loquar) fracta; ſive affirmativa, ſive negativa. Nam, ſicut Analyſtæ, pro aa, aaa, &c. ſcribere ſolent a^2, a^3, &c. ſic ego, pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.a^5}$, &c. ſcribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$; & pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, ſcribo a^{-1}, a^{-2}, a^{-3} (a). Et ſic pro $\frac{aa}{\sqrt{c.\overline{a^3+b^2x}}}$ ſcribo $aa\times\overline{a^3+b^2x}|^{-\frac{1}{3}}$; & pro $\frac{a^2b}{\sqrt{c:\overline{a^3+b^2x}\times\overline{a^3+b^2x}}}$ ſcribo $a^2b\times\overline{a^3+b^2x}|^{-\frac{2}{3}}$: In quo ultimo caſu, ſi $\overline{a^3+b^2x}|^{-\frac{2}{3}}$ concipiatur eſſe $\overline{P+PQ}|^{\frac{m}{n}}$ in Regulà; erit $P=a^3$, $Q=\frac{b^2x}{a^3}$, $m=-2$, & $n=3$. Denique, pro terminis inter operandum inventis in quoto, uſurpo

(a) Vide Arithmet. Univerſ. C. I.

A, B,

Excerptum Primum.

A, B, C, D, &c. nempe A pro primo termino $P^{\frac{m}{n}}$, B pro ſecundo $\frac{m}{n}AQ$, & ſic deinceps (b). Cæterùm uſus Regulæ patebit exemplis.

3. *Exempl.* 1. Eſt $\sqrt{c^2+x^2}$ (ſeu $\overline{c^2+x^2}|^{\frac{1}{2}}$) $= c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256c^9}$, &c. Nam, in hoc caſu, eſt $P = c^2$, $Q = \frac{x^2}{c^2}$, $m = 1$, $n = 2$, A ($= P^{\frac{m}{n}} = \overline{cc}|^{\frac{1}{2}}$) $= c$, B ($= \frac{m}{n} AQ$) $= \frac{x^2}{2c}$, C ($= \frac{m-n}{2n} BQ$) $= -\frac{x^4}{8c^3}$, & ſic deinceps.

4. *Exempl.* 2. Eſt $\sqrt{5: c^5 + c^4x - x^5}$ (i. e. $\overline{c^5+c^4x-x^5}|^{\frac{1}{5}}$) $= c + \frac{c^4x - x^5}{5c^4} - \frac{2c^8x^2 + 4c^4x^6 \mp 2x^{10}}{25c^9}$ + &c. Ut patebit ſubſtituendo in allatam Regulam, 1 pro m, 5 pro n, c^5 pro P, & $\frac{c^4x - x^5}{c^5}$ pro Q. Poteſt etiam $-x^5$ ſubſtituti pro P, & $\frac{c^4x + c^5}{-x^5}$ pro Q; et tunc evadet $\sqrt{5: c^5 + c^4x - x^5}$ $= -x$

Demonſtratio

(b) Hujus formulæ hanc ferè inveſtigationem auctores tradunt.

Caſus 1. Deſignante P unitatem, ut ſit $\overline{P+PQ}|^{\frac{m}{n}} = \overline{1+Q}|^{\frac{m}{n}}$, finge ſeriem infinitam

$$1 + aQ + bQ^2 + cQ^3 + dQ^4 + eQ^5 + . = \overline{1+Q}|^{\frac{m}{n}}.$$

Unde hæc orietur fluxionum æquatio.

$$a\dot{Q} + 2bQ\dot{Q} + 3cQ^2\dot{Q} + 4dQ^3\dot{Q} + 5eQ^4\dot{Q} + . = \frac{m}{n}\dot{Q} \times \overline{1+Q}|^{\frac{m-n}{n}}.$$

Et ex duabus hiſce, diviſione nimirum, tertia fiet.

$$\frac{a + 2bQ + 3cQ^2 + 4dQ^3 + 5eQ^4 + .}{1 + aQ + bQ^2 + cQ^3 + dQ^4 + eQ^5 + .} = \frac{m}{n} \frac{\times \overline{1+Q}|^{\frac{m-n}{n}}}{\overline{1+Q}|^{\frac{m}{n}}} = \frac{m}{n.\overline{1+Q}}.$$

Ex hâc tertiâ, fractionibus per multiplicationes quibus opus eſt ſublatis, quarta demum exſiſtet.

$$\begin{matrix} na + 2nbQ + 3ncQ^2 + 4ndQ^3 + 5neQ^4 + . \\ + naQ + 2nbQ^2 + 3ncQ^3 + 4ndQ^4 + . \end{matrix} = m + maQ + mbQ^2 + mcQ^3 + mdQ^4 + meQ^5 + .$$

Iam vero membrorum homologorum coefficientibus æquatis, elicientur ſeriei principio poſitæ coefficientes. Nempe hoc modo.

I. $na = m$. Ergo $a = \frac{m}{n}$.

II. $2nb + na = ma$. Ergo $b = a \times \frac{m-n}{2n}$.

III. $3nc + 2nb = mb$. Ergo $c = b \times \frac{m-2n}{3n}$.

IV. $4nd + 3nc = mc$. Ergo $d = c \times \frac{m-3n}{4n}$.

V. $5ne + 4nd = md$. Ergo $e = d \times \frac{m-4n}{5}$.

Hinc literâ A in hoc caſu deſignante unitatem, membrum ſecundum ſeriei principio poſitæ, ſive $aQ = \frac{m}{n} AQ$.

M rum

$= -x + \frac{c^4x + c^5}{5x^4} + \frac{2c^8x^2 + 4c^9x + c^{10}}{25x^7} +$ &c. Prior modus eligendus eſt, ſi x valdè parvum ſit; poſterior, ſi valdè magnum.

THEOREMA BINOMINALE.

5. *Exempl.* 3. Eſt $\frac{N}{\sqrt{3:y^3 - a^2y}}$ (hoc eſt, $N \times \overline{y^3 - a^2y}|^{-\frac{1}{3}}$) æqualis $N \times \frac{1}{y} + \frac{a}{3y^3} + \frac{2a^4}{9y} + \frac{14a^6}{81y} +$, &c. Nam $P = y^3$, $Q = \frac{-aa}{yy}$, $m = -1$, $n = 3$. $A\ (= P^{\frac{m}{n}} = y^{3 \times -\frac{1}{3}}) = y^{-1}$, hoc eſt $\frac{1}{y}$. $B\ (= \frac{m}{n} AQ = -\frac{1}{3} \times \frac{1}{y} \times \frac{-aa}{yy}) = \frac{aa}{3y^3}$, &c.

6. *Exempl.* 4. Radix cubica ex quadrato-quadrato ipſius $d+e$, (hoc eſt, $\overline{d+e}|^{\frac{4}{3}}$) eſt $d^{\frac{4}{3}} + \frac{4ed^{\frac{1}{3}}}{3} + \frac{2e^2}{9d^{\frac{2}{3}}} - \frac{4e^3}{81d^{\frac{5}{3}}} +$, &c. Nam $P = d$, $Q = \frac{e}{d}$, $m = 4$, $n = 3$, $A\ (= P^{\frac{m}{n}}) = d^{\frac{4}{3}}$, &c.

7. *Exempl.* 5. Eodem modo ſimplices etiam poteſtates eliciuntur. Ut ſi quadrato-cubus ipſius $d+e$, (hoc eſt, $\overline{d+e}|^5$, ſeu $\overline{d+e}|^{\frac{5}{1}}$) deſideretur: erit, juxta regulam, $P = d$, $Q = \frac{e}{d}$, $m = 5$, & $n = 1$; adeoque $A\ (= P^{\frac{m}{n}}) = d^5$, $B\ (= \frac{m}{n} AQ) = 5d^4e$, & ſic $C = 10d^3e^2$, $D = 10d^2e^3$,

E =

THEOREMATIS BINOMINALIS.

Membrum tertium, ſive $bQ^2 = a \times \frac{m-n}{2n} Q^2 = aQ \times \frac{m-n}{2n} Q = \frac{m-n}{2n} BQ$, deſignante nimirum literâ B quantitatem aQ.

Membrum quartum ſeriei principio poſitæ, ſive $cQ^3 = b \times \frac{m-2n}{3n} Q^3 = bQ^2 \times \frac{m-2n}{3n} Q = \frac{m-2n}{3n} CQ$, deſignante literâ C quantitatem bQ^2.

Membrum quintum ſeriei principio poſitæ, ſive $dQ^4 = c \times \frac{m-3n}{4n} Q^4 = cQ^3 \times \frac{m-3n}{4n} Q = \frac{m-3n}{4n} DQ$, deſignante literâ D quantitatem cQ^3.

In ſummâ, ſeries principio poſita, videlicet, $1 + aQ + bQ^2 + cQ^3 + dQ^4 + eQ^5$, &c. huic æqualis erit $1 + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \frac{m-4n}{5n} EQ +$, Q. E. D.

Caſus 2. Jam verò litera P quantitatem quamvis deſignet ſive majorem ſive minorem unitate. Quantitas $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$. Sed ex caſu priori $\overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n} Q + \frac{m-n}{2n} aQ^2 + \frac{m-2n}{3n} bQ^3 + \frac{m-3n}{4n} cQ^4 + \frac{m-4n}{5n} dQ^5 +$.

Ergo $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m}{n}} Q + \frac{m-n}{2n} P^{\frac{m}{n}} aQ^2 + \frac{m-2n}{3n} P^{\frac{m}{n}} bQ^3 + \frac{m-3n}{4n} P^{\frac{m}{n}} cQ^4 + \frac{m-4n}{5n} P^{\frac{m}{n}} dQ^5$, +.

Hoc eſt, ſi quantitatem $P^{\frac{m}{n}}$ literâ A deſignaveris, quantitatem $P^{\frac{m}{n}} aQ$ literâ B, quantitatem $P^{\frac{m}{n}} bQ^2$ literâ C, quantitatem $P^{\frac{m}{n}} cQ^3$ literâ D, quantitatem $P^{\frac{m}{n}} dQ^4$ literâ E: $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \frac{m-4n}{5n} EQ +$. Q. E. D.

Hujus inveſtigationis auctorem Heyneſium quendam nomine Raphſonus perhibet. (Hiſtory of Fluxions, Cap. 3.)

EXCERPTUM PRIMUM. $E = 5de^4$, $F = e^5$, & $G\ (= \frac{m-5n}{6n} FQ) = 0$. Hoc eſt, $\overline{d+e}|^5 = d^5 + 5d^4e + 10d^3e^2 + 10d^2e^3 + 5de^4 + e^5$.

8. *Exempl.* 6. Quinetiam Diviſio, ſive ſimplex ſit, ſive repetita, per eandem regulam perficitur. Ut ſi $\frac{1}{d+e}$ (hoc eſt, $\overline{d+e}|^{-1}$ ſive $\overline{d+e}|^{-\frac{1}{1}}$) in ſeriem ſimplicium terminorum reſolvendum ſit: Erit, juxta regulam, $P = d$. $Q = \frac{e}{d}$. $m = -1$. $n = 1$. & $A\ (= P^{\frac{m}{n}} = d^{-\frac{1}{1}}) = d^{-1}$ ſeu $\frac{1}{d}$. $B\ (= \frac{m}{n} AQ = -1 \times \frac{1}{d} \times \frac{e}{d}) = -\frac{e}{dd}$. Et ſic $C = \frac{ee}{d^3}$. $D = -\frac{e^3}{d^4}$, &c. Hoc eſt, $\frac{1}{d+e} = \frac{1}{d} - \frac{e}{dd} + \frac{ee}{d^3} - \frac{e^3}{d^4}$, &c.

9. *Exempl.* 7. Sic & $\overline{d+e}|^{-3}$, (hoc eſt, Unitas ter diviſa per $d+e$, vel ſemel per cubum ejus) evadit $\frac{1}{d^3} - \frac{3e}{d^4} + \frac{6ee}{d^5} - \frac{10e^3}{d^6} +$, &c.

10. *Exempl.* 8. Et $N \times \overline{d+e}|^{-\frac{1}{3}}$, (hoc eſt, N diviſum per radicem cubicam ipſius $d+e$) evadit $N \times \overline{\frac{1}{d^{\frac{1}{3}}} - \frac{e}{3d^{\frac{4}{3}}} + \frac{2ee}{9d^{\frac{7}{3}}} - \frac{14e^3}{81d^{\frac{10}{3}}} +, \&c.}|$

11. *Exempl.* 9. Et $N \times \overline{d+e}|^{-\frac{3}{5}}$, hoc eſt, N diviſum per radicem quadrato-cubicam ex cubo ipſius $d+e$, ſive $\frac{N}{\sqrt{5: d^3 + 3dde + 3dee + e^3}}$) evadit $N \times \frac{1}{d^{\frac{3}{5}}} - \frac{3e}{5d^{\frac{8}{5}}} + \frac{12ee}{25d^{\frac{13}{5}}} - \frac{52e^3}{125d^{\frac{18}{5}}} +$, &c.

12. Per eandem regulam, Geneſis poteſtatum, Diviſiones per poteſtates aut per quantitates radicales, & Extractiones radicum altiorum in numeris, etiam commodè inſtituuntur.

13. Extractiones radicum affectarum in ſpeciebus imitantur earum extractiones in numeris. Sed methodus *Vietæ* & *Oughtredi* noſtrì huic negotio minùs idonea eſt. Quapropter aliam excogitare adactus ſum, *cujus ſpecimina, ne repetantur, vide in tractatu de Analyſi, &c.* Cap. IV.

14. Dicam tantùm in genere, quòd radix cujuſvis æquationis ſemel extracta, pro regulâ reſolvendi conſimiles æquationes aſſervari poſſit; quòdque ex pluribus ejuſmodi regulis, regulam generaliorem plerumque efformare liceat; & quòd radices omnes, ſive ſimplices ſint ſive affectæ, modis infinitis extrahi poſſint; de quorum ſimplicioribus itaque ſemper conſulendum eſt.

EXCERPTUM

E X C E R P T U M IIm.

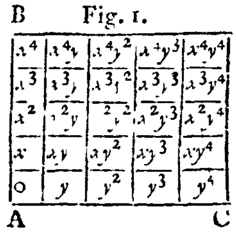
Ex Epiſtolâ Newtoni ad Oldenburgum poſteriore.

[*Commerc. Epiſtolic.*] N°. 55—64.

De Seriebus condendis et invertendis.

[*Quæ ſequuntur ſcripta ſunt in explicationem Epiſtolæ præcedentis.*]

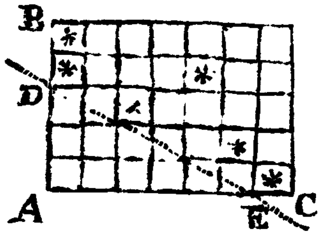
Fig. 1.



QUOD verò attinet ad Inventionem terminorum p, q, r, (vide Analyſ. per Æquat. Infinit. cap. 4.) in extractione radicis affectæ, primum p ſic eruo (c)

2. Deſcripto angulo recto EAC, latera ejus BA, CA divido in partes æquales; & inde normales erigo diſtribuentes angulare ſpatium in æqualia parallelogramma vel quadrata, quæ concipio denominata eſſe à dimenſionibus duarum indefinitarum ſpecierum, puta x & y, regulariter aſcendentium à termino A; prout vides in Fig. 1. inſcriptas. Ubi y denotat radicem extrahendam; et x alteram indefinitam quantitatem, ex cujus poteſtatibus ſeries conficienda eſt. Deinde, cùm æquatio aliqua proponitur, parallelogramma ſingulis ejus terminis correſpondentia inſignio notâ aliquâ: et regulâ ad duo, vel fortè plura, ex inſignitis parallelogrammis applicatâ; quorum unum ſit humillimum in columnâ ſiniſtrâ juxta AB, & alia ad regulam dextrorſum ſita, cæteraque omnia, non contingentia regulam, ſupra eam jaceant; ſeligo terminos æquationis per parallelogramma contingentia regulam deſignatos, & inde quæro quantitatem quotienti addendam.

Fig. 2.



3. Sic ad extrahendam radicem y, ex $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$; parallelogramma hujus terminis reſpondentia ſigno notâ aliquâ *; ut vides in Fig. 2. Dein applico regulam DE ad inferiorem è locis ſignatis in ſiniſtrâ columnâ; eamque ab inferioribus ad ſuperiora dextrorſum gyrare facio,

(c) Vide Geometr. Analyt. c. iii. §. 2.

EXCERPTUM SECUNDUM. donec alium ſimiliter, vel fortè plura, è reliquis ſignatis locis coeperit attingere. Videoque loca ſic attacta eſſe x^3, x^2y^2, & y^6. E terminis itaque, $y^6-7a^2x^2y^2+6a^3x^3$ tanquam nihilo æqualibus (& inſuper ſi placet reductis ad $v^6-7v^2+6=0$, ponendo $y=v\sqrt{ax}$,) quæro valorem y, & invenio quadruplicem, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, & $-\sqrt{2ax}$, quorum quemlibet pro primo termino quotientis accipere licet, prout è radicibus quampiam extrahere decretum eſt.

4. Sic æquatio $y^3+axy+aay-x^3-2a^3=0$, quam reſolvebam in priori Epiſtolâ, dat $-2a^3+aay+y^3=0$, & inde $y=a$ proximè: cùm itaque a ſit primus terminus valoris y, pono p pro cæteris omnibus in infinitum, & ſubſtituo $a+p=y$. (Obvenient hic aliquando difficultates nonnullæ; ſed ex iis, credo, lector ſe proprio marte extricabit.) Subſequentes vero termini q, r, s, &c. eodem modo ex æquationibus ſecundis, tertiis, cæteriſque eruuntur, quo primus p è prima, ſed curâ leviori; quia cæteri valores y ſolent prodire dividendo terminum involventem infimam poteſtatem indefinitæ quantitatis x per coefficientem radicis p, q, r, aut s.

5. Intellexti credo ex ſuperioribus, regreſſionem ab areis curvarum ad lineas rectas, fieri per hanc extractionem radicis affectæ. Sed duo alii ſunt modi quibus idem perficio.

6. Eorum unus affinis eſt computationibus, quibus colligebam approximationes ſub finem alterius Epiſtolæ, & intelligi poteſt per hoc exemplum. Proponatur æquatio ad aream hyperbolæ $z=x+\frac{1}{2}xx+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5$, &c (d). Et partibus ejus multiplicatis in ſe, emerget $z^2=x^2+x^3+\frac{11}{12}x^4+\frac{5}{6}x^5$, &c. $z^3=x^3+\frac{3}{2}x^4+\frac{7}{4}x^5$, &c. $z^4=x^4+2x^5$, &c. $z^5=x^5$, &c. Jam de z aufero $\frac{1}{2}zz$, & reſtat $z-\frac{1}{2}zz=x-\frac{1}{6}x^3-\frac{5}{24}x^4-\frac{13}{60}x^5$, &c. Huic addo $\frac{1}{6}z^3$, & fit $z-\frac{1}{2}z^2+\frac{1}{6}z^3=x+\frac{1}{24}x^4+\frac{3}{40}x^5$, &c. Aufero $\frac{1}{24}z^4$, & reſtat $z-\frac{1}{2}z^2+\frac{1}{6}z^3-\frac{1}{24}z^4=x-\frac{1}{120}x^5$, &c. Addo $\frac{1}{120}z^5$, & fit $z-\frac{1}{2}z^2+\frac{1}{6}z^3-\frac{1}{24}z^4+\frac{1}{120}z^5=x$ quamproximè; ſive $x=z-\frac{1}{2}z^2+\frac{1}{6}z^3-\frac{1}{24}z^4+\frac{1}{120}z^5$, &c.

7. Eodem modo, ſeries de unâ indefinitâ quantitate, in aliam transferri poſſunt. Quemadmodum ſi poſito r radio circuli, x

(d) Vide Geometr. Analyt. cap. IX. §. 34.

ſinu

ſinu recto arcûs z, & $x + \frac{x^3}{6rr} + \frac{3x^5}{40r^4} +$ &c. longitudine arcûs iſtius; atque hanc ſeriem à ſinu recto ad tangentem vellem transferre: quæro longitudinem tangentis $\frac{rx}{\sqrt{rr-xx}}$, & reduco in infinitam ſeriem, $x + \frac{x^3}{2rr} + \frac{3x^5}{8r^4} +$ &c. Vocetur hæc quantitas, t. Colligo poteſtates ejus $t^3 = x^3 + \frac{3x^5}{2rr}$ &c. $t^5 = x^5 +$ &c. Aufero autem t de z, & reſtat $z - t = -\frac{1}{3}x^3 - \frac{3}{10}x^5 -$ &c. Addo $\frac{1}{3}t^3$, & fit $z - t + \frac{1}{3}t^3 = \frac{1}{5}x^5 +$ &c. Aufero $\frac{1}{5}t^5$, & reſtat $z - t + \frac{1}{3}t^3 - \frac{1}{5}t^5 = 0$ quamproximè. Quare eſt $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 -$ &c. Sed ſiquis in uſus trigonometricos me juſſiſſet exhibere expreſſionem arcûs per tangentem; eam non hoc circuitu, ſed directâ methodo quæſiviſſem. INVERSIO SERIERUM.

8. Per hoc genus computi colliguntur etiam ſeries ex duabus vel pluribus indefinitis quantitatibus conſtantes; & radices affectarum æquationum magnâ ex parte extrahuntur. Sed ad hunc poſteriorem uſum adhibeo potiùs methodum in alterâ epiſtolâ deſcriptam tanquam generaliorem, & (regulis pro eliſione ſuperfluorum terminorum habitis) paulo magis expeditam.

9. Pro Regreſſione vero ab areis ad lineas rectas, & ſimilibus, poſſunt hujuſmodi *Theoremata* adhiberi.

10. THEOREMA I. Sit $z = ay + by^2 + cy^3 + dy^4 + ey^5$, &c.

Et viciſſim erit
$$y = \frac{z}{a} - \frac{b}{a^3}z^2 + \frac{2b^2 - ac}{a^5}z^3 + \frac{5abc - 5b^3 - a^2d}{a^7}z^4 + \frac{3a^2c^2 - 21ab^2c + 6a^2bd + 14b^4 - a^3e}{a^9}z^5 + \&c.$$

11. *Exempli gratiâ.* Proponatur æquatio ad aream hyperbolæ, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c (e). Et ſubſtitutis in regula 1 pro a, $-\frac{1}{2}$ pro b, $\frac{1}{3}$ pro c, $-\frac{1}{4}$ pro d, & $\frac{1}{5}$ pro e; viciſſim exurgit, $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 +$ &c.

12. THEOREMA II. Sit $z = ay + by^3 + cy^5 + dy^7 + ey^9 +$ &c.

(e) Vide Geometr. Analyt. c. IX. §. 34.

EXCERPTUM TERTIUM.

Et viciſſim erit $y = \frac{z}{a}$

$$- \frac{b}{a^4} z^3$$

$$+ \frac{3b^2 - ac}{a^7} z^5$$

$$+ \frac{8abc - a^2d - 12b^3}{a^{10}} z^7$$

$$+ \frac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}} z^9 + \&c.$$

13. *Exempli gratiâ.* Proponatur æquatio ad arcum circuli, $z = y + \frac{y^3}{6rr} + \frac{3y^5}{40r^4} + \frac{5y^7}{112r^6} + \&c.$ Et ſubſtitutis in regulâ 1 pro a, $\frac{1}{6r^2}$ pro b, $\frac{3}{40r^4}$ pro c, $\frac{5}{112r^6}$ pro d, &c. orietur $y = z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \&c$ (f).

EXCERPTUM III.

Ex Epiſtolâ Newtoni ad Walliſium, de radicibus æquationum fluxionalium extrahendis.

1. SUB finem epiſtolæ anni 1676 [*hæc ſunt verba* Walliſii] ſcribit [D. *Newtonus*] etiam Problema determinandi curvas per conditiones tangentium in ſuâ poteſtate eſſe, unà cum aliis difficilioribus; ad quæ ſolvenda ſe uſum eſſe dicit duplici methodo, unâ concinniore, alterâ generaliore; & utramque literis tranſpoſitis celat: quæ in ordinem redactæ hanc ſententiam exhibent. *Una methodus conſiſtit in extractione fluentis quantitatis ex æquatione ſimul involvente fluxionem ejus. Altera tantùm in aſſumptione ſeriei*

ÆQUATIONIS INFINITÆ

(f) Duobus hiſce Newtonianis meritò tertium adjungatur Moivræi Theorema, cujus ope radices extrahantur infinitè quæ dicuntur affectæ.

Sit æquatio inter ſeries infinitas

$$\mathrm{a}z + \mathrm{b}z^2 + \mathrm{c}z^3 + \mathrm{d}z^4 + \mathrm{e}z^5 + . = ay + by^2 + cy^3 + dy^4 + ey^5 +$$

Ut extrahatur radix y, ponatur

$$y = \mathrm{A}z + \mathrm{B}z^2 + \mathrm{C}z^3 + \mathrm{D}z^4 + \mathrm{E}z^5 +$$

Hujus coefficientes his æquationibus definientur:

$$\mathrm{A} = \frac{\mathrm{a}}{a}$$

$$\mathrm{B} = \frac{a^2\mathrm{b} - b\mathrm{a}^2}{a^3}$$

$$\mathrm{C} = \frac{a^4\mathrm{c} - ac\mathrm{a}^3 - 2ba^2\mathrm{ab} + 2b^2\mathrm{a}^3}{a^5}$$

$$\mathrm{D} = \frac{a^6\mathrm{d} - 2ba^4\mathrm{ca} + 2bac\mathrm{a}^4 + 6b^2a^2\mathrm{a}^2\mathrm{b} - 5b^3\mathrm{a}^4 - b\mathrm{b}^2a^4 + 3cba\mathrm{a}^4 - 3ca^3\mathrm{a}^2\mathrm{b} - da^2\mathrm{a}^4}{a^7}$$

$a^8\mathrm{e}$ —

riei pro quantitate quâlibet incognitâ ex quâ cætera commodè derivari poſſunt; & in collatione terminorum homologorum æquationis reſultantis ad eruendos terminos aſſumptæ ſeriei. Harum methodorum ſecunda ex verbis jam recitatis abſque ulteriore explicatione intelligi poteſt; priorem ab authore jam accepi ut ſequitur. De Rad. Æquat. Flux. Extrahend.

2. Hæc methodus, ait, ejuſdem eſt generis cum eâ pro extrahendo radices ex æquationibus affectis ſuperiùs deſcriptâ. Pone quòd Problema reſolvendum reducatur ad æquationem fluentes quantitates y & z unà cum earum fluxionibus $\dot{y}$ & $\dot{z}$ involventem, & quòd fluxio ipſius z uniformis ſit. Ut hæc fluxio ex æquatione evaneſcat, pro eâ ponatur unitas, & manebit æquatio ſolas y, z & $\dot{y}$ involvens, *quam reſolvendam* vocat. Proponitur inventio ipſius y in ſerie infinitâ convergente, quæ ſolam z involvet. Hoc in aliquibus æquationibus impoſſibile eſt, in aliis præparationem æquationum requirit, ubi verò directè confici poſſit reſolutio eſt hujuſmodi.

PROBLEMA

3. *Ex æquatione fluxionem radicis involvente radicem extrahere.*

RESOLUTIO.

TERMINI omnes, ex eodem æquationis latere conſiſtentes, æquentur nihilo, & ipſarum y & $\dot{y}$ dignitates (ſi opus ſit) exaltentur vel deprimantur, ſic ut earum indices nec alicubi negativi ſint, nec tamen altiores quàm ad hunc effectum requiritur; & ſit kz^{λ} terminus infimæ dignitatis eorum, qui neque per y, ne-

$$E = \frac{a^8\mathrm{e} - 2ba^6\mathrm{ad} + 6b^2a^4\mathrm{ca}^2 - 21b^2ac\mathrm{a}^5 - 16b^3a^2\mathrm{a}^3\mathrm{b} + 14b^4\mathrm{a}^5 + 2b^2\mathrm{b}^2a^4\mathrm{a} + 20bca^3\mathrm{a}^3\mathrm{b} + 6bda^2\mathrm{a}^5 - 2ba^6\mathrm{cb} + 4b^2a^4\mathrm{b}^2 - 4b^3a^2\mathrm{ba}^2 - 3ca^5\mathrm{ca}^2 + 3c^2a^2\mathrm{a}^5 - 3ca^5b^2\mathrm{a} - 4da^4\mathrm{a}^3\mathrm{b} - ea^3\mathrm{a}^5}{a^9}$$ Reductio Moivræana.

&c.

Vel hoc modo

$$A = \frac{\mathrm{a}}{a}. \quad B = \frac{\mathrm{b} - bA^2}{a}. \quad C = \frac{\mathrm{c} - 2bBA - c.A^3}{a}.$$

$$D = \frac{\mathrm{d} - 2b.AC - bB^2 - 3cA^2B - d.A^4}{a}.$$

$$E = \frac{\mathrm{e} - 2b.AD - 2bBC - 3cA^2C - 3cB^2A - 4dA^3B - eA^5}{a}.$$

Harum inveſtigationem Maclaurinus tradit Algebræ ſuæ Part II. Cap. X.

que

Excerptum Tertium.

que per ejus fluxionem $\dot{y}$, neque per earum dignitatem quamvis multiplicantur. Sit $lz^{\mu}y\,\dot{y}^{\beta}$ terminus alius quilibet, & omnes ordine terminos percurrendo collige ex ſingulis ſeorſim numerum $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ ſic, ut tot habeas ejuſmodi numeros quot ſunt termini. Horum numerorum maximus vocetur ν, & z^{ν} erit dignitas primi termini ſeriei. Pro ejus coefficiente ponatur a, & in æquatione quæ *reſolvenda* dicitur, ſcribe az^{ν} pro y, & $\nu az^{\nu-1}$ pro $\dot{y}$; ac termini omnes reſultantes in quibus z ejuſdem eſt dignitatis ac in termino kz^{λ}, ſub propiis ſignis collecti, ponantur æquales nihilo. Nam hæc æquatio debitè reducta dabit coefficientem a. Sic habes az^{ν} terminum primum ſeriei.

Operatio ſecunda.

4. Pro reliquis omnibus hujus ſeriei terminis nondum inventis pone p, & habebis æquationem $y=az^{\nu}+p$, & inde etiam æquationem $\dot{y}=\nu az^{\nu-1}+\dot{p}$. In reſolvenda, pro y & $\dot{y}$ ſcribe hos eorum valores, & habebis reſolvendam novam, ubi p officium præſtat ipſius y: & ex hâc reſolvendâ primum extrahes terminum ſeriei p eodem modo atque terminum primum ſeriei totius, $y=az^{\nu}+p$, ex reſolvendâ primâ extraxiſti.

Operatio tertia et ſequentes.

5. Dein tertiam reſolvendam eâdem ratione invenias atque ſecundam inveniſti, & ex eâ terminum tertium ſeriei totius extrahes. Et ſimiliter reſolvendam quartam invenies, & ex eâ quartum ſeriei terminum, & ſic in infinitum. Series autem ſic inventa erit radix æquationis quam extrahere oportuit.

EXEMPLUM.

6. Ex æquatione $y^2\dot{z}^2-z^2\dot{z}\dot{y}-d^2z\dot{z}+dz\dot{z}^2=0$, extrahenda ſit radix y. Pone $\dot{z}=1$, & æquatio evadet $y^2-z^2\dot{y}-dd+dz=0$, quæ eſt reſolvenda. Jam verò terminus infimus, in quo nec y, neque $\dot{y}$ reperitur, eſt dd, qui ipſi kz^{λ} æquatus dat $\lambda=0$. Terminis reliquis y^2, $-z^2\dot{y}$ pone $lz^{\mu}y^{\alpha}\dot{y}^{\beta}$ æqualem ſucceſſivè, & inde in primo caſu habebis $\mu=0$, $\alpha=2$, $\beta=0$; in ſecundo $\mu=2$, $\alpha=0$, & $\beta=1$.

Et

Et hinc $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ ſit in primo caſu o, in ſecundo -1. Unde ν eſt o, & az^{ν} & $\nu az^{\nu-1}$ ſunt a & o; quarum ultimæ duæ a & o in reſolvendâ pro y & $\dot{y}$ ſcriptæ, producunt $aa+oz^2-dd+dz$; & termini aa & $-dd$, in quibus index dignitatis z eſt λ ſeu o, poſiti æquales nihilo dant $a=d$. Unde primus ſeriei terminus az^{ν} evadit d. De Rad. Æquat. Flux. Extrahend.

Operatio ſecunda.

7. Pro terminis reliquis pone p, & habebis æquationem $y=d+p$, & inde $\dot{y}=\dot{p}$; qui valores in reſolvendâ pro y & $\dot{y}$ ſubſtituti dant reſolvendam novam $2dp+pp-zz\dot{p}+dz=0$, ubi p & $\dot{p}$ vices ſubeunt ipſarum y & $\dot{y}$. Terminus unicus in quo nec p neque $\dot{p}$ reperitur eſt dz, qui cum termino kz^{λ} collatus dat $\lambda=1$. Terminis reliquis $2dp$, pp & $-zz\dot{p}$ pone $lz^{\mu}p^{\alpha}\dot{p}^{\beta}$ æqualem ſucceſſivè, & inde in primo caſu habebis $\mu=0$, $\alpha=1$, & $\beta=0$; in ſecundo $\mu=0$, $\alpha=2$, & $\beta=0$; & in tertio $\mu=2$, $\alpha=0$, & $\beta=1$. Et hinc $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ evadit primo caſu 1, in ſecundo $\frac{1}{2}$, in tertio o. Unde ν eſt 1, & az^{ν} & $\nu az^{\nu-1}$ ſunt az & a. Termini duo ultimi az & a in reſolvendâ pro p & $\dot{p}$ reſpectivè ſcripti, producunt $2daz+a^2z^2-az^2+dz$. Et termini $2daz$ & dz, in quibus index dignitatis z eſt λ ſeu 1, poſiti æquales nihilo, dant $a=-\frac{1}{2}$. Unde az, terminus primus ſeriei, p fit $-\frac{1}{2}z$.

Operatio tertia.

8. Pro terminis reliquis nondum inventis pone q, & habebis æquationem $p=-\frac{1}{2}z+q$, & inde $\dot{p}=-\frac{1}{2}+\dot{q}$: qui valores pro p & $\dot{p}$ in reſolvendâ noviſſimâ ſubſtituti producunt reſolvendam novam $2dq-zq+qq+\frac{3}{4}zz-zz\dot{q}=0$. Ubi q & $\dot{q}$ vices ſupplent ipſorum y & $\dot{y}$. Terminus unicus in quo neque q nec $\dot{q}$ reperitur eſt $\frac{3}{4}zz$, qui cum kz^{λ} collatus dat $\lambda=2$. Terminis reliquis $2dq$, $-zq$, $+qq$, $-zz\dot{q}$ pone $lz^{\mu}q^{\alpha}\dot{q}^{\beta}$ æqualem ſucceſſivè; & inde in primo caſu habebis $\mu=0$, $\alpha=1$, & $\beta=0$; in ſecundo, $\mu=1$, $\alpha=1$, $\beta=0$; in tertio, $\mu=0$, $\alpha=2$, $\beta=0$: in quarto, $\mu=2$, $\alpha=0$, $\beta=1$: & inde $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ evadit in primo caſu 2, in ſecundo, tertio, & quarto, 1.

Excerptum Tertium.

Et hinc ν eſt 2, vel az^{ν} & $\nu az^{\nu-1}$ ſunt az^2 & $2az$: qui valores in reſolvendâ pro q & $\dot{q}$ ſubſtituti dant $2daz^2 - az^3 + aaz^4 + \frac{3}{4}zz - 2az^3$; & termini $2dazz + \frac{3}{4}zz$, in quibus index dignitatis z eſt λ ſeu 2, poſiti æquales nihilo, dant $a = -\frac{3}{8d}$. Unde az^{ν} terminus primus ſeriei q evadit $-\frac{3zz}{8d}$.

Operatio quarta.

9. Pro reliquis ſeriei terminis nondum inventis pone r, & habebis æquationes $q = -\frac{3zz}{8d} + r$, & $\dot{q} = -\frac{3z}{4d} + \dot{r}$; & inde reſolvendam novam $2dr + \frac{9z^3}{8d} - zr + \frac{9z^4}{64dd} - \frac{3zz\dot{r}}{4d} + rr - zzr = 0$; & ex eâ per methodum ſuperiorem habebis $-\frac{9z^3}{16dd}$ terminum primum ſeriei r. Et ſic pergitur in infinitum.

10. Eſt igitur radix extrahenda $y = d + p = d - \frac{1}{2}z + q = d - \frac{1}{2}z - \frac{3zz}{8d} + r = d - \frac{1}{2}z - \frac{3zz}{8d} - \frac{9z^3}{16dd} - \&c.$ Et operationem continuando producere licet radicem ad terminos plures ([g]).

11. Et eâdem methodo, dicit *Newtonus*, radices æquationum, fluxiones ſecundas, tertias, quartas, ($\ddot{y}$, $\dddot{y}$, $\ddddot{y}$,) aliaſque involventium, extrahi poſſe ([h]).

12. His utitur radicum extractionibus ubi aliæ methodi nil proſunt. Nam in epiſtolâ prædictâ anni 1676 docet, quòd in ſolutione problematum de tangentibus inverſorum, caſus aliqui dantur in quibus hæc methodus generalis non requiritur: & particulariter, ſi in triangulo rectangulo quod ab ordinatâ, tangente, & interjacente parte abſciſſæ conſtituitur, relatio duorum quorumlibet è lateribus tribus per æquationem quamvis definiatur, problema abſque methodo hâcce generali ſolvi poterit.

13. Methodi autem hæ omnes, tam particulares quàm generales, collectum

([g]) Harum operationum rationem Maclaurinus optimè explicavit Algebræ ſuæ Part II. Cap. X.

([h]) Aliam etiam viam reſolvendi æquationes, quæ fluxionibus ſunt implicitæ, Newtonus in Geometriâ Analyticâ docet capite ejus Libri 4°. Sect. 2. quæ cùm non minùs generalis ſit quàm illa quæ in hoc excerpto tradita eſt, & computationum facilitate, meo equidem judicio, longè illi antecellat, miror profectò eam ab omnibus ferè relictam, cùm hæc, ſubtilior ſanè ſed uſu difficilior, diligentiſſimè exculta ſit.

collectim ſumptæ, ſolutionem exhibent ſecundæ partis problematis, quod *Newtonus* ſub initio iſtius epiſtolæ his verbis propoſuit. *Datâ æquatione quotcunque fluentes quantitates involvente fluxiones invenire, & vice verſâ.* Nam tota fluxionum methodus in hujus directâ & inverſâ ſolutione conſiſtit. Arcus ex Sinu.

EXCERPTUM IV.

Ex epiſtolâ Newtoni ad Oldenburgum primâ.

De Problematis per Series Infinitas Reſolvendis.

QUOMODO ex æquationibus, ſic ad infinitas ſeries reductis, Areæ & Longitudines curvarum, Contenta & Superficies ſolidorum, vel quorumlibet ſegmentorum figurarum quarumvis, eorumque centra gravitatis determinantur; & quomodo etiam curvæ omnes mechanicæ ad ejuſmodi æquationes infinitarum ſerierum reduci poſſint, indeque problemata circa illas reſolvi, perinde ac ſi Geometricæ eſſent; nimis longum foret deſcribere: ſufficiat ſpecimina quædam talium problematum recenſuiſſe: inque iis, brevitatis gratiâ, literas A, B, C, D, &c. pro terminis ſeriei, ſicut ſub initio, nonnunquam uſurpabo.

1. *Si ex dato ſinu recto, vel ſinu verſo, arcus deſideretur:* ſit radius r, & ſinus rectus x: eritque arcus $= x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6}$ &c. hoc eſt, $= x + \frac{1 \times 1 \times xx}{2 \times 3 \times rr} A + \frac{3 \times 3xx}{4 \times 5rr} B + \frac{5 \times 5xx}{6 \times 7rr} C + \frac{7 \times 7xx}{8 \times 9rr} D +$ &c.

Vel, ſit d diameter, ac x ſinus verſus; & erit arcus æqualis $d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c. hoc eſt, $= \sqrt{dx}$ in $1 + \frac{x}{6d} + \frac{3x^2}{40d^2} + \frac{5x^3}{112d^3} +$ &c (i).

2. Si viciſſim, *ex dato arcu deſideretur ſinus:* ſit radius r, & arcus

(i) 1. Puncto A manente (vid. fig. p. 299.) circumduci puta rectam AB datæ longitudinis, quæ termino ſuo mobili B circuli *BBc* circumflexum ſcribat. Sit *AB* ſitus quivis rectæ revolventis, et à puncto *B* in ſitum ejus primum AB deducatur ad perpendiculum recta *BE*: et per idem punctum *B* ducatur *BF* quæ circulum in *B* contingat, rectæque AB productæ in F occurrat. Fluxio arcûs B*B* ad fluxionem rectæ *BE* rationem habet eam quam *BF* ad *BE* (Introduct. ad Quadrat. Curv. § 4.) ſive eam quam AB ad AE. Hinc ſi rectam datam AB litera r deſignet, arcum B*B* litera z, ſinum ejus rectum *BE* litera x, erit $\dot{z} : \dot{x} = r :$ AE. Hoc eſt Arcu

 $\dot{z} : \dot{x} =$

EXCERPTUM QUARTUM. arcus z: eritque ſinus rectus $= z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{362880r^8} -$ &c. hoc eſt, $= z - \frac{zz}{2 \times 3rr}\,\text{A} - \frac{zz}{4 \times 5rr}\,\text{B} - \frac{zz}{6 \times 7rr}\,\text{C} -$ &c. Et ſinus verſus

EX SINU. $\dot{z} : \dot{x} = r : \sqrt{r^2 - x^2}$. Quare $\dot{z} = \frac{\dot{x}r}{\sqrt{r^2 - x^2}}$. Hoc eſt, ſi fractio $\frac{r}{\sqrt{r^2 - x^2}}$ convertatur in ſeriem infinitam (quod perficietur, ſi primùm extrahatur radix quadratica quantitatis $r^2 - x^2$, tum quantitas r radice illâ dividatur)

$$\dot{z} = \dot{x} \times 1 + \frac{x^2}{2r^2} + \frac{3x^4}{8r^4} + \frac{5x^6}{16r^6} + \frac{35x^8}{128x^8} +$$

$$\text{Ergo } z = x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6} +$$

SIN. EX ARC. Et invertendo, ope Theorematis 2. Excerpti II.

$$x = z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{362880r^8} +$$

2. *Cor.* Fluxio arcûs circularis ad ſinûs fluxionem rationem habet quam radius ad ſinum complementi.

ARC. EX SIN. VERS. 3. (k) FLUXIO arcûs B*B* ad fluxionem rectæ BE rationem habet quam *B*F ad FE (Introduct. Quad. Curv. §4.) hoc eſt, quam A*B* ad *B*E. Si igitur litera d circuli diametrum BC deſignet, litera z arcum *B*B, ſinum ejus verſum BE litera y, erit $\dot{z} : \dot{y} = \frac{1}{2}d : B\text{E} = d : 2B\text{E} = d : 2\sqrt{dy - y^2}$. Quare $\dot{z} = \frac{d\dot{y}}{2\sqrt{dy - y^2}}$.

Vel ſi fractio $\frac{d}{2\sqrt{dy - y^2}}$ convertatur in ſeriem infinitam, $\dot{z} = \dot{y} \times \frac{1}{2}d^{\frac{1}{2}}y^{-\frac{1}{2}} + \frac{1}{4}\frac{y^{\frac{1}{2}}}{d^{\frac{1}{2}}} + \frac{3y^{\frac{3}{2}}}{16d^{\frac{3}{2}}} + \frac{5y^{\frac{5}{2}}}{32d^{\frac{5}{2}}} + \frac{35y^{\frac{7}{2}}}{256d^{\frac{7}{2}}} +$. Ergo $z = d^{\frac{1}{2}}y^{\frac{1}{2}} + \frac{y^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3y^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5y^{\frac{7}{2}}}{112d^{\frac{5}{2}}} + \frac{35y^{\frac{9}{2}}}{1152d^{\frac{7}{2}}} +$. Quæ eſt formula generalis arcûs è dato ſinu verſo.

SIN. VERS. EX ARC. 4. Et hanc ſeriem invertendo ope Theorematis 2. Excerpti II.

$$y^{\frac{1}{2}} = \frac{1}{d^{\frac{1}{2}}}z - \frac{1}{6d^{\frac{5}{2}}}z^3 + \frac{1}{120d^{\frac{9}{2}}}z^5 - \frac{1}{5040d^{\frac{13}{2}}}z^7 +$$

Unde, quadraticè multiplicando, conficitur

$$y = \frac{1}{d}z^2 - \frac{1}{3d^3}z^4 + \frac{2}{45d^5}z^6 - \frac{1}{315d^7}z^8 +$$

Vel pro literâ d ſymbolo $2r$ ſubſtituto

$$y = \frac{1}{2r}z^2 - \frac{1}{24r^3}z^4 + \frac{1}{720r^5}z^6 - \frac{1}{40320r^7}z^8 +$$

Quæ eſt formula generalis ſinûs verſi ex arcu dato. Unde ſtatim elicitur ſinus complementi $r - y = r - \frac{z^2}{2r} + \frac{z^4}{24r^3} - \frac{z^6}{720r^5} + \frac{z^8}{40320r^7} +$.

5. *Cor.* Fluxio arcûs cujuſpiam circularis ad fluxionem ſinûs complementi, ſive ſinûs ſui verſi, rationem habet quam radius ad ſinum.

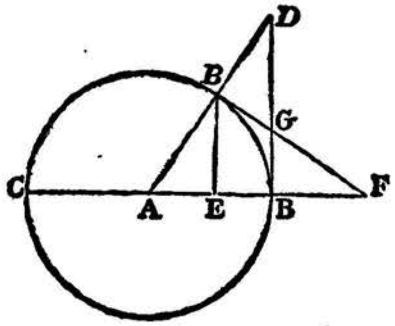
SERIES GREGORIANÆ. 6. HAUD diſſimilis eſt inventio ſerierum, quibus arcûs longitudinem ex tangente datâ, et tangentis viciſſim ex arcu, Jacobus Gregorius omnium primus æſtimare docuit.

7. Ducatur enim per punctum B recta BD quæ circulum B*B*C in loco B contingat, rectæque *B*F circulum utique in *B* contingenti in G occurrat. Cùm igitur recta A*B*, circa punctum A volubilis, curvæ B*B* rectæque ED in locis *B*, D, occurrat, recta vero *B*G curvam in *B* contingat, rectæque BD in G occurrat, fluxio rectæ BD erit ad fluxionem curvæ B*B* ut rectangulum AD × DG ad rectangulum A*B* × *B*G. (Introduct. ad Quad. Curv. § 9.) Sed propter angulos D*B*G, DBA, rectos, et angulum *B*DG triangulis duobus ADB, GD*B*, communem, triangula illa erunt inter ſe ſimilia, et latera eorum angulos æquales ambientia eaſdem inter ſe proportiones habebunt. Quare AD eſt ad AB, vel ad A*B*,

 ut

verſus $= \frac{z^2}{2r} - \frac{z^4}{24r^3} + \frac{z^6}{720r^5} - \frac{z^8}{40320r^7} +$ &c. hoc eſt, $= \frac{zz}{1 \times 2r} - \frac{zz}{3 \times 4r} A - \frac{zz}{5 \times 6rr} B - \frac{zz}{7 \times 8rr} C -$ &c (k). Sinus ex Arcu.

3. *Si*



ut GD ad G*B*. Rectangula igitur AD × DG, A*B* × *B*G inter ſe ſimilia ſunt. Figuræ autem ſimiles ſunt inter ſe ſicut quadrata è lateribus homologis. Erit igitur rectangulum AD × DG ad rectangulum A*B* × *B*G ſicut quadratum ex AD ad quadratum ex A*B*. Quare fluxio rectæ BD erit ad fluxionem arcûs B*B* ut quadratum ex AD ad quadratum ex A*B*. Hoc eſt, ſi litera t rectam BD deſignet, $\dot{t} : \dot{z} = t^2 + r^2 : r^2$. Ergo $\dot{z} = \frac{r^2}{r^2 + t^2}\dot{t}$. Hoc eſt, fractione $\frac{r^2}{r^2+t^2}$ in ſeriem infinitam per diviſionis operam reſolutâ, Series Gregorianæ.

$$\dot{z} = \dot{t} - \frac{t^2\dot{t}}{r^2} + \frac{t^4\dot{t}}{r^4} - \frac{t^6\dot{t}}{r^6} + \frac{t^8\dot{t}}{r^8} - \frac{t^{10}\dot{t}}{r^{10}} +$$

Ergo $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$, Arc. ex Tang.

Et hanc ſeriem invertendo ope Theorematis 2. Excerp. II.

$$t = z + \frac{1}{3r^2}z^3 + \frac{2}{15r^4}z^5 + \frac{17}{315r^6}z^7 + \frac{62}{2835r^8}z^9 +$$ Tang. ex Arc.

(vide Commerc. Epiſt. N°. XX.)

8. *Cor.* Fluxio tangentis ad fluxionem arcûs rationem habet ſecantis ad radium duplicatam.

9. Formula quâ Gregorius ſecantem definivit ex arcu, ex illâ Newtoni quæ ex arcu ſinum complementi repreſentat, facilè deducenda eſt. Cùm enim horum trium, ſecantis, radii et ſinûs complementi continua ſit analogia, ad ſecantem æſtimandam id nimirum opus erit, quadratum è radio dividere complementi ſinu. Jam verò inventus eſt ſinus complementi arcûs $z = r - \frac{z^2}{2r} + \frac{z^4}{24r^3} - \frac{z^6}{720r^5} + \frac{z^8}{40320r^7} -$. (§. 4 hujus) Quare ſecans, ſive $\frac{r^2}{r - \frac{z^2}{2r} + \frac{z^4}{24r^3} +}$, $= r + \frac{z^2}{2r} + \frac{5}{24r^3}z^4 + \frac{61}{720r^5}z^6 + \frac{277}{8064r^7}z^8 + \frac{50521}{3628800r^9}z^{10} +$; et hæc eſt formula generalis Gregoriana ſecantis ex arcu. (Vide Commercium Epiſt. N°. XX.) Sec. ex Arc.

10. Atque hæ quidem ſeries vulgò notæ ſunt, et quotidiano ferè Geometrarum uſu frequentatæ. Sunt autem aliæ quædam minùs pervagatæ, ab eodem illo Gregorio profectæ, ſubtiliſſimæ indaginis; quarum utique compendio Secantem et Tangentem arcûs cujuſque, uti loquuntur, Logarithmicam, ſive logarithmum potiùs rationis quam ſecans quæque vel tangens ad radium habeat, vir acutiſſimus generaliter definivit ex arcu. Ad harum ſcrierum inventionem Theoremate Geometrico uno atque altero via nobis munienda eſt.

THEOREMA I.

11. *Fluxio ſecantis eſt ad fluxionem arcûs ut rectangulum ſub ſecante et tangente ad quadratum ex circuli radio.*

Recta AB, circa punctum A volubili, circumflexum circuli ſcribente, ducatur recta BD, quæ circulum in B contingat, rectæque volubili, ſitum quemvis novum A*B* obtinenti, in D occurrat. Dico fluxionem rectæ AD eſſe ad fluxionem arcûs B*B* ut rectangulum AD × DB ad quadratum ex AB. Fluxio rectæ AD eſt ad fluxionem rectæ DB ut DB ad DA (Introduct. ad Quad. Curv. § 7.) ſive ut rectangulum DB × DA ad quadratum ex DA. Fluxio autem rectæ DB eſt ad fluxionem arcûs B*B*, ut quadratum ex DA ad quadratum ex AB. (§. 8. hujus.) Erit igitur ex æquo fluxio rectæ AD ad fluxionem arcûs B*B* ut rectangulum AD × DB ad quadratum ex AB. Q. E. D.

EXCERPTUM QUARTUM.

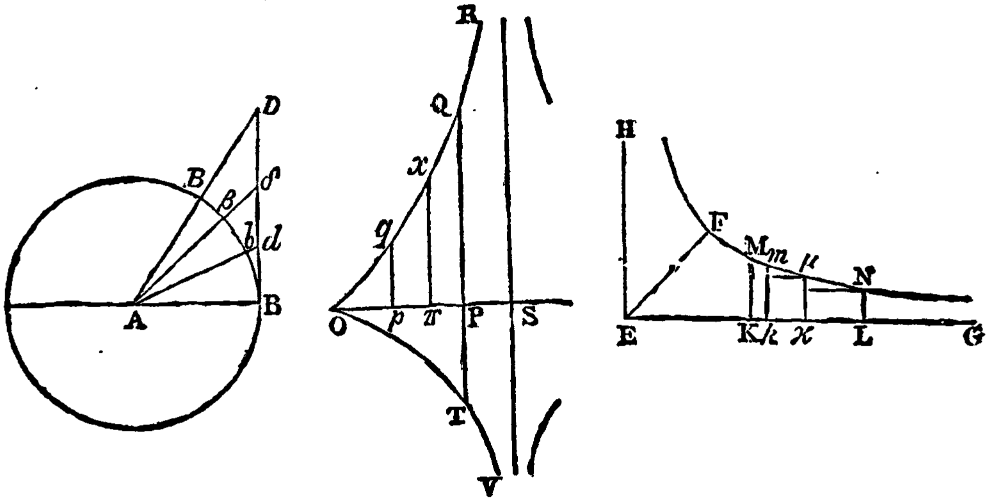
3. *Si arcus capiendus ſit in ratione datâ ad alium arcum :* eſto diameter

THEOREMA II.

DE SERIEBUS

12. *Sit* BB *arcus circuli, cujus centrum* A *radius* AB. *Sit hyperbola æquilatera* FMN, *cujus centrum* E, *ſemiaxis* EF, *aſymptotæ rectæ* EG, EH *ad perpendiculum compoſitæ. Harum in alterutrâ, puta* EG, *capiatur* EK *radio circuli, ſive rectæ* AB *æqualis ; et* EL *æqualis rectæ* AD *arcûs* BB *ſecanti : et à punctis* K, L *educantur ad perpendiculum rectæ* KM, LN, *hyperbolam uſque ; cui occurrant illæ in punctis* M, N. *Sit etiam curva* OQR, *cujus ea ſit natura, ut ſi in axe ejus,* OS, *capiatur* OP *arcui cuipiam circuli, æqualis, recta* PQ, *à puncto* P *ad perpendiculum educta curvam uſque, æqualis ſit ejuſdem arcûs tangenti. His poſitis, ſi* OP *capiatur in axe curvæ æqualis arcui illi* BB, *cujus ſecanti,* AD, *recta* EL *accepta eſt æqualis in aſymptotâ hyperbolæ, area hyperbolica* KLMN *ad aream* OPQ *datam rationem geret : nempe eam, quam dimidium quadrati ex* EF, *ſemiaxe hyperbolæ, habet ad quadratum ex circuli radio* AB.

In figurâ harum trium mediâ omiſſæ ſunt rectæ per puncta Q, *x*, *q*, o ducendæ, ut rectangula QP × Pπ, *x*π × π*p*, *qp* × *p*o perficerentur.



DIVIDATUR enim arcus B*B* in partes quotlibet, ſive æquales ſive inæquales B*b*, *b*β, β*B*. Junctæ A*b*, Aβ tangenti BD in punctis *d*, δ occurrant. Capiantur in hyperbolæ aſymptotâ EG rectæ E*k*, Eκ ſecantibus A*d*, Aδ, et in curvæ OR axe os capiantur o*p*, oπ arcubus B*b*, Bβ æquales. Ita rectarum KL, OP in partes numero æquales diviſio fiet; quarum illæ K*k*, *k*κ, κL, ſecantium AB, A*d*, Aδ, AD differentiis, hæ o*p*, *p*π, πP, arcubus B*b*, *b*β. β*B*, ordine ſumendis, æquales erunt. A diviſionum punctis, *k*, κ ; *p*, π, educantur ad perpendiculum curvas uſque, illic quidem *km*, κμ, hic vero *pq*, π*x* : compleanturque rectangula K*m*, *k*μ, κN ; o*q*, *p*x, πQ. Rectangulum κN ad rectangulum πQ rationem habet compoſitam è rationibus rectæ NL ad PQ rectæque κL ad πP ; ſive è rationibus rectæ NL ad rectam BD, rectæque κL ad arcum *B*β. Rectanguli quoque μ*k* ad rectangulum *xp* compoſita eſt ratio è rationibus rectæ μκ ad rectam π*x* rectæque κ*k* ad π*p* : hoc eſt, cùm recta π*x*, ex curvæ OR naturâ, tangenti Bδ ſit æqualis, è rationibus rectæ μκ ad tangentem Bδ, rectæque κ*k* ad arcum β*b*. Ratio denique rectanguli *m*K ad rectangulum o*q* compoſita eſt è rationibus rectæ *mk* ad tangentim B*d*, rectæque K*k* ad arcum *b*B. Jam verò ſi numerus arcuum B*b*, *b*β, β*B* infinitè augeatur, magnitudinibus ſingulorum infinitè decreſcentibus, unde partium quoque rectæ KL videlicet ipſarum K*k*, *k*κ, κL, necnon partium rectæ OP, videlicet ipſarum o*p*, *p*π, πP numerus pariter infinitè creſcet, magnitudinibus ſingularum infinitè imminutis, hoc inquam ſi fiat, ratio ultima evaneſcentis rectæ Lκ ad arcum evaneſcentem *B*β, ea erit quæ eſt rectanguli AD × DB ad quadratum ex AB : (per Theor. I.) nempe cùm illa Lκ ſecantium quidem, AD, Aδ, ſit differentia, hæc *B*β arcuum verò ; et differentiæ quantitatum evaneſcentes fluxionum proportiones ultimò inter ſe aſciſcant. (Geomet. Flux. Prop. I.) Quæ igitur è rationibus rectæ LN ad rectam BD rectæque Lκ ad arcum *B*β compoſita eſt ratio, hæc inquam, rectâ Lκ arcuque Bβ evaneſcentibus, fit illi quidem ultimò æqualis, quæ componitur e rationibus rectæ LN ad rectam BD rectanguliquc AD × DB ad quadratum ex AB. Vel è rationibus rectanguli LN × EL, ad EL × BD vel AD × BD, ejuſdemque

AD

diameter $= d$, chorda arcûs dati $= x$, & arcus quæſitus ad arcum illum Arcus ex ex Arcu x.

AD × BD ad quadratum ex AB. Ex hiſce verò rectanguli LN × EL ad quadratum ex AB compoſita eſt ratio. (El. Lib. VI. Def. 5.) Quæ igitur è rationibus rectæ LN ad rectam BD, rectæque LK ad arcum $B\beta$, ultimò ſcilicet evaneſcente arcu illo, compoſita eſt ratio, quæ ex priùs oſtenſis rectangulorum evaneſcentium NK, πQ, ultima inter ipſa ratio eſt, hæc eadem rectanguli LN × EL ad quadratum ex AB eſt ratio. Sed è naturâ hyperbolæ æquilateræ rectangulum LN × EL, et dimidium quadrati ex EF inter ſe ſunt æqualia. Rectanguli igitur LN × EL et dimidii quadrati ex EF ad quadratum ex AB eadem eſt ratio (Elem. V. 7.) Rectangulorum igitur evaneſcentium NK, Qπ, ea quidem inter ipſa ultima eſt ratio, quam dimidium quadrati ex EF habet ad quadratum ex AB. Eodemque prorſùs modo evaneſcentium μk, $x\rho$ necnon mK, qO eadem efficietur ratio ultima communis; nimirum data illa, quæ eſt dimidii quadrati ex EF, ſemiaxe hyperbolæ, ad quadratum ex AB circuli radio. Quare et arearum KMNL, OPQ eadem inter ipſas ratio erit (De Rat. Prim. et Ult. Lem. 4.) Q. E. D. Gregoriani.

13. *Cor.* In rectâ QP ſumatur PT ad quam PQ rationem habeat quam quadratum ex AB ad dimidium quadrati ex EF, et duci intelligatur curva OTV, quam punctum T perpetuò tangat. Si OP arcui illi BB ſit æqualis, cujus ſecanti AD recta EL accepta eſt æqualis, area OPT areæ hyperbolicæ MKLN æqualis erit. Etenim cùm duæ ſint curvæ, OQR, OTV, axe communi, OS, quarum ordinatæ, binas utique conferendo ab iiſdem axis punctis exeuntes, datam inter ſe rationem habent, areæ etiam ipſæ, OPQ, OPT, inter ſe datam ordinatarum rationem ſervant (De Rat. Prim. et Ult. Prop. IV. Cor. H. 1.) Area igitur OPT erit ad aream OPQ ut PT ad PQ; ſive ut dimidium quadrati ex EF ad quadratum ex AB (ex Conſtruct.) ſive denique ut area KLNM ad aream OPQ. Arearum igitur KLNM, OPT ad aream OPQ eadem eſt ratio. Areæ igitur illæ KLNM, OPT ſunt inter ſe æquales (El. 5. 9.) Q. E. D.

14. Area KLNM Logarithmus eſt rationis rectæ EL ad rectam EK, ſive ſecantis AD ad radium AB, pro modulo $\frac{1}{2}$EF^2. Cùm igitur æquales inter ſe ſint areæ illæ KLNM, OPT; ſi poſterioris menſura ad formulam aliquam generalem, ex arcu BB tanquam radice procreatam, revocari poſſit, per eandem formulam logarithmus quidem ille pro modulo ſuo generaliter definietur ex arcu. Talem verò areæ OPT formulam hæc ferè computatio dabit, in promptu ſanè poſita et leviſſimi laboris.

Literâ z deſignante arcum BB, vel illi æqualem rectam OP, literâ t tangentem BD vel illi æqualem rectam PQ, literâ r circuli radium AB, literâ denique B deſignante dimidium quadrati ex EF; cùm ex curvæ OTV naturâ PT ſit ad PQ ut dimidium quadrati ex EF ad quadratum ex AB, hoc eſt cùm PT ſit ad t ut B ad r^2: hinc ſanè efficitur PT $= \frac{B}{r^2}t$. Et cùm ſemper inter ſe æquales ſint arcus BB rectaque OP, fluxio quoque hujus illius fluxioni ſemper æqualis erit. (Geometr. Flux. Prop. II.) Hoc eſt $\dot{\overline{OP}} = \dot{z}$. Rectangulum igitur PT × $\dot{\overline{OP}}$, ſive fluxio areæ OPT, $= \frac{B}{r^2}t\dot{z}$. Vel pro t ſubſtituendo ſeriem illam, $z + \frac{1}{3r^2}z^3 + \frac{2}{15r^4}z^5 +$, cujus ſumma, ſi infinitè illa producatur, rectæ t ultimò ſit æqualis, (§.7. hujus) fluxio areæ OPT $= \frac{B}{r^2} \times \overline{z\dot{z} + \frac{1}{3r^2}z^3\dot{z} + \frac{2}{15r^4}z^5\dot{z} + \frac{17}{315r^6}z^7\dot{z} + \frac{62}{2835r^8}z^9\dot{z} +}$.

Ergo area OPT $= B \times \overline{\frac{1}{2r^2}z^2 + \frac{1}{12r^4}z^4 + \frac{1}{45r^6}z^6 + \frac{17}{2520r^8}z^8 + \frac{31}{14175r^{10}}z^{10} +}$. Atque hæc eſt formula generalis logarithmi rationis ejus quam ſecans arcûs cujuſpiam ad radium habet pro modulo utique B. Quòd ſi ipſius radii logarithmus pro eodem modulo deſignetur literâ R, hæc erit Secantis Logarithmicæ formula generalis ex arcû, $R + \frac{B}{2r^2}z^2 + \frac{B}{12r^4}z^4 + \frac{B}{45r^6}z^6 + \frac{17B}{2520r^8}z^8 + \frac{31B}{14175r^{10}}z^{10} +$. (Vide Commerc. Epiſtol. N° XX.) Sec. Log. ex Arc.

15. Hinc etiam formulam generalem ſinûs complementi logarithmici ex arcu effingere in promptu eſt. Nempe cùm quadratum è radio æquale ſit rectangulo ſub ſecante & ſinu complementi, erit ſinus complementi logarithmicus dupli logarithmi radii & ſecantis logarithmicæ differentia; cujus hæc erit formula generalis ex arcu; $R - \frac{B}{2r^2}z^2 - \frac{B}{12r^4}z^4 - \frac{B}{45r^6}z^6$. Cosin. Log. ex Arc.

THE

Excerptum Quartum.

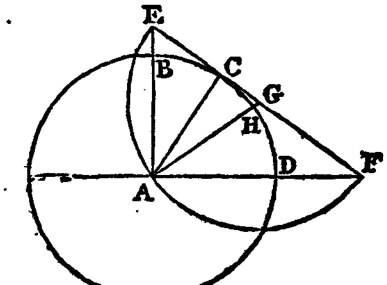
illum datum ut n ad 1; eritque arcûs quæſiti chorda $= nx +$
$1 - nn$

THEOREMA III.

De Seriebus

16. *Tangentis arcûs cujuſpiam circularis et tangentis ejuſdem arcûs complementi ſumma quidem dimidiata ſecanti, differentia verò dimidiata tangenti, complementi arcûs dupli eſt æqualis.*

Circuli cujus centrum A, radius AB, ſit arcus quiſpiam BC. A centro A educatur AD ad perpendiculum ſupra AB, quæ circulo in D occurrat. Per C ducatur recta quæ circulum in C contingat; et utrinque producta rectis AB, AD productis in punctis E, F occurrat. Recta EF media dividatur in puncto G, et jungatur AG.



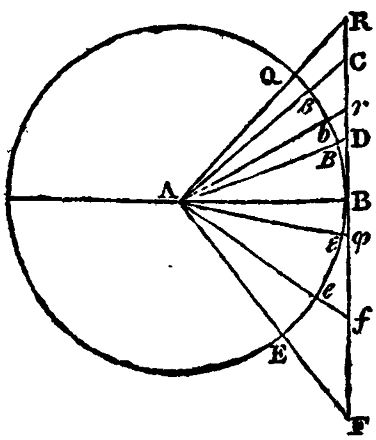
Propter angulum ad A rectum arcus BD totius circuitûs quadrans erit. Ergo arcus CD complementum erit arcûs CB, et recta CF tangens complementi arcûs BC, cùm recta CE ipſius quidem BC tangens ſit. Erit igitur GE vel GF, tangentis arcûs BC et tangentis complementi ejuſdem ſumma dimidiata; erit autem GC dimidiata earundem differentia. Dico rectam GE quidem, vel GF, ſecanti, rectam verò GC tangenti complementi dupli arcûs BC æqualem eſſe. Recta AG circulo in H occurrat. Triangulo autem EAF circumſcribatur circulus. Propter angulum ad A rectum (talis enim factus eſt) recta EF circuli, triangulo EAF circumſcripti, diameter erit. Quare punctum G, cùm medium ſit rectæ EF, circuli illius centrum erit. Tres igitur GE, GA, GF inter ſe æquales. Propter æquales verò GA, GF, anguli GAF, GFA inter ſe æquales ſunt. Sed propter angulos ad A, C rectos, anguli etiam CAE, CFA, vel GFA, inter ſe ſunt æquales (El. vi. 8.) Angulus igitur GAF angulo CAE æqualis; ſive angulus HAD angulo CAB. Arcus igitur HD, BC inter ſe æquales. Duo igitur BC, HD, ſimul ſumpti, duplo arcûs BC æquales ſunt. Tres verò BC, HD, HC quadrantem abſolvunt. Ergo HC complementum eſt dupli arcûs BC. Eſt autem recta GA, cui GF, GE oſtenſæ ſunt æquales, eſt, inquam, GA ſecans arcûs HC, recta verò GC ejuſdem tangens. Ergo recta GE vel GF ſecanti, recta CG verò tangenti complementi dupli arcûs BC eſt æqualis. Q. E. D.

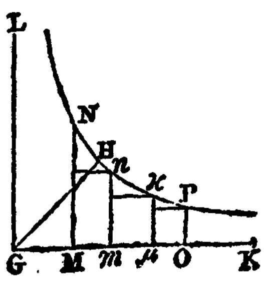
17. *Cor.* Rectangulum ſub tangente arcûs cujuſpiam, et ſecante complementi arcûs dupli, dimidio quadrati ex ſecante eſt æquale.

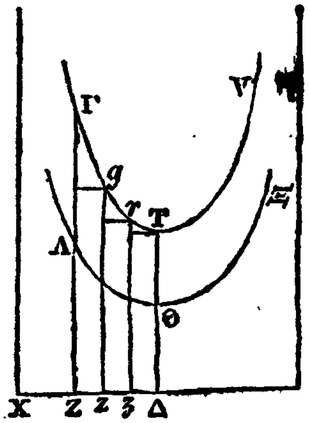
Rectangulum enim ſub EC, GA æquale eſt rectangulo ſub EC, EG; quod dimidium eſt rectanguli ſub EC, EF, cui æquale eſt quadratum ex ſecante EA (El. vi. 8.) Rectangulum igitur ſub EC, GA hoc eſt ſub tangente arcûs BC et ſecante complementi dupli arcûs BC, dimidio quadrati ex ſecante EA æquale eſt. Q. E. D.

THEOREMA IV.

18. *Circuli, cujus centrum* A, *radius* AB, *ſit arcus quilibet* BB. *Rectæ* BD, *circulum in* B *contingenti, juncta* AB *in* D *occurrat, ut ſit* BD *tangens arcûs* BB. *Sit arcus* BE *complementum dupli arcûs* BB, *junc-*







taque

$$\frac{1-nn}{2\times 3dd}xxA+\frac{9-nn}{4\times 5dd}xxB+\frac{25-nn}{6\times 7dd}xxC+\frac{49-nn}{8\times 9dd}xxD+\frac{81-nn}{10\times 11dd}xxE+\&c.$$

Arcus *xz* ex Arcu *z*.

Ubi

taque AE *rectæ* DB *in* F *occurrat, ut ſit recta* AF *ſecans complementi dupli arcûs* B*B*. *Sit hyperbola æquilatera poſitione data,* NHP, *cujus centrum* G, *ſemiaxis recta* GH *datæ cujuſvis longitudinis, aſymptotæ rectæ* GK, GL *poſitione datæ, et ad perpendiculum inter ſe compoſitæ. Harum in alterutrâ, puta in* GK *capiatur* GO *circuli radio* AB *æqualis, et* GM *æqualis rectæ* BD *arcûs* B*B tangenti; et è punctis* O, M *educantur ad perpendiculum* OP, MN *hyperbolam uſque; cui illæ in punctis* P, N *occurrant. Sit etiam curva* ΓTV, *cujus ea ſit natura, ut ſi in rectâ poſitione datâ,* XΔ, *à dato in eâ puncto* X *initio ſumpto, capiatur* XZ *arcui cuilibet circuli æqualis, recta* ZΓ, *à puncto* Z *ad perpendiculum educta curvam uſque, ſecanti complementi dupli arcûs æqualis ſit. His poſitis, ſi in axe curvæ capiatur* XΔ *æqualis dimidio quadrantis,* XZ *verò æqualis arcui illi* B*B, cujus tangenti,* BD, *accepta eſt* GM *in aſymptotâ hyperbolæ æqualis, eductis ad perpendiculum* ΔT, ZΓ, *area hyperbolica* POMN *ad aream* ΔTΓZ *datam rationem habebit: nimirum duplicatam ejus quam ſemiaxis hyperbolæ* GH *habet ad circuli radium* AB. Gregorianis.

19. Capiatur enim arcus BQ æqualis dimidio quadrantis; diviſo autem arcu *BQ*, differentiâ utique arcûs B*B* et dimidii quadrantis, in partes quotlibet ſive æquales ſive inæquales, *Bb*, *bβ*, *β*Q junctæ A*b*, Aβ, AQ productæque tangenti BD in punctis *r*, C, R occurrant. Sumantur arcus B*e*, Bε duplorum arcuum B*b*, Bβ complementa; junctæque A*e*, Aε et productæ rectæ BF in punctis *f*, φ, occurrant. In aſymptotâ hyperbolæ GK, ſumantur G*m*, Gμ tangentibus B*r*, BC; in axe verò curvæ ΓTV, ſumantur X*z*, Xζ arcubus B*b*, Bβ æquales. Ita rectarum OM, ΔZ in partes numero æquales diviſio fiet, quarum illæ Oμ, μ*m*, *m*M tangentium differentiis RC, C*r*, *r*D; hæ Δζ, ζ*z*, *z*Z arcubus Qβ, β*b*, *bB* æquales erunt. A diviſionum punctis, μ, *m*; ζ, *z*, educantur ad perpendiculum curvas uſque, illic quidem μ*x*, *mn*, hîc verò ζγ, *zg*, compleanturque rectangula Pμ, *xm*, *n*M; Tζ, γ*z*, *g*Z. Rectangulum Pμ ad rectangulum Tζ rationem habet compoſitam è rationibus rectæ PO ad rectam ΔT, rectæque Oμ ad rectam Δζ: vel, cùm recta ΔT, ex naturâ curvæ ΓTV, circuli radio AB ſit æqualis, è rationibus rectæ PO ad rectam AB rectæque RC ad arcum Qβ. Rectangulum quoque *mx* ad rectangulum γ*z* rationem habet compoſitam è rationibus rectæ *x*μ ad rectam γζ rectæque μ*m* ad rectam ζ*z*; vel è rationibus rectæ *x*μ ad rectam Aφ rectæque C*r* ad arcum β*b*. Recta enim γζ, ex naturâ curvæ STV, rectæ Aφ eſt æqualis; nempe cùm Aφ ſecans ſit arcus Bε qui complementum eſt dupli arcûs Bβ, cui abſciſſa Xζ ponitur æqualis. Ratio denique rectanguli *n*M ad rectangulum *g*Z compoſita eſt è rationibus rectæ *nm* ad rectam A*f*, rectæque *r*D ad arcum *bB*.

Jam verò ſi arcurum Qβ, β*b*, *bB* numerus infinitè augeatur, magnitudinibus ſingulorum infinitè decreſcentibus, unde partium quoque rectæ OM, videlicet ipſarum Oμ, μ*m*, *m*M; necnon partium rectæ ΔZ, videlicet ipſarum Δζ, ζ*z*, *z*Z numerus pariter infinitè creſcet, magnitudinibus ſingularum infinitè decreſcentibus; hoc inquam ſi fiat, ratio ultima evaneſcentis RC ad evaneſcentem Qβ ea erit, quæ eſt quadrati ex AR ad quadratum ex AQ vel AB. (§ 8. hujus.) Quæ igitur è rationibus rectæ PO ad rectam AB rectæque RC ad arcum Qβ compoſita eſt ratio, ultimò ſcilicet, arcu Qβ rectâque RC evaneſcente, ea illi erit æqualis, quæ componitur è rationibus rectæ PO ad rectam AB et quadrati ex AR ad quadratum ex AB; ſive è rationibus rectæ PO ad rectam AB numerique binarii ad unitatem. Nam propter angulum ABR rectum, et ſemirectum RAB, quadratum ex AR duplum eſt quadrati ex AB. Quæ igitur è rationibus rectæ OP ad rectam AB, rectæque RC ad arcum Qβ compoſita eſt ratio, ultimò ſcilicet arcu Qβ evaneſcente, quæ ex priùs oſtenſis rectanguli evaneſcentis Pμ, ad evaneſcens Tζ ultima eſt ratio, ea illi erit æqualis, quæ eſt duplæ rectæ PO ad rectam AB; ſive dupli rectanguli PO × AB ad quadratum ex AB; ſive denique (propter GO, AB inter ſe æquales) dupli rectanguli PO × OG ad quadratum ex AB. Sed, ex hyperbolæ æquilateræ naturâ, duplum rectanguli PO × OG quadrato ex GH eſt æquale. Rectanguli igitur evaneſcentis Pμ ad evaneſcens Tζ ratio ultima rectæ GH ad circuli radium AB duplicata erit.

Rurſum, ratio quæ componitur è rationibus rectæ *x*μ ad rectam Aφ rectæque C*r* ad arcum β*b*, ultimò ſcilicet arcu β*b* evaneſcente, ſive rectanguli evaneſcentis *xm* ad evaneſcens γ*z* ratio ultima; hæc, inquam, illi erit æqualis, quæ componitur è rationibus rectæ *x*μ ad rectam Aφ, et quadrati ex AC ad quadratum ex Aβ (§ 8. hujus), vel è rationibus rectanguli *x*μ × μG vel *x*μ × BC ad Aφ × BC et quadrati ex AC ad quadratum ex AB. Sed cùm Aφ ſecans ſit arcûs Bε, qui complementum eſt dupli arcûs Bβ, cujus tangens quidem eſt BC, AC verò ſecans, propter hæc rectangulum Aφ × BC dimidio quadrati ex AC æquale eſt. (Cor. Theor. 3.) Quare rectanguli evaneſcentis *xm* ad evaneſcens γ*z* ratio ultima æqualis eſt compoſitæ è rationibus rectanguli *x*μ × BC ad dimidium quadrati ex AC, et quadrati ex AC ad quadratum ex AB; ſive compoſitæ è rationibus

dupli

Ubi nota, quòd cùm n eſt numerus impar, ſeries deſinet eſſe infinita,

De Seriebus

dupli rectanguli $\varkappa\mu \times$ BC, vel dupli $\varkappa\mu \times \mu$G, ad quadratum ex AC, et quadrati ex AC ad quadratum ex AB. Sed ex hiſce compoſita eſt ratio dupli rectanguli $\varkappa\mu \times \mu$G ad quadratum ex AB. Et è naturâ hyperbolæ æquilateræ, duplum rectanguli $\varkappa\mu \times \mu$G quadrato ex GH eſt æquale. Ratio igitur ultima rectanguli evaneſcentis $\varkappa\mu$ ad evaneſcens γz ea eſt quam quadratum ex GH habet ad quadratum ex AB, ſive rectæ GH ad circuli radium AB duplicata. Ac ſimilibus prorſus argumentis rectangulorum evaneſcentium $\varkappa$M, $g z$ eadem ratio ultima efficietur. Quare et arearum POMN, TΔZΓ eadem inter ipſas ratio erit. (De Rat. Prim. et Ult. Prop. IV.) Q. E. D.

20. *Cor.* In rectâ ZΓ capiatur ZΛ ad quam illa ZΓ rationem habeat duplicatam ejus quam AB ad GH; et duci intelligatur linea curva ΛΘΞ, quam punctum Λ perpetuò tangat, cui recta ΔT in puncto Θ occurrat. Area ΘΔZΛ hyperbolicæ POMN æqualis erit. (Hoc verò ad exemplum Corollarii Theor. 2. ex Corollario H. 1. Prop. IV. Tractatûs Newtoni De Rat. Prim. et Ult. demonſtrandum eſt.)

21. Cùm igitur rationis ejus quam GO habet ad GM, hoc eſt, quam AB radius circuli QBE habet ad BD tangentem arcûs BB, area POMN pro modulo $\frac{1}{2}$GH^2 ſit logarithmus, ad logarithmum illum generaliter definiendum opus erit aream ΘΔZΛ generali formulâ expoſuiſſe. Id verò hæc ferè computatio præſtabit.

22. Deſignet litera r circuli radium AB, ſeu illi æqualem rectam GO: litera q quadrantem circuitûs circularis, cujus ſemiſſis, recta ΔX, ſymbolo $\frac{1}{2}q$ deſignanda erit. Deſignet litera z arcum BB, ſive rectam XZ arcui illi æqualem; litera t arcûs BB tangentem BD, vel illi æqualem rectam GM; litera $ſ$ rectam AF ſecantem complementi dupli arcûs BB, ſive æqualem illi rectam ZΓ. Deſignet denique litera B dimidium quadrati ex GH. Jam cùm ZΓ ſit ad ZΛ ut quadratum ex AB ad quadratum ex GH, hoc eſt, $ſ : \mathrm{Z\Lambda} = r^2 : 2\mathrm{B}$, hinc ſcilicet efficietur $\mathrm{Z\Lambda} = \frac{2\mathrm{B}ſ}{r}$. Recta verò $\Delta\mathrm{Z} = \Delta\mathrm{X} - \mathrm{ZX} = \frac{1}{2}q - z$. Quare fluxio rectæ $\Delta\mathrm{Z} = -\dot{z}$: et fluxio areæ $\Theta\Delta\mathrm{Z}\Lambda = \frac{2\mathrm{B}ſ}{r^2} \times -\dot{z}$. Arcum BE deſignet litera y. Et cùm arcus BE is ſit, quo duplus BB abeſt à quadrante, erit $y = q - 2z$. Ergo $\dot{y} = -2\dot{z}$, et $\frac{\dot{y}}{2} = -\dot{z}$. Subſtituendo igitur hoc ſymbolum $\frac{\dot{y}}{2}$ pro illo $-\dot{z}$, efficietur fluxio areæ $\Theta\Delta\mathrm{Z}\Lambda = \frac{\mathrm{B}ſ}{r^2}\dot{y}$. Sed cùm $ſ$ ſecans ſit arcûs y, hujus ſeriei $r + \frac{y^2}{2r} + \frac{5}{24r^3}y^4 + \frac{61}{720r^5}y^6 +$, ſi infinitè ea producatur, ſumma rectæ $ſ$ ultimò ſit æqualis. (§ 9. hujus.) Hæc igitur ſeries ſi in ſymbolo fluxionis areæ ΘΔZΛ pro literâ $ſ$ ſubſtituatur, efficitur fluxio illa $= \mathrm{B} \times \overline{\frac{\dot{y}}{r} + \frac{y^2\dot{y}}{2r^3} + \frac{5y^4\dot{y}}{24r^5} + \frac{61y^6\dot{y}}{720r^7} + \frac{277y^8\dot{y}}{8064r^9} + \frac{50521y^{10}\dot{y}}{3628800r^{11}} +}$. Ergo area ipſa $\Theta\Delta\mathrm{Z}\Lambda = \mathrm{B} \times \overline{\frac{y}{r} + \frac{y^3}{6r^3} + \frac{y^5}{24r^5} + \frac{61y^7}{5040r^7} + \frac{277y^9}{72576r^9} + \frac{50521y^{11}}{39916800r^{11}} +}$. Atque hæc eſt formula generalis logarithmi rationis ejus quam tangens arcûs cujuſpiam ad radium habet, pro modulo B (vide Commerc. Epiſtol. N° XX.) Quòd ſi ipſius radii logarithmus, pro eodem modulo, deſignetur literâ R, hæc erit Tangentis Logarithmicæ formula generalis $\mathrm{R} \mp \mathrm{B} \times \overline{\frac{y}{r} + \frac{y^3}{6r^3} + \frac{y^5}{24r^5} + \frac{61y^7}{5040r^7} +}$, ſignorum, +, −, hoc vel illo uſurpato, prout arcus z dimidio quadrante minor majorve fuerit.

Tang. Log. ex Arc.

23. Jam ſi ex datâ Tangente Logarithmicâ Arcum definire cupias, deſignet litera T datam tangentem logarithmicam arcûs cujuſdam, quem litera z deſignat. Deſignet autem litera τ logarithmorum R, T differentiam, ſive rationis ejus logarithmum, quam tangens arcûs z ad circuli radium habet. Tum literâ y arcum eum deſignante quo duplus z abeſt à quadrante, efficitur ex iis quæ jam oſtenſa ſunt.

$$\tau = \frac{\mathrm{B}y}{r} + \frac{\mathrm{B}y^3}{6r^3} + \frac{\mathrm{B}y^5}{24r^5} + \frac{61\mathrm{B}y^7}{5040r^7} + \frac{277\mathrm{B}y^9}{72576r^9} + .$$

24. Et hanc ſeriem invertendo ope Theorematis 2. Excerpti II. veniet y, ſive $q - 2z$, $= \frac{r}{\mathrm{B}}\tau - \frac{r}{6\mathrm{B}^3}\tau^3$

+

finita, & evadet eadem quæ prodit per Vulgarem Algebram ad multi- ARCUS nz EX ARCU z.

$+ \frac{r}{24B^5}\tau^5 - \frac{61r}{5040B^7}\tau^7 + \frac{277r}{72576B^9}\tau^9 -$. Unde efficietur $z = \frac{1}{2}q - \frac{r}{2B}\tau + \frac{r}{12B^3}\tau^3 - \frac{r}{48B^5}\tau^5 +$ ARC. EX. TANG. LOG.
$\frac{61r}{10080B^7}\tau^7 - \frac{277r}{145152B^9}\tau^9 +$.

25. SI arcus ex ſecante logarithmicâ definiendus ſit, deſignante literâ z arcum, deſignet litera Σ datam ſecantem logarithmicam; litera autem σ logarithmorum Σ, R differentiam deſignet; ſive ejus rationis logarithmum, quam ſecans arcûs z ad radium habet. Tum ex iis quæ jam ſuprà ex ſecundo Theoremate oſtenſa ſunt, hæc efficietur æquatio

$$\sigma = \frac{B}{2r^2}z^2 + \frac{B}{12r^4}z^4 + \frac{B}{45r^6}z^6 + \frac{17B}{2520r^8}z^8 +,$$

Hanc autem, ope Theorematis 2. Excerpti II. invertendo, hæc alia veniet;

$$z^2 = \frac{2r^2}{B}\sigma - \frac{2r^2}{3B^2}\sigma^2 + \frac{4r^2}{45B^3}\sigma^3 + \frac{2r^2}{315B^5}\sigma^4 +,$$

Ex hâc denique, partis utriuſque radicem quadraticam extrahendo, conficietur

$$z = \frac{\sqrt{2}.r}{B^{\frac{1}{2}}}\sigma^{\frac{1}{2}} - \frac{r}{3.\sqrt{2}.B^{\frac{3}{2}}}\sigma^{\frac{3}{2}} + \frac{r}{60.\sqrt{2}.B^{\frac{5}{2}}}\sigma^{\frac{5}{2}} + \frac{r}{2520.\sqrt{.2}.B^{\frac{7}{2}}}\sigma^{\frac{7}{2}},$$ ARC. EX. SEC. LOG.

quâ quidem formulâ, è logarithmo rationis ejus quam ſecans ad radium habeat, arcus generaliter definietur.

26. ALIAM vero arcûs formulam, è logarithmo rationisquam ſecans ejus ad ſecantem dimidii quadrantis habeat, Gregorius concinnavit. Hujus inventio neſcio an ex alio fonte promptiùs arceſſenda ſit, quàm ex tertii Theorematis Corollario.

Literis R, B, Σ, σ, z, y eadem quæ ſuprà deſignantibus, ſecans logarithmica dimidii quadrantis pro modulo B deſignetur literâ S. Secantis autem dimidii quadrantis ad ſecantem arcûs z rationis logarithmus, hoc eſt logarithmorum S, Σ differentia, literâ l deſignetur: literâ denique D numeri binarii logarithmus pro modulo illo B.

Itaque cùm ſecans logarithmica arcûs y per hanc formulam exponatur,

$$R + \frac{B}{2r^2}y^2 + \frac{B}{12r^4}y^4 + \frac{B}{45r^6}y^6 + \frac{17B}{2520}y^8 +,$$ (§ 14. hujus) hâc aliâ verò tangens logarithmica arcûs z (§ 22. hujus)

$$R - \frac{By}{r} - \frac{By^3}{6r^3} - \frac{By^5}{24r^5} - \frac{61By^7}{5040r^7} -,$$

ex harum quidem ſummâ conflabitur logarithmus rectanguli ſub ſecante arcûs y et tangente arcûs z. Huic igitur logarithmo hujus ſeriei,

$$2R - \frac{By}{r} + \frac{B}{2r^2}y^2 - \frac{B}{6r^3}y^3 + \frac{B}{12r^4}y^4 - \frac{B}{24r^5}y^5 + \frac{B}{45r^6}y^6 - \frac{61B}{5040r^3}y^7 +,$$

ſi infinitè ea producatur, ſumma ultimò fit æqualis.

Rectangulum autem ſub ſecante arcûs y et tangente arcûs z dimidio quadrati ex ſecante arcûs z æquale eſt (per Cor. Theor. 3.) Dimidii igitur quadrati hujus logarithmo ſeries noviſſima æqualis eſt. Additoque illi binarii logarithmo, integri quidem quadrati logarithmus veniet, hoc eſt duplum ipſius ſecantis logarithmicæ arcûs z. Unde hæc efficitur æquatio.

$$2\Sigma = D + 2R - \frac{By}{r} + \frac{B}{2r^2}y^2 - \frac{B}{6r^3}y^3 + \frac{B}{12r^4}y^4 -,$$

Et binarii diviſione,

$$\Sigma = \frac{1}{2}D + R - \frac{By}{2r} + \frac{B}{4r^2}y^2 - \frac{B}{12r^3}y^3 + \frac{B}{24r^4}y^4 - \frac{B}{48r^5}y^5 + \frac{B}{90r^6}y^6 - \frac{61B}{10080r^7}y^7 +,$$

Atqui quantitas $\frac{1}{2}D + R$ eſt ſecans logarithmica dimidii quadrantis. Cùm enim ille R radii ſit logarithmus, erit 2R logarithmus quadrati è radio. Ergo cùm D logarithmus ſit binarii, horum ſumma, 2R + D, dupli quidem quadrati è radio logarithmus erit. Sed quadratum ex ſecante dimidii quadrantis duplum eſt quadrati è radio (per Elem. I. 47.) Quadrati igitur ex dimidii qua-

 drantis

multiplicandum datum angulum per iſtum numerum n [1].

4. Si

drantis ſecante ille $2\text{R}+\text{D}$ logarithmus erit. Quare illius ſemiſſis, nimirum $\text{R}+\frac{1}{2}\text{D}$, dimidii quadrantis ipſa ſecans erit logarithmica, ſive logarithmo s æqualis. Ex æquatione igitur noviſſimâ, demptis hinc inde æqualibus, s, $\text{R}+\frac{1}{2}\text{D}$, efficietur

$$-l = -\frac{\text{B}y}{2r}+\frac{\text{B}}{4r^2}y^2-\frac{\text{B}}{12r^3}y^3+\frac{\text{B}}{24r^4}y^4-\frac{\text{B}}{48r^5}y^5+\frac{\text{B}}{90r^6}y^6-,$$

$$\text{Ergo } l = \frac{\text{B}y}{2r}-\frac{\text{B}}{4r^2}y^2+\frac{\text{B}}{12r^3}y^3-\frac{\text{B}}{24r^4}y^4+\frac{\text{B}}{48r^5}y^5-\frac{\text{B}}{90r^6}y^6+,$$

Hujus autem inverſio, ope Theorematis 1. Excerpt. II. perficienda, hanc aliam excitabit;

$$y,\ \text{ſive } q-2z, = \frac{2r}{\text{B}}l-\frac{2r}{\text{B}^2}l^2+\frac{8r}{3\text{B}^3}l^3-\frac{14r}{3\text{B}^4}l^4+\frac{28r}{3\text{B}^5}l^5-,$$

$$\text{Ergo } 2z = q-\frac{2r}{\text{B}}l+\frac{2r}{\text{B}^2}l^2-,$$

Ex hâc verò, per binarii diviſionem, exſiſtet tandem generalis arcûs formula

Arc. ex Sec. Log.

$$z = \tfrac{1}{2}q-\frac{r}{\text{B}}l+\frac{r}{\text{B}^2}l^2-\frac{4r}{3\text{B}^3}l^3+\frac{7r}{3\text{B}^4}l^4-\frac{14r}{3\text{B}^5}l^5+,$$

ex ejus nimirum rationis conflata logarithmo, quam ſecans arcûs z ad ſecantem dimidii quadrantis habet.

Arcus nz ex Arcu z.

[1] Cum x chorda ſit arcûs dati, arcus datus erit

$$x+\frac{x^3}{6d^2}+\frac{3x^5}{40d^4}+\frac{5x^7}{112d^6}+$$

(per § 1. hujus Excerpti) nam arcus circularis per eandem formulam ex chordâ cum diametro definitur, atque ex ſinu recto cum radio.

Quare arcus exquirendus $= nx+\frac{nx^3}{6d^2}+\frac{3nx^5}{40d^4}+\frac{5x^7}{112d^6}+$

Subſtituatur hæc ſeries pro z in formulam chordæ ex arcu dato, & proveniet formula chordæ arcûs exquirendi è datâ x. Nempe hoc modo. Deſignet z arcum exquirendum; chorda igitur arcûs exquirendi $= z-\frac{z^3}{6d^2}+\frac{z^5}{120d^4}-\frac{z^7}{5040d^6}+$ (§ 2. hujus Excerpti)

$$\begin{aligned}
\text{Sed}\quad z &= nx+\frac{nx^3}{6d^2}+\frac{3nx^5}{40d^4}+\frac{5nx^7}{112d^6}+\\
\text{Ergo} -\frac{z^3}{6d^2} &= -\frac{n^3x^3}{6d^2}-\frac{n^2x^5}{12d^4}-\frac{37n^3x^7}{720d^6}-\\
\text{et} +\frac{z^5}{120d^4} &= +\frac{n^5x^5}{120d^4}+\frac{5n^5x^7}{720d^6}\\
\text{et} -\frac{z^7}{5040d^6} &= -\frac{n^7x^7}{5040d^6}
\end{aligned}$$

Quare hujus ſeriei, infinitè quidem productæ, ſumma chordæ arcûs exquirendi ultimò æqualis erit. Atque hæc ſeries, ſi coefficientium poteſtatis cujuſque ſummæ ſcitè colligantur, formam induet Newtonianam.

ALITER

2. Sit x chorda arcûs dati, y chorda arcûs exquirendi;

Erit arcus datus $= x+\frac{x^3}{6d^2}+\frac{3x^5}{40d^4}+\frac{5x^7}{112d^6}+$

Arcus autem exquirendus $= y+\frac{y^3}{6d^2}+\frac{3y^5}{40d^4}+\frac{5y^7}{112d^6}+$

Hinc rurſum arcus datus $= \frac{y}{n}+\frac{y^3}{6nd^2}+\frac{3y^5}{40nd^4}+\frac{5y^7}{112nd^6}+$

Ergo $x+\frac{x^3}{6d^2}+\frac{3x^5}{40d^4}+\frac{5x^7}{112d^6}+ = \frac{y}{n}+\frac{y^3}{6nd^2}+\frac{3y^5}{40nd^4}+\frac{5y^7}{112nd^6}+$

Et radice y infinitè affectâ per formulam generalem Moivræanam extractâ

$y=$

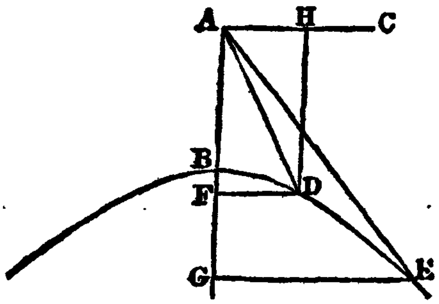
4. Si in axe alterutro AB, ellipſeos ADB (cujus centrum C, & axis alter DH) detur punctum aliquod E, circa quod recta EG, occurrens PROBLEMA KEPLERI.

$$y = nx + \frac{n-n^3}{6d^2}x^3 + \frac{9n-10n^3+n^5}{120d^4}x^5 + \frac{225n-259n^3+35n^5-n^7}{5040d^6}x^7 + \&c.$$

Eâdem vero formulâ ſinus rectus arcûs exquirendi exponetur, ſi litera x dati arcûs ſinum rectum ſignificet; litera d circuli radium.

3. ILLUD verò minimè prætereundum eſt; per eandem formulam, ſignis tantummodo mutatis, Æquilateræ etiam Hyperbolæ, non arcum quidem, ſed ſectorem definiri, qui ad datum ejuſdem hyperbolæ ſectorem datam rationem habeat. Nempe hoc modo.

Centro A vertice B ſemiaxibus æqualibus AB, AC ſcriptam puta hyperbolam æquilateram BDE; cujus ſint BAD, BAE ſectores; quorum ille BAE ad alterum BAD rationem datam habeat quam n ad 1. A punctis D, E in axem AB deductis ad perpendiculum DF, EG; ſi recta DF, ſinus arcûs hyperbolici DB, cui ſectorum alter ADB inſiſtit, deſignetur literâ x, literâ r ſemiaxem AB deſignante, recta quidem EG, ſinus arcûs EB, cui inſiſtit ſector alter AEB,



hâc formula exponetur, $nx + \frac{nn-1}{2\times 3.r^2}xx\text{A} + \frac{nn-9}{4\times 5.r^2}xx\text{B} + \frac{nn-25}{6\times 7.r^2}xx\text{C} + \frac{nn-49}{8\times 9.r^2}xx\text{D} + \frac{nn-81}{10\times 11.r^2}xx\text{E} + \&c.$ SECT. HYP. nz EX SECT. z.

4. Hoc autem naturali quâdam nititur Hyperbolæ Æquilateræ cum Circulo cognatione, quam accuratiùs explicare haud abs re erit; præſertim cùm rem neque injucundam ſcitu neque inutilem, minimè verò difficilem, nimio calculorum apparatu apud optimos quidem ſcriptores non tam illuſtratam, quàm obrutam planè atque merſam cernimus.

5. A puncto D in axem ſecundum AC deducatur ad perpendiculum DH. Quadratum ex DH duobus ex AH et ex AC ſimul ſumptis eſt æquale (Hamilton. Conic. Lib. I. Prop. XLI.) et recta AH = DF = x. Quare DH = $\sqrt{r^2+x^2}$, et area hyperbolica DHAB $= rx + \frac{x^3}{6r} - \frac{x^5}{40r^3} + \frac{x^7}{112r^5} - \frac{5x^9}{1152r^7} + \&c.$ (Analyſ. per Æquat. Infinit. Cap. III. §. 5.)

Sed triangulum DHA $= \frac{1}{2}x\sqrt{r^{-}+x^2} = \frac{1}{2}rx + \frac{x^3}{4r} - \frac{x^5}{16r^3} + \frac{x^7}{32r^5} - \frac{5x^9}{256r^7} + \&c.$ (Analyſ. per Æquat. Infinit. Cap. III. §. 5.)

Quare ſector BAD = areæ DHAB − triang. DHA $= \frac{1}{2}rx - \frac{x^3}{12r} + \frac{3x^5}{80r^3} - \frac{5x^7}{224r^5} + \frac{35x^9}{2304r^7} + \&c.$

Hoc eſt, ſector BAD $= \frac{1}{2}r \times \overline{x - \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} - \frac{5x^7}{112r^6} + \frac{35x^9}{1152r^7}}$.

Atque hæc erit formula generalis ſectoris Hyperbolici ex ſinu recto ejus arcûs hyperbolæ quem ſector baſin habet: quæ à formulâ ſectoris Circularis haud aliâ re quàm ſignis diſcrepat.

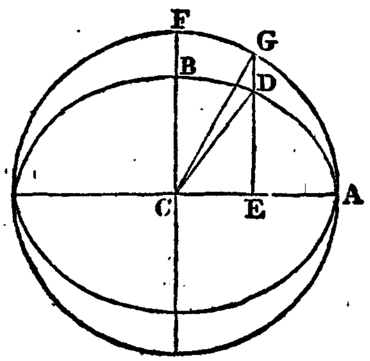
6. CUM enim arcûs circularis formula ex ſinu recto talis ſit, qualis à Newtono eſt expoſita (§. 1. hujus Excerpti) et ſector circularis æqualis ſit rectangulo ſub dimidio radii et arcu, idcirco ſectoris circularis formula ex ſinu recto hæc erit; $\frac{1}{2}r \times \overline{x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x'}{112r^6}}$ membris omnibus affirmatis; cùm in formulâ ſectoris hyperbolici alterna negata ſint; ſingulorum verò utriuſque formulæ membrorum inde ab initio eadem eſt conſtitutio.

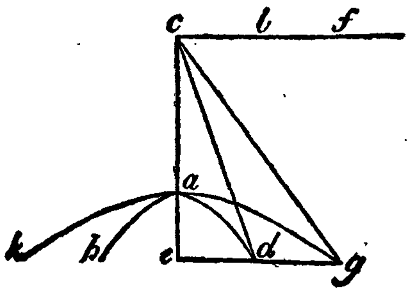
7. NEQUE verò proprium eſt illud Hyperbolæ Æquilateræ Circulique, quòd figuræ utriuſque ſectores formulis exponantur ad omnia præter ſigna, +, −, ſimilibus, ſed ad omnes quidem Hyperbolas omneſque Ellipſes pertinet.

EXCERPTUM QUARTUM.

occurrens ellipſi in G, motu angulari feratur; & ex datâ areâ ſectoris elliptici BEG, quæratur recta GF, quæ à puncto G ad axem AB normaliter demittitur: eſto BC $= q$, DC $= r$, EB $= t$, ac duplum

MENSURARUM ELLIPSEOS





8. Centro enim C, ſemiaxibus CA, CB, ſcriptam puta ellipſin ADB, cujus ſit ſector ACD. Centro verò *c*, ſemiaxibus *ca*, *cb* inæqualibus, ſcriptam puta hyperbolam *dab*, cujus ſector ſit *acd*. A puncto D ellipſeos in axem ejus CA deducatur DE ad perpendiculum; et à puncto *d* hyperbolæ in axem *ca*, qui hyperbolæ tranſverſus ſit, deducatur ad perpendiculum *de*; ut ſit DE ſinus arcûs elliptici AD, quem ſector ellipſeos ACD baſin habet; *de* ſimiliter ſinus arcûs hyperbolici *ad*, quem ſector hyperbolæ *cad* baſin habet. Centro C ellipſeos, radio CA ellipſeos ſemiaxe, ſcribatur circulus, cui recta DE in G occurrat, et jungatur CG. Centro *c* ſemiaxibus æqualibus *ca*, *cf*, ſcribatur hyperbola æquilatera *gak*, cui recta *ed* ad *g* occurrat, et jungatur *cg*.

9. Propter circulum et ellipſin axe AC communi, recta GE, erit ad DE ut FC ad BC, vel ut CA ad BC. Atque hæc conſtans erit rectarum GE, DE inter ipſas ratio, in omni ſitu puncti D. Areis igitur GAE, DAE inter ipſas eadem intercedet (De Rat. Prim. et ult. Prop. IV. Cor. H. I.) Verùm eadem triangulis CGE, DCE (per El. 6. I.) Igitur eadem ſectoribus etiam CGA, CDA (El. 5. 12.) Hoc eſt ſector circularis CGA ad ſectorem ellipſeos CDA rationem habet eam quam CA ad CB. Quare ſector ellipſeos CDA $= \frac{CB}{CA} \times$ CGA.

Similiter cùm ſit hyperbola æquilatera *gak*, aliaque *dab*, ſemiaxe tranſverſo *ca* communi, erit *ge* ad *de* ut ſemiaxis tranſverſus ille communis, *ca*, ad *cb* ſemiaxem ſecundum hyperbolæ *dab*. Hæc vero rectarum *ge*, *de*, in omni ſitu puncti *d*, conſtans erit inter ipſas ratio. Areis igitur *age*, *ade* eadem intercedet. Eadem verò triangulis *cge*, *cde*. Quare eadem ſectoribus *cga*, *cda*. Hoc eſt hyperbolæ æquilateræ ſector *cga*, erit ad ſectorem *cda* alterius hyperbolæ, ut *ca* ad *cb*. Quare ſector $cda = \frac{cb}{ca} \times cga$.

Si igitur ſemiaxes ellipſeos, CA, CB; vel hyperbolæ *dab*, *ca*, *cb*; literæ *r*, *a* deſignent, arcûs verò circularis, GA, ſinum GE, vel arcûs hyperbolici, *ga*, ſinum *ge*, ſi litera *x* deſignet, cùm circuli hyperbolæque æquilateræ ſectores CGA, *cga* formulis exponantur quas ſuprà poſuimus, erit ſector ellipticus CDA $= \frac{1}{2}a \times \overline{x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6} +}$,

Sector verò hyperbolicus $cda = \frac{1}{2}a \times \overline{x - \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} - \frac{5x^7}{112r^6} +}$,

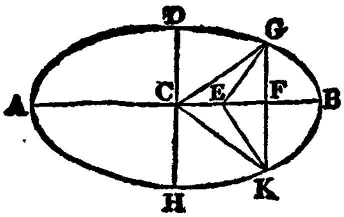
Quòd ſi litera χ deſignet DE ſinum arcûs elliptici DA, vel *de* ſinum arcûs hyperbolici *da*, cùm ſit GE : DE $=$ CA : CB, et $ge : de = ca : cb$; hoc eſt, cùm ſit $x : \chi = r : a$, erit $x = \frac{r\chi}{a}$. Hoc igitur ſymbolo, $\frac{r\chi}{a}$, in formulas modò expoſitas pro literâ *x* ſubſtituto, venient formulæ ſectorum, elliptici hyperbolicique, ex ſinu cujuſque recto, ad omnia præter ſigna ſimiles.

SECT. ELL. EX SINU. Nimirum elliptici ſectoris hæc; CDA $= \frac{1}{2}r \times \overline{\chi + \frac{\chi^3}{6a^2} + \frac{3\chi^5}{40a^4} + \frac{5\chi^7}{112a^6} +}$,

SECT. HYP. EX SINU. Hæc verò hyperbolici $cda = \frac{1}{2}r \times \overline{\chi - \frac{\chi^3}{6a^2} + \frac{3\chi^5}{40a^4} - \frac{5\chi^7}{112a^4} +}$,

quæ, ſi ponantur *r*, *a*, æquales, in priores illas circuli hyperbolæque æquilateræ abibunt.

10. CÆTERUM in hyperbolâ æquilaterâ, ut ad illam revertamur, ſi ſector BAE ad ſectorem BAD eam rationem habeat quam *n* ad I, ſi rectæ DF, EG arcuum DB, EB, quibus ſectores inſiſtunt, ſinus literis

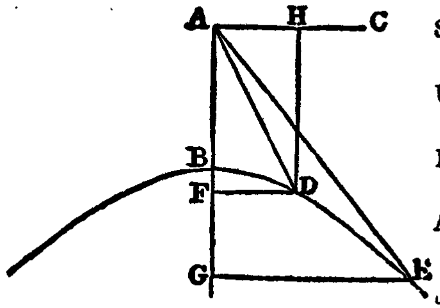


duplum areæ BEG $= z$; et erit GF $= \frac{1}{t} z - \frac{q}{6r^2t^4} z^3 + \frac{10q^2 - 9qt}{120r^4t^7} z^5 - \frac{280q^3 + 504q^2t - 225qt^2}{5040r^6t^{10}} z^7 +$ &c. PROBLEMA KEPLERI.

Sic itaque Aſtronomicum illud *Kepleri* Problema reſolvi poteſt (m).

5. In

literis x, y, deſignentur: cùm ſectorum hyperbolæ æquilateræ ea ſit ex ſinubus rectis formula, quam ſuprà ſcripſimus, erit ſector hyperbolicus BAD $= \frac{1}{2} r \times \overline{x - \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} - \frac{5x^7}{112r^6} +}$, HYPERBOLÆQUE HARMONIA.



Sector hyperbolicus BAE $= \frac{1}{2} r \times \overline{y - \frac{y^3}{6r^2} + \frac{3y^5}{40r^4} - \frac{5y^7}{112r^6} +}$,

Unde rurſum ſector BAD $= \frac{1}{2} r \times \overline{\frac{y}{n} - \frac{y^3}{6nr^2} + \frac{3y^5}{40nr^4} - \frac{5y^7}{112nr^6} +}$,

Ergo $\frac{1}{2} r \times \overline{x - \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} -}, = \frac{1}{2} r \times \overline{\frac{y}{n} - \frac{y^3}{6nr^2} + \frac{3y^5}{40nr^4} -}$,

Ac propterea $x - \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} -, = \frac{y}{n} - \frac{y^3}{6nr^2} + \frac{3y^5}{40nr^4} -$,

Ad inveniendam igitur rectam EG, literâ y deſignatam, nacti ſumus æquationem, ejus omnia præter ſigna ſimilem, quâ è dato ſinu arcûs circularis z ſinum arcûs nz eruendum eſſe pridem oſtendimus. Etenim in æquatione illâ circulari ſigna in utrâque parte poſitiva erunt omnia, in hâc hyperbolicâ alternè negativa ſunt utrinque. Hujus igitur, ſicut prioris, radice y infinitè affectâ per formulam generalem Moivræanam extractâ, veniet formula literæ y prioris omnia ſimilis, niſi quòd ſigna membrorum alternorum contraria habeant. Id quod coefficientium Theorematis Moivræanam compoſitionis modum attentiùs conſideranti cuivis credo manifeſtum erit. Ergo y, ſive EG $= nx + \frac{nn-1}{2 \times 3 . r^2} xxA + \frac{nn-9}{4 \times 5 . r^2} xxB +$. In majorem rei evidentiam calculos qui volet ſubducat.

11. Cæterùm ſi n ſit numerus impar, hæc formula, ſicut prior illa, infinita eſſe deſinet; et finitas quidem æquationes promet, ad ſectorum hyperbolicorum, hoc eſt ad logarithmorum cum numeris imparibus multiplicationes, vel diviſiones, earum omnia æquationum præter ſigna ſimiles, quibus ſimiles arcuum circularium vel angulorum multiplicationes diviſioneſque continentur. Nimirum hoc ipſum eſt, quo omnis ferè nititur mirabilis illa Angulorum Rationumque Menſurarum Harmonia, cujus contemplatione magnus ille olim Coteſius tantopere delectatus eſt. Nec immeritò ſcilicet: cùm tantam illa ſummo viro attuliſſet inventorum atque famæ frugem: utpote cujus ope tot tantaque tam eleganter perfecerat. Quorum tamen ſummam EDVARDUS WARING ὁ πάνυ, in eo contineri ſcribit "quòd æquationis $x^n \pm 1 = 0$ radices invenit, ex quibus detexit fluentem fluxionis $\frac{\dot{x}}{x^{2n} \pm ax^n + 1}$ *." At verò ea eſt Coteſii laus, non quòd *æquatiunculam* illam explicaverit quidem, ſed quòd modo illo ſuo peculiari hoc effecerit; tam ſimplici, tam ſubtili, tam perſpicuo: quòd illam viâ geometricâ explicando, calculorum laborem multùm minuerit; tot tantaque problemata ad menſuras aut angulorum aut rationum revocaverit: unde factum eſt, ut ex analyſi facillimâ compoſitiones problematum elegantiſſimæ quaſi ſponte enaſcerentur; compoſitionum demonſtrationes, in eo quidem genere quod ſit cum brevitate firmiſſimum, ultrò ſe offerrent. Has quidem ſtatuo magni Coteſii laudes, quas vereor ut ſatis amplè Waringius, apud æquos judices, celebraſſe exiſtimandus erit.

(m) Recta GF producta ellipſi iterum, ab alterâ parte axis, in K occurrat et jungantur CG, CK, EK. PROBLEMA KEPLERI.
Itaque ſi litera x rectam GF deſignet, ſectori elliptico GCB hujus ſeriei, ſi infinitè ea producatur, ſumma ultimò æqualis erit; $\frac{1}{2} q \times \overline{x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6} +}$, (not. [1] § 9.)

Sector igitur GCK $= 2$GCB $= qx + \frac{qx^3}{6r^2} + \frac{3qx^5}{40r^4} + \frac{5qx^7}{112r^6} +$,

Sed recta CE $= q - t$. Quare duplum trianguli GEC, ſive duo triangula GEC, KEC ſimul ſumpta $= qx$

* Vid. Meditat. Analyt. Præfat.

ARC. ELLIP. E SINU.

5. In eâdem ellipſi, ſi ſtatuatur CD $= r$, $\frac{CB^2}{CD} = c$, & CF $= x$: erit arcus ellipticus

$$(^n)\ DG = x + \frac{1}{6c^2}x^3 + \frac{1}{10rc^3}x^5 + \frac{1}{14r^2c^4}x^7 + \frac{1}{18r^3c^5}x^9 + \frac{1}{22r^4c^6}x^{11} + \&c.$$
$$\qquad\qquad - \frac{1}{40c^4} \qquad - \frac{1}{28rc^5} \qquad - \frac{1}{24r^2c^6} \qquad - \frac{1}{22r^3c^7}$$
$$\qquad\qquad\qquad\qquad + \frac{1}{112c^6} \qquad + \frac{1}{48rc^7} \qquad + \frac{3}{88r^2c^8}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad - \frac{5}{1152c^8} \qquad - \frac{5}{352rc^9}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{7}{2816c^{10}}$$

Hîc numerales coefficientes ſupremorum terminorum ($\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{14}$, &c.) ſunt in muſicâ progreſſione: et numerales coefficientes omnium inferiorum in unaquâque columnâ prodeunt multiplicando continuò numeralem coefficientem ſupremi termini per terminos hujus progreſſionis $\frac{\frac{1}{2}n-1}{2}$, $\frac{n-3}{4}$, $\frac{\frac{5}{4}n-5}{6}$, $\frac{\frac{7}{5}n-7}{8}$, $\frac{\frac{9}{6}n-9}{10}$, &c. Ubi n ſignificat

PROBLEMA KEPLERI.

$= qx - tx$. Sed area elliptica BGEK differentia eſt ſectoris elliptici GCK, duorumque triangulorum GCE, KCE. Quare area elliptica BGEK, ſive 2GEB, ſive z, $= tx + \frac{qx^3}{6r^2} + \frac{3qx^5}{40r^4} + \frac{5qx^7}{112r^6} +$, Et hanc ſeriem invertendo, ope Theorematis 2. Excerpti II. veniet formula rectæ GF Newtoniana, $x = \frac{1}{t}z - \frac{q}{6r^2t^4}z^3 + \frac{10q^2-9qt}{120r^4t^7}z^5 - \frac{280q^3-504q^2t+225qt^2}{5040r^6t^{10}}z^7 +$,

2. In coefficiente membri quarti ſigna nominum ſingulorum calculis ſuadentibus emendavi, cùm coefficientem illam Newtonus ipſe in Commercio Epiſtolico, & Jonesius hoc modo exhibuiſſent. $-\frac{280q^3+504q^2t-225qt^2}{5040r^6t^{10}}$. Vitioſè certè. Nam per Theorema ſecundum Newtoni ad regreſſiones coefficiens ille $= + \frac{504q^2t-225t^2q-280q^3}{5040r^6t^{10}} = - \frac{280q^3+225t^2q-504q^2t}{5040r^6t^{10}}$.

3. Credibile eſt errorem ex eo natum, quòd Newtonus formulam ſuam, primâ ſcriptione, hoc modo diſpoſuiſſet $x = \frac{1}{t}z - \frac{q}{6r^2t^4}z^3 + \frac{q^2}{12r^4t^7}z^5 - \frac{q^3}{18r^6t^{10}}z^7 +$,

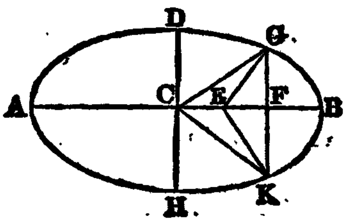
$$\qquad\qquad - \frac{3q}{40r^4t^6} \quad + \frac{q^2}{10r^6t^9} -,$$
$$\qquad\qquad\qquad\qquad - \frac{5q}{112r^6t^8} +,$$

Poſtea verò cùm alia diſpoſitio magis ei placuiſſet, ut coefficientis cujuſque nomina ad denominatorem revocarentur communem, et numeratoribus ſignis +, −, algebraicè conjunctis, in unius fractionis ſpeciem coaleſcerent; hæc, inquam, formulæ ſuæ diſpoſitio cùm Newtono potior viſa eſſet, integris quidem coefficientibus ea, credo, ſigna præfixit, quæ in primâ illâ diſpoſitione ſummi ordinis nominibus à ſe præfixa inveniſſet: nominum verò inferiorum ſigna, poſitiva pro negativis, negativa pro poſitivis mutare, quoties nomen ſummum negativum eſſet, quod fieri omninò debuit, neglexit. Sed quocunque tandem modo natus fuerit error, error certè eſt. Atque mecum hâc in re ſenſiſſe video virum harum rerum peritiſſimum Carolum Huttonum, qui in opere ſuo de Menſionibus Anglicò ſermone Anno 1770 edito, huic formulæ, cujus etiam inveſtigationem tradidit, ſimilem emendationem adhibuit, (Treatiſe of Menſuration, Part 3. Sect. 3. Problem VIII. Cor. 9.)

ARC. ELLIP. (^{n}) RECTA GF deſignetur literâ y. Tum è natura ellipſeos erit CD^2 : CB^2 $=$ GF^2 : AF $\times$ FB; hoc eſt $r : c = y^2 : rc - x^2$. Ergo $y^2 = \frac{r}{c}\,\overline{rc - x^2}$. Ergo $\dot{y} = -\frac{rx\dot{x}}{c \times y} = -\frac{rx\dot{x}}{r} \times \sqrt{\frac{c}{rc - rx^2}}$.

Quare

nificat numerum dimenſionum ipſius c in denominatore iſtius ſupremi termini. E. g. ut terminorum infra $\frac{1}{22r^4c^6}$, numerales coefficientes inveniantur, pono $n=6$, ducoque $\frac{1}{22}$ (numeralem coefficientem ipſius $\frac{1}{22r^4c^6}$) in $\frac{\frac{1}{2}n-1}{2}$, hoc eſt, in 1; & prodit $\frac{1}{22}$, numeralis coefficiens termini proximè inferioris: dein duco hunc $\frac{1}{22}$ in $\frac{\frac{1}{2}n-3}{4}$, ſive in $\frac{n-3}{4}$, hoc eſt, in $\frac{3}{4}$; & prodit $\frac{3}{88}$ numeralis coefficiens tertii termini in iſtâ columnâ. Atque iſta $\frac{3}{88}\times\frac{\frac{1}{2}n-5}{6}$ facit $\frac{5}{352}$ numeralem coefficientem quarti termini; & $\frac{5}{352}\times\frac{\frac{3}{2}n-7}{8}$ facit $\frac{7}{2816}$ numeralem coefficientem infimi termini. Idem in aliis ad infinitum uſque columnis præſtari poteſt: adeoque valor ipſius DG per hanc regulam pro lubitu produci. ARCUS ELLIPTICUS



6. Adhæc, ſi BF dicatur x; ſitque r latus rectum ellipſeos, & $e=\frac{r}{AB}$; erit arcus ellipticus (°) BG= EX SINU.

Quare $\dot{y}\dot{y}=\frac{r^2x^2\dot{x}\dot{x}}{c^2}\times\frac{c}{r^2c-rx^2}=\frac{rx^2\dot{x}\dot{x}}{rc^2-cx^2}$. Et $\dot{y}\dot{y}+\dot{x}\dot{x}=\dot{x}\dot{x}\times\overline{1+\frac{rx^2}{rc^2-cx^2}}$.

Hoc eſt, ſi fractio $\frac{rx^2}{rc^2-cx^2}$, diviſionis operâ, convertatur in ſeriem infinitam,

$\dot{y}\dot{y}+\dot{x}\dot{x}=\dot{x}\dot{x}\times\overline{1+\frac{x^2}{c^2}+\frac{x^4}{rc^3}+\frac{x^6}{r^2c^4}+\frac{x^8}{r^3c^5}+\frac{x^{10}}{r^4c^6}+}$. Ergo $\sqrt{\dot{y}\dot{y}+\dot{x}\dot{x}}$, ſive fluxio arcûs elliptici DG, $=\dot{x}\times\sqrt{1+\frac{x^2}{c^2}+\frac{x^4}{rc^3}+\frac{x^6}{r^2c^4}+\frac{x^8}{r^3c^5}+\frac{x^{10}}{r^4c^6}+}$.

Vel, ſi ope formulæ noſtræ generalis, (Logiſtic. Infinit.) radix quadratica ſeriei infinitæ extrahatur, fluxio arcûs elliptici DG =

$$=\dot{x}\times 1+\frac{1}{2.c^2}x^2+\left.\begin{array}{l}\frac{1}{2.r.c^3}\\-\frac{1}{2^2.1\times 2.c^4}\end{array}\right\}x^4\left.\begin{array}{l}+\frac{1}{2.r^2.c^4}\\-\frac{2}{2^2\times 1\times 2.r.c^5}\\+\frac{2}{2^3\times 1\times 2\times 3.c^6}\end{array}\right\}x^6\left.\begin{array}{l}+\frac{1}{2.r^3.c^5}\\-\frac{3}{2^2\times 1\times 2.r^2.c^6}\\+\frac{3\times 3}{2^3\times 1\times 2\times 3.r.c^7}\\-\frac{3\times 5}{2^4\times 1\times 2\times 3\times 4.c^8}\end{array}\right\}x^8+$$

Et fluenti hujus ſeriei arcus ipſe DG æqualis erit. Hujus autem ſeriei fluens eſt ipſa quidem ſeries Newtoniana.

(°) DESIGNETUR enim recta GF, ut priùs, literâ y. ARC. ELLIP.
E naturâ ellipſeos $CD^2 : CB^2 = GF^2 : AF\times FB$, hoc eſt, $r : AB = y^2 : ABx-xx$.

Ergo $y^2=\overline{ABx-xx}\times\frac{r}{AB}=rx-ex^2$. Ergo $2y\dot{y}=\dot{x}\times\overline{r-2ex}$. Et $\dot{y}=\dot{x}\times\frac{r-2ex}{2\sqrt{rx-ex^2}}$.

Ergo $\dot{y}\dot{y}=\dot{x}\dot{x}\times\frac{r^2-4rex+4e^2x^2}{4x\times\overline{r-ex}}$. Et $\dot{y}\dot{y}+\dot{x}\dot{x}=\dot{x}\dot{x}\times\overline{1+\frac{r^2-4rex+4e^2x^2}{4x\times r-ex}}=\frac{\dot{x}\dot{x}}{4x}\times$

$\frac{r^2+\left.\begin{array}{l}4r\\-4re\end{array}\right.x+\left.\begin{array}{l}4e^2\\-4e\end{array}\right.x^2}{r-ex}$. Et, ſi fractio diviſionis operâ convertatur in ſeriem infinitam, $\dot{y}\dot{y}+\dot{x}\dot{x}=$

$\dot{x}\dot{x}$

Ex Sinu Verso.

$$BG = \sqrt{rx} \text{ in } \overline{1 + \frac{2 - \frac{3}{2}e}{3r}x + \frac{-2 + 3e - \frac{5}{8}e^2}{5r^2}x^2 + \frac{+4 - 9e + \frac{23}{4}e^2 - \frac{7}{16}e^3}{7r^3}x^3 + \frac{-10 + 30e - \frac{123}{4}e^2 + \frac{91}{8}e^3 - \frac{45}{128}e^4}{9r^4}x^4, + \&c.}$$

7. Quare, ſi ambitus totius ellipſeos deſideretur; biſeca CB in F, & quære arcum DG, per prius Theorema, & arcum BG per poſterius.

8. Si, vice verſâ, ex dato arcu elliptico DG, quæratur ſinus ejus CF; tum dicto $CD = r$, $\frac{CB^2}{CD} = c$, & arcu illo $DG = z$; erit

(Γ) CF

Ex Sin. Vers.

$\frac{\dot{x}\dot{x}}{4x} \times r + \overline{4-3e}.x + \frac{e^2x^2}{r} + \frac{e^3x^3}{r^2} + \frac{Ax^4}{r^3} +$. Ergo $\sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}}$, ſive fluxio arcûs elliptici BG $= \frac{\dot{x}}{2\sqrt{x}} \times \sqrt{r + \overline{4-3e}.x + \frac{e^2.x^2}{r} + \frac{e^3x^3}{r^2} + \frac{Ax^4}{r^3} +}$

Vel, ſi ope formulæ noſtræ generalis (Logiſtic. Infin.) radix quadratica ſeriei infinitæ extrahatur, extractaque dividatur quantitate $x^{\frac{1}{2}}$,

$$\text{Fluxio arcûs } B\dot{G} = \frac{\dot{x}}{2} \times r^{\frac{1}{2}}x^{-\frac{1}{2}} + \frac{2 - \frac{3e}{2}}{r^{\frac{1}{2}}}x^{\frac{1}{2}} + \frac{+\frac{e^2}{2} - \frac{16 - 24e + 9e^2}{2^2 + 1 \times 2}}{r^{\frac{3}{2}}}x^{\frac{3}{2}} +$$

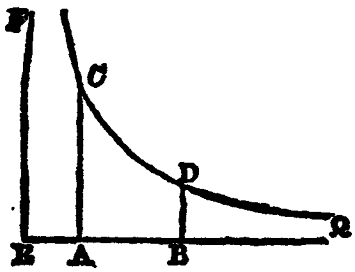
$$\frac{+\frac{e^3}{2} - \frac{8e^2 - 6e^3}{2^2 \times 1 \times 2} + \frac{\overline{4-3e}|^3 \times 3}{2^3 \times 1 \times 2 \times 3}}{r^{\frac{5}{2}}}x^{\frac{5}{2}} \quad \frac{+\frac{e^4}{2} - \frac{8e^3 - 5e^4}{2^2 \times 1 \times 2} + \frac{3 \times \overline{16e^2 - 24e^3 + 9e^4} \times 3}{2^3 \times 1 \times 2 \times 3} - \frac{\overline{4-3e}|^4 \times 3 \times 5}{2^4 \times 1 \times 2 \times 3 \times 4}}{r^{\frac{7}{2}}}x^{\frac{7}{2}}$$

$$\text{Hinc arcus } BG = r^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{2 - \frac{3e}{2}}{3r^{\frac{1}{2}}}x^{\frac{3}{2}} + \frac{+\frac{e^2}{2} - \frac{16 - 24e + 9e^2}{2^2 \times 1 \times 2}}{5r^{\frac{3}{2}}}x^{\frac{5}{2}} + \frac{+\frac{e^3}{2} - \frac{8e^2 - 6e^3}{2^2 \times 1 \times 2} + \frac{\overline{4-3e}|^3 \times 3}{2^3 \times 1 \times 2 \times 3}}{7r^{\frac{5}{2}}}x^{\frac{7}{2}} +$$

$= \surd$

$$\text{(P) } CF = z - \frac{1}{6c^2} z^3 \begin{array}{l} -\frac{1}{10rc^3} \\ +\frac{13}{120c^4} \end{array} z^5 \begin{array}{l} -\frac{1}{14r^2c^4} \\ +\frac{17}{420rc^5} \\ -\frac{493}{5040c^6} \end{array} z^7 - \&c.$$ SIN. ELLIP. EX ARC.

Quæ autem de Ellipſi dicta ſunt, omnia facilè accommodantur ad Hyperbolam; mutatis tantùm ſignis ipſorum c & e ubi ſunt imparium dimenſionum (q).



7. Præterea, ſi ſit CD Hyperbola, cujus aſymptoti EB, EF rectum angulum BEF conſtituant; & ad EB erigantur utcunque perpendicula

$$= \sqrt{rx} \times 1 + \frac{\left.\begin{array}{l} 2 \\ -\frac{3e}{2} \end{array}\right\}}{3r} x + \frac{\left.\begin{array}{l} +\frac{e^2}{2} \\ -\frac{16-24e+9e^2}{2^2 \times 1 \times 2} \end{array}\right\}}{5r^2} x^2 + \frac{\left.\begin{array}{l} +\frac{e^3}{2} \\ -\frac{8e^2-6e^3}{2^2 \times 1 \times 2} \\ +\frac{\overline{4-3e}|^3 \times 3}{2^3 \times 1 \times 2 \times 3} \end{array}\right\}}{7r^3} x^3 +$$ ARC. ELLIPT. EX SIN. VERS.

Unde, coefficientis cujuſque membris ſecundum poteſtates literæ e diſpoſitis, veniet ſeries Newtoniana.

(p) FORMULAM arcûs DG priùs inventam (§. 5. hujus Excerpti) ſi ope Theorematis 2. Excerpt. II. invertas, obtinebis formulam ſinûs CF ex arcu illo, qualis à Newtono hîc deſcripta eſt.

(q) ET ad Parabolam, membris iis omnibus deletis, quæ, in formulâ arcûs ex ſinu verſo, litera e ingreſſa eſt; vel iis, quorum denominatores litera r, in aliis illis arcûs ex ſinu recto, ſinûſve ex arcu formulis. Ut hæc ſcilicet ſit formula Arcûs Parabolici ex ſinu recto

$$x + \frac{1}{6c^2} x^3 - \frac{1}{40c^4} x^5 + \frac{1}{112c^6} x^7 - \frac{5}{1152c^8} x^9 + \frac{7}{2816c^{10}} x^{11} -,$$ ARC. PARAB. EX SIN.

Sinûs autem ex arcu

$$z - \frac{1}{6c^2} z^3 + \frac{13}{120c^4} z^5 - \frac{493}{5040c^6} z^7 +,$$ SIN. PARAB. EX ARC.

Deſignante nimirum literâ x ſinum rectum; z, arcum; c ſemiparametrum parabolæ.
Arcûs etiam Parabolici ex ſinu verſo ut hæc ſit formula

$$r^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{2}{3r^{\frac{1}{2}}} x^{\frac{3}{2}} - \frac{2}{5r^{\frac{3}{2}}} x^{\frac{5}{2}} + \frac{4}{7r^{\frac{5}{2}}} x^{\frac{7}{2}} - \frac{10}{9r^{\frac{7}{2}}} x^{\frac{9}{2}} +,$$ ARC. PAR. EX SIN. VERS.

Deſignante x ſinum verſum; r, parametrum parabolæ.

2. HARUM vero tranſmutationum in eo poſita eſt ratio, quòd Algebræ, proprio ſermone curvarum proprietates ediſſerenti, Hyperbolam tanquam Ellipſin negativo axe, et viciſſim, conſiderare receptum eſt; Parabolam verò vel tanquam Ellipſin quidem illam, vel tanquam Hyperbolam, axe infinito. Hoc autem quale ſit clariùs dicam.

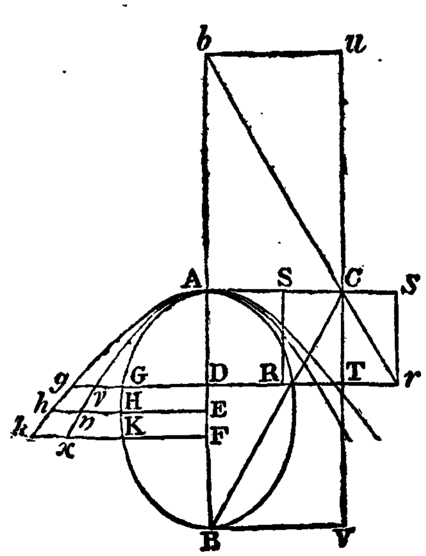
3. SINT duæ rectæ AB, AC magnitudine, et poſitione datæ et ad perpendiculum inter ſe compoſitæ. Harum in alterutrâ, puta in AB, capiantur pro arbitrio inter puncta extrema A, B, alia aliquot D, E, F. A quibus eductæ ad perpendiculum, DG, EH, FK, ejus capiantur quæque longitudinis, ut rectangula ſegmentis rectæ datæ comprehenſa, rectangula dico AD × DB, AE × EB, AF × FB, ad quadrata ex eductis, DG, EH, FK, ordine ſumendis, datam illam rationem habeant, quam recta AB ad rectam AC. Linea curva quæ per eductarum DG, EH, FK, aliarumque omnium, quæ, à punctis QUA MENTE ALGEBRA

BASIS SPATII

pendicula AC, BD occurrentia hyperbolæ in C & D: & EA dicatur *a*; AC, *b*; & area CADB, *z*;

Erit

NEGATIVOS LOQUITUR



tis in rectâ AB, pro arbitrio ſumendis, ſimili lege educi poterunt, extrema tranſeat, Ellipſis erit; cujus axis AB, parameter AC. Juncta enim BC rectæ GD, ſi opus ſit, productæ, in R occurrat; actiſque RS, CT cum rectâ AB parallelis, actâque rectâ BV cum AC parallelâ, rectangula AR, AT, AV compleantur. Jam propter DR, AC parallelas, erit BD ad DR ut BA ad AC. Unde BD × DA ad DR × DA, hoc eſt ad DS, ut AB ad AC. (Elem. VI.1.) Eſt autem rectangulum BD×DA ad quadratum ex DG ut AB ad AC. (Id enim poſuimus.) Rectanguli igitur BD × DA ad rectangulum DS et ad quadratum ex DG eadem eſt ratio. Quadratum igitur ex DG rectangulo DS æquale erit. Rectangulum verò DS deficit ab illo DC, quod datâ AC abſciſſâque AD comprehenſum eſt, alio illo ST, ejus nimirum quod datis AB, AC comprehenſum eſt, AV, ſpecie poſituque ſimili. Punctum igitur G ad ellipſin erit, cujus axis recta AB, parameter illa AC. Similibus argumentis efficietur puncta H, K ad eandem ellipſin eſſe, & extremum cujuſvis itidem rectæ, quæ, à puncto inter extrema illa A, B pro arbitrio ſumpto, ad priorum legem educta fuerit. Linea igitur curva quæ per omnium illo modo eductarum extrema tranſeat, haud alia erit atque illa quam modò diximus Ellipſin.

4. SED retrorſùm ducatur recta AB, ſumaturque A*b* datæ AB æqualis. Jam punctis D, E, F, non inter extrema *b*, A ſed infra A pro arbitrio ſumptis, interceptæ illæ AD, AE, AF, contrario quidem modo atque priùs ad AB ſe habuerunt, poſituræ ſcilicet ratione nunc ad A*b* habent; ut quæ alterius partes fuerint, eæ hujus ſint additamenta. Eductæ verò ad perpendiculum D*g*, E*h*, F*k*, ſi capiantur ejus quæque longitudinis, ut rectangulum datâ *b*A cum additamento, additamentoque comprehenſum ad quadratum ex eductâ datam illam rationem habeat, quam recta A*b* ad rectam AC; hoc eſt, ſi rectangula *b*D × DA, *b*E × EA, *b*F × FA, ad quadrata ex eductis D*g*, E*h*, F*k*, ordine ſumendis, datam illam rationem habeant; linea curva, quæ per eductarum D*g*, E*h*, F*k*, aliarumque omnium, quæ, à punctis infra A, pro arbitrio ſumendis, ſimili lege educi poterunt, extrema tranſeat, hæc inquam curva, non jam ut priùs Ellipſis, ſed Hyperbola quidem erit; parametro eâdem AC atque ellipſis habuit, et axe tranſverſo A*b* ellipſeos axi AB æquali.

Juncta enim *b*C rectæ *g*D ſi opus ſit productæ in *r* occurrat; aganturque *rs*, *bu* cum illis AD, AC parallelæ; quæ rectangula A*r*, A*u* perficiant. Itaque propter parallelas D*r*, AC, erit *b*D ad D*r* ut *b*A ad AC. Atque idcirco *b*D × DA ad D*r* × DA, hoc eſt ad D*S*, ut *b*A ad AC. Eſt autem rectangulum *b*D × DA ad quadratum ex D*g* ut *b*A ad AC (Id enim poſuimus.) Rectanguli igitur *b*D × DA ad rectangulum D*S* eadem atque ad quadratum ex D*g* eſt ratio. Quadratum igitur ex D*g* rectangulo D*S* eſt æquale. Rectangulum verò D*S* exuperat illud DC, quod datâ AC abſciſſâque AD comprehenſum eſt, alio illo T*S*, ejus nimirum, quod datis A*b*, AC comprehenſum eſt, A*u*, ſpecie poſituque ſimili. Punctum igitur *g* ad hyperbolam erit, cujus axis tranſverſus A*b* parameter AC. Similibuſque argumentis de punctis *h*, *k*, omniumque adeo rectarum extremis, quæ à punctis infra A, pro arbitrio ſumendis, ad priorum legem eductæ fuerint, efficietur ea omnia ad hyperbolam eſſe, lateribus quibus diximus, AC, A*b*, recto tranſverſoque. Linea igitur curva per eductarum extrema tranſiens, quæ modò Ellipſis erat nunc fit Hyperbola, parametro eâdem quam ellipſis habuit, axeque æquali quidem ſed ſitu contrario. Et ex everſo axis ſitu figuræ immutatæ ratio pendet. Cùm enim in priori caſu recta BC, in hoc verò *b*C, ab eductâ quâlibet, verbi gratiâ, à DG, vel D*g*, partem quandam, DR, vel D*r*, abſcindat, quæ cum datâ abſciſſâ AD rectangulum claudit quadrato ex eductâ æquale, harum verò altera BC, à puncto B egreſſa, eductam DG medio quaſi itinere objectam tranſeat, priuſquam punctum C attingat, altera autem *b*C ultra locum C progrediens eductæ *g*D in occurſum eat; horum conſequens erit, duarum DR, D*r* illam quidem rectâ AC minorem, hanc verò majorem eſſe: ac propterea quadrata ex eductis DG, D*g*, à rectangulo DC, illud defectu, hoc exceſſu aberunt. Quod ſanè cùm primarium ſit et præcipuum Ellipſeos et Hyperbolæ

Erit AB $= \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5}$ + &c. Ubi coeffici- HYPERB. EX AREA, SIVE
entes denominatorum prodeunt multiplicando terminos hujus
arithmeticæ

perbolæ diſcrimen, ex ſitu axis, ſicut modò diximus, everſo videtur omnis figuræ fieri tranſmutatio. Algebra igitur, quæ contrarios linearum ductus ſignorum diverſitate indicare ſolet, haud ineptè Hyperbolam tanquam Ellipſin, et viciſſim, conſiderat, negato axe. Atque his quidem congruenter, poſitâ æquatione ad Ellipſin $y^2 = px - \frac{p}{d}x^2$, deſignante nimirum literâ p parametrum, d axem, y ordinatam, x abſciſſam ab axe, ſi in æquatione illâ pro d ſcribatur $-d$, veniet æquatio ad Hyperbolam: $y^2 = px + \frac{p}{d}x^2$. E contrario, poſitâ æquatione ad Hyperbolam $y^2 = px + \frac{p}{d}x^2$, ſi in illâ pro d ſcribatur $-d$, veniet æquatio ad Ellipſin. CURVARUM AXES.

5. CÆTERUM manentibus ellipſi et hyperbolâ modò ſcriptis, axibus quidem æqualibus AB, Ab, parametro verò AC communi, eâdem parametro AC, vertice autem A, axeque rectâ AB inde ab A in infinitum protenſâ, ſcriptam puta Parabolam Aνx, cui ordinatæ DG, EH, FK in punctis ν, n, x occurrant. Hæc Parabola Ellipſin in vertice communi A contingens, eam intra ſe totam continebit. Quadratum enim ex Dν rectangulo DC æquale eſt, propter parabolam. Quadratum autem ex DG rectangulo DS, propter ellipſin. Sed rectangulum DS rectangulo DC minus eſt. Quare quadratum ex DG minus erit quadrato ex Dν. Recta igitur DG minor quàm illa Dν; ac propterea, cùm punctum ν ſit ad parabolam, (cujus in axe punctum D) punctum G intra parabolam erit. Eodemque modo oſtendatur de alio quolibet ellipſeos puncto, præter verticem communem A, intra parabolam illud eſſe. Quòd ſi manente parametro AC, axis ellipſeos AB ſenſim augeatur, ambitus ellipſeos paulatim dilatando, ductu quidem parabolæ interiorem ſe continebit; eum tamen, quo major axis fuerit, eo magis accedet; et, creſcente infinitè axe, propiùs quidem quàm pro datâ aliquâ diſtantiâ. Congruunt igitur ultimò, et parabola figura ultima eſt ellipſeos. Nam propter parallelas AC, DR, erit AC ad DR, vel AS, ut BA ad BD. Sed rectangulum DC eſt ad rectangulum DS ut AC ad AS. Quare BA erit ad BD ut rectangulum DC ad rectangulum DS, hoc eſt, ut quadratum ex Dν ad quadratum ex DG. Creſcente autem axe AB, et manente abſciſſâ AD, rectæ BD ad majorem BA proportio perpetim fit major, uſquedum illâ AB infinitè auctâ ipſam æqualium rationem, propiùs quàm pro datâ aliquâ differentiâ, aſſecuta fuerit. Creſcente igitur axe ellipſeos AB proportio quadrati ex DG ad quadratum è Dν continuò major fit, et æqualium rationem ultimò aſſequitur. Ipſæ igitur rectæ DG, Dν fiunt ultimò æquales, et punctum G ellipſeos illud ν parabolæ ſenſim accedendo, ultimò attigerit. Simili modo aliud omne Ellipſeos punctum, axe ejus in infinitum creſcente, Parabolam ultimò deveniet.

6. Similibus prorſus argumentis efficere licet, Hyperbolam Parabolâ, cujus eadem parameter, quamque in vertice communi A contingit, exteriorem eſſe: eam tamen, axe tranſverſo Ab infinitè aucto, propiùs accedere quàm pro datâ aliquâ diſtantiâ. Hiſce vero illud etiam congruit, quòd ſi, reliquis manentibus, axis, literâ d deſignatus, infinitè uſque increſcere fingatur, æquatio ad ellipſin vel hyperbolam $y^2 = px \mp \frac{p}{d}x^2$, evaneſcente membro $\mp \frac{p}{d}x^2$, et ad nihilum ultimò redacto, in æquationem ad parabolam, $y^2 = px$, ultimò abierit.

7. HAUD igitur nihil eſt quod dicunt Algebraici, cùm Parabolam pro Ellipſi vel Hyperbolâ haberi velint infinito axe. Nimirum, ut totius diſputationis ſummam paucis complectar, trium illarum Sectionum Conicarum multa ita ſunt communia, ut quicquid diſcriminis interſit, aut ex variâ axis magnitudine, aut ex ſitu ejus contrario natum ſit. Si igitur hujuſmodi aliquid, notis ſibi conſuetis, de utrâque curvarum, Ellipſi vel Hyperbolâ, Algebra enunciaverit, illud ad alteram ſtatim accommodatur, ſigno tantummodò axis, qui hyperbolæ tranſverſus eſſe debet, quod quidem poſituræ ſignum eſt non quantitatis, in contrarium mutato. Ad Parabolam verò idem accommodetur, ſi pro formulâ generali ſumatur ea in quam ultimò illa abierit, axe ellipſeos, vel hyperbolæ, modò hujus ſit tranſverſus, infinitè aucto. Hoc equidem principium, ut ad concinnandas formulas Algebraicas libens eo utar, utpote qui omnibus modis effugiendum putem computandi tædium, demonſtrationibus tamen componendis minùs aptum judico. Quibus ut ſua ſive elegantia ſive evidentia conſtet, ex eo potiùs deducendæ ſunt, quod figura quæque proprium habeat, quâm ex ſimilitudinibus quæ diverſis intercedat. Quarum licèt jucundiſſima et philoſopho

 digniſſima

Numerus e Logarithmo.

arithmeticæ progreſſionis, 1, 2, 3, 4, 5, &c. in ſe continuò. Et hinc ex logarithmo dato poteſt numerus ei competens invenire.

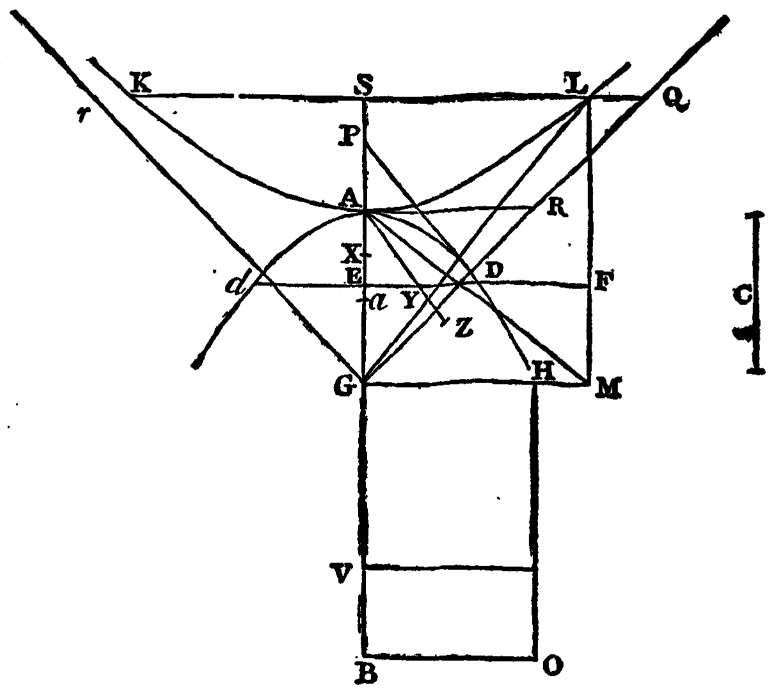
8. Eſto

digniſſima ſit contemplatio, utpote ex quâ id maximè percipiendum ſit, quod quo magis quis perceperit, eo quidem acutior exiſtimandus erit et ingenio pollentior, ſcilicet quàm ſit unum illud omne quod verum eſt; eos tamen, quibus nimium illæ captari atque credi ſolent, ſæpiſſimè fallunt. Nec mirum ſanè. Cùm haud in aliâ re magis cernatur mentis humanæ imbecillitas, quàm in eo quòd rerum ſimilitudines, quæ nullæ ſunt, ſæpiſſimè ſomniamus, quæ verè ſunt eas ægrè cernimus.

8. Sed ut ad noſtra revertamur; ſiquis, nullâ formulæ arcûs elliptici ratione habitâ, Parabolici longitudinem è naturâ ipſius parabolæ definire velit, is, ex eo quòd quadratum ordinatæ rectangulo comprehenſo abſciſſâ ab axe et parametro æquale ſit, ſubductis fluxionum calculis, in hanc æquationem incidet; deſignante nimirum literâ *z* arcum, *y* ordinatam, ſive ſinum arcûs rectum, *p* parametrum parabolæ: $\dot{z}=\frac{\dot{y}}{p}\sqrt{p^2+4yy}$. Vel in hanc, $\dot{z}=\frac{\dot{x}}{2\sqrt{x}}\sqrt{p+4x}$: deſignante *x* abſciſſam ab axe. Hiſce verò formulis, per radicum extractiones, in ſeries infinitas reſolutis, fluentium æſtimationem per Regulam 3^m^. Analyſ. per æquationes infinitas ineundo, formulas aſſequetur quas ſuprà expoſuimus.

Cognatio Arcus Parab.

9. Cæterum ex hâc æquatione, $\dot{z}=\frac{\dot{y}}{p}\sqrt{p^2+4yy}$, manifeſtà fit cognatio illà, quam Arcubus Parabolicis cum Areis quibuſdam Hyperbolicis intercedere, omnium credo primus Walliſius intellexit; quæ quidem talis eſt, ut Arcûs cujuſlibet Parabolici dimenſio ad quadraturam Hyperbolæ, et viciſſim, revocari poſſit.



Sit parabola cujus axis recta infinita AB poſitione data, vertex, A, parameter data C. Sumatur in parabolâ punctum quodvis D. A puncto D in axem deducatur ad perpendiculum recta DE, et in ED, extrorſùm productâ, capiatur DF æqualis ordinatæ DE; ut ſit EF duplo illius DE æqualis. In axe AB capiatur AG datæ C æqualis; eductâque ad perpendiculum rectâ GH, in eâ capiatur GH datæ etiam C æqualis. Semiaxe tranſverſo GA, ſecundo GH, ſcribatur hyperbola æquilatera KAL. Per F ducatur FL, cum hyperbolæ axe tranſverſo, GA, parallela; quæ hyperbolæ in L occurrat, axi ejus ſecundo in M. Ad ipſum ſemiaxem ſecundum, GH, applicetur rectangulum GO ſpatio hyperbolico AGML æquale. Dimidium lateris GB arcui AD æquale erit; vel, productâ DE uſque dum parabolæ rurſum in *d* occurrat, latus GB arcui DA*d*.

Cùm enim area AGML rectangulo GO ſit æqualis, erunt etiam areæ illius et rectanguli fluxiones ſemper inter ſe æquales. (Geometr. Flux. Prop. 11.) Rectangula igitur $LM\times G\dot{M}$, $GH\times G\dot{B}$ inter ſe æqualia. Fluxio igitur rectæ GB eſt ad fluxionem rectæ GM ut LM ad GH. Ducatur recta DP, quæ parabolam in D contingat, axi verò ejus in P occurrat. Erunt EA, AP inter ſe æquales. (Hamilton.

8. Eſto VDE *Quadratrix*, cujus vertex eſt v, exiſtente A centro & AE ſemi-diametro circuli ad quem aptatur, & angulo VAE recto: demiſſoque Arcus et Area:

milton. Conic. Lib. I. Prop. XLVII.) Quare EA ſemiſſis eſt rectæ EP, et EA ad EP ut ED ad EF. Jam verò cùm ED ſit ad EA, ut GH ad ED (propter parabolam) et EA ad EP ut ED ad EF, erit ex æquo ED ad EP ut GH ad EF, vel GM: et quadratum ex ED ad quadratum ex EP ut quadratum ex GH ad quadratum ex GM. Invertendo et componendo, quadrata ex EP et ex ED ſimul ſumpta ad quadratum ex ED, ut quadrata ex GM et ex GH ſimul ſumpta ad quadratum ex GH. At verò quadratis ex EP et ex ED ſimul ſumptis quadratum ex DP eſt æquale (propter angulum ad E trianguli PED rectum). Et quadratis ex GM et ex GH ſimul ſumptis quadratum ex ML eſt æquale (propter hyperbolam æquilateram.) (Hamilton. Conic. Lib. I. Prop. XXXII.) Quare quadratum ex DP erit ad quadratum ex DE, ut quadratum ex ML ad quadratum ex GH: quare et recta DP ad rectam DE ut recta ML ad rectam GH. Sed ut recta DP ad rectam DE ita eſt fluxio arcus AD ad fluxionem rectæ DE. (Introduct. Quadrat. Curv. § 4.) Et oſtenſum eſt, fluxionem rectæ GB eſſe ad fluxionem rectæ GM ut recta ML ad rectam GH. Erit igitur fluxio arcûs AD ad fluxionem rectæ ED ut fluxio rectæ GB ad fluxionem rectæ GM. Permutando, fluxio arcûs AD ad fluxionem rectæ GB ut fluxio rectæ ED ad fluxionem rectæ GM. Rectæ autem ED ad rectam GM ratio data eſt, illa utique quam unitas ad binarium habet. (ex conſtruct.) Fluxionis igitur arcûs AD ad fluxionem rectæ GB eadem data ratio, unitatis ad binarium, erit. Et cùm hæc conſtans ſit fluxionum ratio, fluentium etiam, curvæ AD rectæque GB, cùm ſimul illæ à nihilo generari inceperint, eadem inter ipſas ratio erit. (Geom. Flux. Prop. IV.) Arcus igitur AD rectæ GB ſemiſſis; et arcus DA*d* rectæ GB æqualis erit. Q. E. D. Cum Area Hyperb.

10. Ex hiſce ſtatim enaſcitur problematis, de longitudine Arcûs Parabolici inveniendâ, compoſitio elegantiſſima Coteſii. Ea verò eſt hæc.

MENSURA ARCUS PARABOLICI COTESIANA.

Inveniatur umbilicus parabolæ *d*AD qui ſit X. A puncto A ducatur in ordinatam ED, recta AY, quæ ordinatam illam mediam dividat. Et rectæ AY addatur YZ, quæ æqualis ſit ejus rationis, pro modulo, AX, logarithmo, quam trianguli AYE latera duo AY, AE, ſimul ſumpta, ad tertium EY habent. Recta AZ arcui AD æqualis erit. (Harmon. Menſurar. Pars I. Scholion. Generale.) Hujus demonſtrationem, quam nullam quidem ipſe Coteſius attulit, Theorema noſtrum facillimam ſuppeditat.

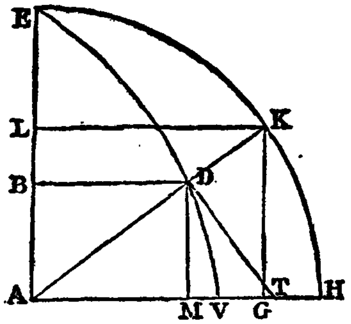
Jungantur enim GL, AM. Ad rectam GH applicetur rectangulum HV triangulo GML æquale. Sint GR, G*r* hyperbolæ KAL aſymptotæ; per puncta A, L ducantur rectæ AR, LQ cum ſemiaxe ſecundo GH parallelæ; aſymptotarum alteri, in punctis R, Q occurrentes. Harum illa AR hyperbolam in A continget; hæc LQ, in alio ei puncto præter L occurret. Sit occurſus ille alter K. Eadem verò LQ axi tranſverſo GA in S occurrat. Recta denique AG in puncto *a* media ſit diviſa. Itaque cùm DP ſit ad DE ut ML ad GH (id enim in demonſtrando theoremate noviſſimo oſtendimus,) rectangula GH × DP, DE × ML erunt inter ſe æqualia. Rectangulum verò DE × ML triangulo GML eſt æquale, cùm utrumque dimidium ſit rectanguli GL. Rectangulum igitur GH × DP triangulo GML eſt æquale. Sed eidem triangulo rectangulum HV eſt æquale. (Ita factum enim.) Rectangula igitur GH × DP, et HV, ſive GH × GV, ſunt inter ſe æqualia. Rectæ igitur DP, GV inter ſe æquales. Cùm verò rectæ EP, ED in punctis, A, Y, ſimiliter ſint diviſæ, media namque earum utraque dividitur, rectæ AY, PD parallelæ erunt, et triangula EPD, EAY inter ſe ſimilia. Quare AY erit ad PD ut EA ad EP; hoc eſt, AY ſemiſſi rectæ PD, ac propterea ſemiſſi rectæ GV, æqualis erit.

Rurſùm cùm rectangulum GO ſpatio hyperbolico AGML ſit æquale (ita enim factum eſt) et rectangulum HV triangulo GML (nam id quoque factum) erit rectangulum VO æquale ſectori hyperbolico AGL; hoc eſt, logarithmo rationis rectæ AR ad LQ pro modulo $\frac{1}{2}$GH², ſive GH × G*a*. Harum verò trium KQ, AR, LQ continua eſt proportionum ſimilitudo. (Hamilton. Conic. Lib. I. Prop. XXXVII.) Rationis igitur illius quam KQ habet ad AR, idem erit, atque ejus quam AR habet ad LQ, logarithmus. Rectangulum igitur VO logarithmus eſt rationis quam KQ habet ad AR pro modulo GH × G*a*. Recta verò AR ſemiaxi ſecundo GH eſt æqualis. (Hamilton. Lib. I. Prop. XXXVIII.) rectaque KQ duabus ML, MG ſimul ſumptis eſt æqualis. Nam, propter hyperbolam æquilateram, triangulum rectangulum QSG eſt iſoſceles; illi utique RAG ſimile. Sunt igitur QS, SG, ac proinde

QS,

QUADRATRICIS.

demiſſoque ad AE perpendiculo quovis DB, & actâ quadratricis



tangente DT occurrente axi ejus AV in T: dic AV $=a$, & AB $=x$; eritque DB $=a-\frac{x^2}{3a}-\frac{x^4}{45a^3}-\frac{2x^6}{945a^5}-$ &c. Et VT $=\frac{x^2}{3a}+\frac{x^4}{15a^3}+\frac{2x^6}{189a^5}+$ &c. Et Area AVDB $=ax-\frac{x^3}{9a}-\frac{x^5}{225a^3}-\frac{2x^7}{6615a^5}-$ &c. Et Arcus VD $=x+\frac{2x^3}{27a^2}+\frac{14x^5}{2025a^4}+\frac{604x^7}{893025a^6}+$ &c (r). Unde viciſſim, ex dato BD, vel VT, aut areâ AVDB, arcuve VD, per reſolutionem affectarum æquationum erui poteſt x, ſeu AB.

9. Eſto denique AEB *Sphæroides*, revolutione ellipſeos AEB circa axem AB genita, & ſecta planis quatuor, AB per axem tranſeunte,

DG

ARC PARAB. COTES.

QS, ML, inter ſe æquales: et propter rectam LK ad axem tranſverſum ordinatim applicatam, KS, LS, ac proinde KS, GM inter ſe æquales ſunt. E duabus igitur, KS, SQ, compoſita illa KQ, duabus ML, MG ſimul ſumptis, æqualis erit. Rectangulum igitur VO illius utique rationis eſt logarithmus, pro modulo HG × Ga, quam duæ ML, MG ſimul ſumptæ ad GH habent; vel rationis ejus quam ad GA habent duæ MA, MG ſimul ſumptæ. (Sunt enim AG, GH inter ſe æquales, factæ enim; et propter hyperbolam æquilateram, æquales etiam ML, MA.) Sed recta VB eſt ad rectam Ga ut rectangulum VO, vel GH × VB, ad rectangulum GH × Ga. Quare cùm rectangulum VO logarithmus ſit rationis quam duæ MA, MG ſimul ſumptæ habent ad AG, pro modulo rectangulo GH × Ga, erit recta VB ejuſdem rationis logarithmus pro modulo rectâ Ga.

Porrò, cùm ED ſit ad EP ut GH ad GM, id enim in demonſtrando theoremate noviſſimo oſtendimus, erit etiam ED ad EP, ut AG ad GM. Ac proinde EY ad EA ut AG ad GM. Anguli vero YEA, AGM inter ſe æquales ſunt; rectus enim uterque. Triangula igitur YEA, AGM inter ſe ſimilia; ac propterea duæ AY, AE ſimul ſumptæ erunt ad YE, ut duæ illæ AM, MG ſimul ſumptæ ad AG. Idem igitur utriuſque rationis logarithmus erit. Rationis autem illius quam duæ AY, AE ſimul ſumptæ habent ad EY, recta YZ logarithmus eſt pro modulo rectâ AX. Ita factum enim. Quare et illius quam duæ MA, MG ſimul ſumptæ ad AG habent, erit eadem YZ pro modulo utique ſuo AX logarithmus. Ejuſdem vero rationis recta VB logarithmus pro modulo rectâ Ga. Id enim oſtenſum. Sed datæ alicujus rationis, pro modulis diverſis, logarithmi ſunt inter ſe ſicut moduli. Recta igitur VB ad rectam YZ erit ut Ga ad AX. Sed AX ſemiſſis eſt ipſius Ga. Nempe cùm parametri parabolæ hæc Ga ſemiſſis, illa AX pars quarta ſit. Recta igitur YZ rectæ VB ſemiſſis. Sed oſtenſa eſt AY ſemiſſis rectæ GV. Recta igitur AZ, è duabus AY, YZ compoſita, rectæ GB, è duabus GV, VB, compoſitæ, ſemiſſis erit. Sed arcus parabolicus AD ejuſdem GB eſt ſemiſſis. Recta igitur AZ arcui AD æqualis erit. Q. E. D.

Hujus problematis, de arcu parabolico dimetiendo, alia eſt compoſitio Hugenii; (De Horolog. Oſcillatorio. Pars. 3. inter additamenta. Prop. 9.) quæ tamen Coteſianæ multùm elegantiâ et ſimplicitate cedit.

ARC. ET AREA

(r) PONATUR AE $=1$. AV $=a$. AB $=x$. E naturâ Quadratricis erit EA : AV $=$ arc. KH : x. Quare arc. KH $=\frac{x}{a}$.

Quare etiam GK (ſive ſinus arcûs HK) $=\frac{x}{a}-\frac{x^3}{6a^3}+\frac{x^5}{120a^5}-\frac{x^7}{5040a^7}+$,

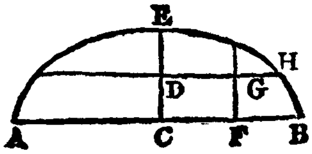
$$GK^2=\frac{x^2}{a^2}-\frac{x^4}{3a^4}+\frac{2x^6}{45a^6}-\frac{x^8}{315a^8}+,$$

$$AG=\sqrt{1-\frac{x^2}{a^2}+\frac{x^4}{3a^4}-\frac{2x^6}{45a^6}+\frac{x^8}{315a^8}+}=1-\frac{x^2}{2a^2}+\frac{x^4}{24a^4}-\frac{x^6}{720a^6}+.$$

Propter

DG parallelo AB, CDE perpendiculariter biſecante axem, & FG parallelo CE: ſitque recta $CB=a$, $CE=c$, $CF=x$, & $FG=y$. Et ſphæroideos ſegmentum CDGF dictis quatuor planis comprehenſum, erit (f) SEGMENTUM SPHÆROID.

$$+2cxy - \frac{x}{3c}y^3 - \frac{x}{20c^3}y^5 - \frac{x}{56c^5}y^7 - \frac{5x}{576c^7}y^9 - \&c.$$
$$-\frac{cx^3}{3a^2} - \frac{x^3}{18ca^2} - \frac{x^3}{40c^3a^2} - \frac{5x^3}{336c^5a^2} - \&c.$$
$$-\frac{cx^5}{20a^4} - \frac{x^5}{40ca^4} - \frac{3x^5}{160c^3a^4} - \&c.$$
$$-\frac{cx^7}{56a^6} - \frac{5x^7}{336ca^6} - \&c.$$
$$-\frac{5cx^9}{576a^8} - \&c.$$
$$-\&c.$$



Ubi

Propter triangula AKG, ADM ſimilia, $KG : GA = DM : MA = AB : BD$. QUADRATRICIS

Ergo $BD = \frac{AB \times GA}{GK} = \dfrac{x - \frac{x^3}{2a^2} + \frac{x^5}{24a^4} - \frac{x^7}{720a^6} +,}{\frac{x}{a} - \frac{x^3}{6a^3} + \frac{x^5}{120a^5} - \frac{x^7}{5040a^7} +,} = a - \frac{x^2}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5}$.

Hinc area $ABDV = ax - \frac{x^3}{9a} - \frac{x^5}{225a^3} - \frac{2x^7}{6615a^5} -,$

Præterea $\dot{\overline{DB}} = \frac{2x\dot{x}}{3a} + \frac{4x^3\dot{x}}{45a^3} + \frac{12x^5\dot{x}}{945a^5} +,$

Sed $\dot{\overline{AB}} : \dot{\overline{DB}} = DM : MT$. Ergo $MT = \frac{\dot{\overline{DB}} \times DM}{\dot{\overline{AB}}} = \frac{\dot{\overline{DB}} \times x}{\dot{x}}$. Ergo $MT = \frac{2x^2}{3a} + \frac{4x^4}{45a^3} + \frac{12x^6}{945a^5} +$.

Et $MV = a - DB = \frac{x^2}{3a} + \frac{x^4}{45a^3} + \frac{2x^6}{945a^5} +$. Hinc $VT = \frac{x^2}{3a} + \frac{x^4}{15a^3} + \frac{2x^6}{189a^5} +$.

Quadratum ex $\dot{\overline{DB}} = \dot{x}\dot{x} \times \overline{\frac{2x}{3a} + \frac{4x^3}{45a^3} + \frac{12x^5}{945a^5} +}\Big|^2 = \dot{x}\dot{x} \times \frac{4x^2}{9a^2} + \frac{16x^4}{135a^4} +$.

Quare $\dot{\overline{DB}}^2 + \dot{\overline{AB}}^2 = \dot{x}\dot{x} \times \overline{1 + \frac{4x^2}{9a^2} + \frac{16x^4}{135a^4} +}$

Et $\sqrt{\dot{\overline{DB}}^2 + \dot{\overline{AB}}^2} = \dot{x} \times \overline{1 + \frac{2x^2}{9a^2} + \frac{14x^4}{405a^4} +}$

Hujus fluens $= VD = x + \frac{2x^3}{27a^2} + \frac{14x^5}{2025a^4} +$.

(f) FIGURA Sphæroides centro c, circumactu ellipſeos cujus ſemiaxes CB, CE, circa axem AB, generata, ſecari intelligatur quatuor planis. Quorum tribus poſitione datis, ſit quartum mobile. Ex tribus poſitione datis, duo per centrum tranſeant; atque horum alterum quidem per axem AB, quod ſuperficiem ſphæroideos ſecando ellipſin effecerit generanti ſimilem et æqualem. Alterum verò è duobus per centrum, illud ipſum ſit, ſuper quo axis AB ad perpendiculum eſt erectus; quod ſuperficiem ſecando circulum effecerit, illum ipſum utique quem planetæ ſpheroideos, circa axem AB verſatilis, Aſtronomi Æquinoctialem vocarent. Cum hoc verò reliquum ex planis tribus poſitione datis parallelum ſit; ut, ſuperficiem ſecando, illud etiam circulum efficiat, minorem ſcilicet, cum æquinoctiali parallelum. Quartum denique ex iis quibus ſphæroidem ſecari intelligimus, quod mobile poſuimus, eâ quidem lege ſit mobile, ut cum illo, quod per axem AB ductum eſt, ſemper parallelum maneat. Ex quo fiet, ut quocunque loco id conſiſtat, ſuperficiem ſphæroideos ſecando, ellipſin effecerit, magnitudine quidem in aliis à centro figuræ ſphæroideos diſtantiis aliam, ſpecie verò generantis ſemper ſimilem. Hiſce poſitis, magnitudinem ſegmenti SEGMENT. SPHÆROID.

figuræ

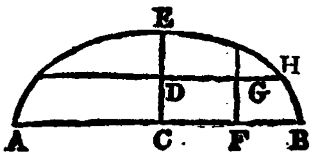
Segmentum Sphæroid.

Ubi numerales coefficientes ſupremorum terminorum ($2, -\frac{1}{3}, -\frac{1}{20}, -\frac{1}{56}, -\frac{5}{576}$, &c.) in infinitum producuntur multiplicando primum

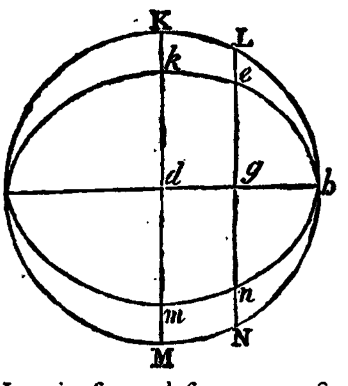
Segmentum

figuræ ſphæroideos, quatuor illis planis intercepti, generaliter definire oportet, pro omni ſitu plani mobilis.

Hunc igitur in finem puta figuram ſphæroidem jam quintùm ſecari, plano quidem per axem AB, quod ſuper illo per axem AB priùs ducto ad perpendiculum erectum ſit. Hoc igitur planum, ſicut omnia quæ per axem AB ducantur, ſuperficiem ſphæroideos ſecando, ellipſin effecerit, generanti ſimilem et æqualem, quam ellipſis quidem AEB lineationis Newtonianæ repreſentat. Idem verò planum circulum æquinoctialem ſecundum rectam CD ſecet; circulum minorem cum æquinoctiali parallelum, ſecundam rectam FG; planum denique mobile ſecundum DG. Itaque rectæ CF, DG, quas duo plana parallela tertium ſecando effecerint, inter ſe parallelæ erunt. (Elem. XI. 16.) Simili ratione duæ CD, FG erunt inter ſe parallelæ. Figura igitur CFGD parallelogramma erit. Angulus autem FCD eſt rectus (per Elem. XI. Def. 3.) Nempe cùm recta CD ſit in plano circuli æquinoctialis, ſuper quo axis AB ad perpendiculum erectus eſt. Quare rectangula eſt figura parallelogramma CFGD. Rectanguli autem CFGD latera CF, DG magnitudine data ſunt, propter rectas CD, FG poſitione datas, quibus illæ CF, DG ad perpendiculum ſunt interpoſitæ. Reliqua verò rectanguli latera, CD, FG magnitudine mutabilia ſunt, nimirum cùm CD, et illi æqualis FG, ſit plani mobilis à ſphæroideos centro mutabilis diſtantia. Rectæ CD, DG ellipſin AEB productæ in punctis E, H, ſecent.



Recta CE ſemiaxis erit ellipſeos AEB, cujus ſemiaxis alter eſt CB, quem recta CE, in centro ſcilicet, ad perpendiculum inſiſtit. Exponatur recta db rectæ DH æqualis. Huic ad perpendiculum apponatur dk, ad quam db rationem habeat quam CB ad CE. Et ſemiaxibus db, dk ſcribatur ellipſis; quæ illi utique ſimilis et æqualis erit, quam planum mobile, ſi ad diſtantiam CD à centro ſphæroideos id conſtiterit, ſuperficiem ſecando, effecerit. Et ſi in db capiatur dg rectæ DG æqualis, et à puncto g educatur ad perpendiculum recta gl, quæ ellipſin bkm in l ſecet; tum ſi producantur kd, lg, uſquedum ellipſin utraque rurſum ſecet, hæc in n, illa in m, area $kmnl$ æqualis erit ellipſeos, quam planum mobile ſuperficiem ſphæroideos ſecando effecerit, ſegmento; illi inquam hujus ellipſeos ſegmento, quod duobus planis parallelis, æquinoctialis circuli alteriuſque minoris, interceptum eſt, iſta area $kmnl$ æqualis erit.



Quare fluxio ſolidi, cujus magnitudo exquirenda eſt, exponetur per ſolidum quoddam, cujus altitudo erit linea recta, ſive data, ſive mutabilis, per quam fluxio rectæ CD exponitur, baſes autem parallelæ, area mutabilis $kmnl$, alteraque huic ipſi ſimilis et æqualis: cujus quidem ſolidi hoc erit ſymbolum algebraicum, $Z \times \dot{y}$; deſignante literâ Z aream mutabilem $kmnl$, literâ y rectam mutabilem CD. Ad fluentis igitur æſtimationem illud opus eſt, ut litera Z in ſeries infinitas reſolvatur à radice y procreatas. Hunc in finem deſignentur rectæ datæ CB, CE, CF, literis a, c, x. Centro d radio db ſcribatur circulus, qui rectis mk, ln, in utramque partem productis, in punctis K, M, L, N occurrat. Propter ellipſin AEB, quadratum ex DH erit ad differentiam quadratorum ex CE, CD, ut quadratum ex C ad quadratum ex CE. Hoc eſt $DH^2 : c^2 - y^2 = a^2 : c^2$. Quare DH, vel db, $= \frac{a}{c}\sqrt{c^2 - y^2}$.

Hinc ſi formula Areæ Circularis, quæ à Newtono in Analyſi per Æquationes Infinitas tradita eſt, (Cap. III. §. 6) cum binario multiplicetur, tum in formulam multiplicatione illâ factam ſi pro a ſubſtituatur $\frac{a}{c}\sqrt{c^2 - y^2}$, veniet formula generalis Areæ Circularis KLMN pro radio mutabili DH, vel db. Nempe

$$\text{KLMN} = \frac{2a}{c}\sqrt{c^2 - y^2}\cdot x - \frac{c}{3a\sqrt{c^2 - y^2}}x^3 - \frac{c^3}{20 a^3 \overline{c^2 - y^2}|^{\frac{3}{2}}}x^5 - \frac{c^5}{56 a^5 \overline{c^2 - y^2}|^{\frac{5}{2}}}x^7 -,$$

Area verò Circularis KLMN ad Aream Ellipticam $klmn$ rationem habet eam, quam db ad dk, ſive eam quam CB ad CE (per Rat. Prim. et Ult. Prop. IV. Cor. H. I.) nempe cùm hæc conſtans ſit rectarum Lg, lg inter ſe ratio, in omni magnitudine abſciſſæ dg. Quare KLMN : Z $= a : c$. Ergo Z =

mum coefficientem a continuò per terminos hujus progreſſionis $-\frac{1\times1}{2\times3}, \frac{1\times3}{4\times5}, \frac{3\times5}{6\times7}, \frac{5\times7}{8\times9}, \frac{7\times9}{10\times11}$, &c. Et numerales coefficientes terminorum SPHÆROID.

$$Z = \frac{c}{a}\,\text{KLMN} = 2\sqrt{c^2-y^2}\,x - \frac{c^2}{3a^2.\sqrt{c^2-y^2}}x^3 - \frac{c^4}{20.a^4.\overline{c^2-y^2}|^{\frac{3}{2}}}x^5 - \frac{c^6}{56a^6.\overline{c^2-y^2}|^{\frac{5}{2}}}x^7 -,$$ SPHÆROID.

Itaque, ſi pro poteſtatibus hujus binomii, c^2-y^2, quæ indicibus $\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -$, deſignantur, ſubrogentur ſeries, in quas poteſtates illæ, per formulam generalem à Newtono in Excerpto Primo traditam, ſunt reſolvendæ; reſolvetur quidem Z in ſeries infinitas à radice y procreatas, vel potiùs à geminâ radice x et y. Etenim licet x datam poſuimus, quo calculi ſcilicet redderentur leviores, reverâ tamen x æquè atque y indefinita cenſenda eſt, cùm utraque pro arbitrio ſumenda ſit. Majoris autem evidentiæ gratiâ calculos ſubduco.

$$\sqrt{c^2-y^2} = c - \frac{y^2}{2c} - \frac{y^4}{8c^3} - \frac{y^6}{16c^5} - \frac{5y^8}{128c^7} -,$$

$$\overline{c^2-y^2}|^{-\frac{1}{2}} = \frac{1}{c} + \frac{y^2}{2c^3} + \frac{3y^4}{8c^5} + \frac{5y^6}{16c^7} + \frac{35y^8}{128c^9} +,$$

$$\overline{c^2-y^2}|^{-\frac{3}{2}} = \frac{1}{c^3} + \frac{3y^2}{2c^5} + \frac{15y^4}{8c^7} + \frac{35y^6}{16c^9} + \frac{315y^8}{128c^{11}}$$

$$\overline{c^2-y^2}|^{-\frac{5}{2}} = \frac{1}{c^5} + \frac{5y^2}{2c^7} + \frac{35y^4}{8c^9} + \frac{105y^6}{16c^{11}} + \frac{1155y^8}{128c^{13}}$$

$$\overline{c^2-y^2}|^{-\frac{7}{2}} = \frac{1}{c^7} + \frac{7y^2}{2c^9} +,$$

Hinc $2\sqrt{c^2-y^2}.\,x = 2cx - \frac{y^2}{c}x - \frac{y^4}{4c^3}x - \frac{y^6}{8c^5}x - \frac{5y^8}{64c^7}x$

$$\frac{c^2}{3a^2\sqrt{c^2-y^2}}x^3 = \frac{c}{3a^2}x^3 + \frac{y^2}{6a^2c}x^3 + \frac{y^4}{8a^2c^3}x^3 + \frac{5y^6}{48c^5a^2}x^3 + \frac{35y8}{384a^2c^7}x^3 +,$$

$$\frac{c^4}{20a^4.\overline{c^2-y^2}|^{\frac{3}{2}}}x^5 = \frac{c}{20.a^4}x^5 + \frac{3y^2}{40a^4c}x^5 + \frac{3y^4}{32.a^4c^3}x^5 + \frac{7y^6}{64a^4c^5}x^5 + \frac{63y^8}{512a^4c^7}x^5 +,$$

$$\frac{c^6}{56a^6.\overline{c^2-y^2}|^{\frac{5}{2}}}x^7 = \frac{c}{56a^6}x^7 + \frac{5y^2}{112a^6c}x^7 + \frac{5y^4}{64a^6c^3}x^7 + \frac{15y^6}{128c^5a^6}x^7 + \frac{165y^8}{1024a^0c^7}x^7 +,$$

$$\frac{5c^8}{576a^8.\overline{c^2-y^2}|^{\frac{7}{2}}}x^9 = \frac{5c}{576a^8}x^9 + \frac{35y^2}{1152a^8c}x^9 +,$$

Horum ordini ſummo inferiorum omnium ordinum ſummas auferendo, conficietur

$$\begin{aligned}
Z = 2cx &- \frac{x}{c}y^2 - \frac{x}{4c^3}y^4 - \frac{x}{8c^5}y^6 - \frac{5x}{64c^7}y^8 - \frac{7x}{128c^9}y^{10} -,\\
-\frac{cx^3}{3a^2} &- \frac{x^3}{6a^2c}y^2 - \frac{x^3}{8a^2c^3}y^4 - \frac{5x^3}{48c^5a^2}y^6 - \frac{35x^3}{384a^2c^7}y^9 -,\\
-\frac{cx^5}{20.a^4} &- \frac{3x^5}{40a^4c}y^2 - \frac{3x^5}{32a^4c^3}y^4 - \frac{7x^5}{64a^4c^5}y^6 -,\\
-\frac{cx^7}{56a^6} &- \frac{5x^7}{112a^6c}y^2 - \frac{5x^7}{64a^6c^3}y^4 -,\\
-\frac{5cx^9}{576a^8} &- \frac{35x^9}{1152a^8c}y^2 -,\\
-\frac{7c.x^{11}}{1408a^{10}} &-,
\end{aligned}$$

Nimirum ordo quiſque eo uſque produci debet, ut in membro cujuſque extremo indices literarum x, et y, ſimul ſumpti, eundem numerum conficiant. Sicut hìc factum eſt. Etenim in membro extremo ordinis ſummi. index literæ x eſt 1, literæ y, 10; qui numeri ſimul ſumpti 11 conficiunt. In membro itidem extremo ordinis proximè inferioris, index literæ x eſt 3, literæ y, 8; atque hi numeri ſimul ſumpti rurſum 11 conficiunt. In membro tertii ordinis extremo index literæ x eſt 5, literæ v, 6; et hi numeri ſimul ſumpti 11 reddunt. Sic in ordinibus etiam inferioribus,

 mem-

EXCERPTUM QUARTUM. terminorum in unâquâque columnâ deſcendentium in infinitum producuntur multiplicando continuò coefficientem ſupremi termini in primâ columnâ per eandem progreſſionem, in ſecundâ autem per terminos hujus $\frac{1\times 1}{2\times 3}$, $\frac{3\times 3}{4\times 5}$, $\frac{5\times 5}{6\times 7}$, $\frac{7\times 7}{8\times 9}$, &c. in tertiâ per terminos hujus $\frac{3\times 1}{2\times 3}$, $\frac{5\times 3}{4\times 5}$, $\frac{7\times 5}{6\times 7}$, $\frac{9\times 7}{8\times 9}$, &c. in quartâ per terminos hujus $\frac{5\times 1}{2\times 3}$, $\frac{7\times 3}{4\times 5}$, $\frac{9\times 5}{6\times 7}$, &c. in quintâ per terminos hujus $\frac{7\times 1}{2\times 3}$, $\frac{9\times 3}{4\times 5}$, $\frac{11\times 5}{6\times 7}$, &c. Et ſic in infinitum.

10. Et eodem modo ſegmenta aliorum ſolidorum deſignari, & valores eorum aliquando commodè per ſeries quaſdem numerales in infinitum produci poſſunt.

11. Ex his videre eſt, quantum fines analyſeos per hujuſmodi infinitas æquationes amplientur: quippe quæ, earum beneficio, ad omnia pæne dixerim problemata (ſi numeralia *Diophanti* & ſimilia excipias) ſeſe extendit.

12. Non tamen omninò univerſalis evadit, niſi per ulteriores quaſdam methodos eliciendi ſeries infinitas. Sunt enim quædam Problemata, in quibus non liceat ad ſeries infinitas, per diviſionem vel

membrorum cujuſque extremorum indices ſimul ſumpti, 7 et 4, 9 et 2, 11 & 0, numerum eundem 11 reddunt. Hoc autem idcirco diligentiùs expoſuimus, quia in formulis hujuſmodi condendis, quæ ex ſeriebus infinitis, numero infinitis, è duplici radice procreatis conflandæ ſunt, idem ſemper tenendum eſt; nempe ut ſeries omnes ſimiliter producantur. Similiter autem produci intelligo, quarum in extremis membris indices radicum duarum, ſimul ſumpti, eundem numerum, vel eandem fortè quantitatem fractam conficiunt.

Itaque quantitate Z in ſeries infinitas, quas modò expoſuimus, reſolutâ, multiplicentur ſeries illæ omnes in fluxionem $\dot{y}$.

Ita veniet

$$\begin{array}{llll}
Z\dot{y} = 2cx & -x & -\frac{x}{4} & -\frac{x^8}{8} \\
-\frac{cx^3}{3a^2} & -\frac{x^3}{6a^2} & -\frac{x^3}{8a^2} & -\frac{5x^3}{48a^2}\, y^6\dot{y} \\
-\frac{cx^5}{20a^4} & -\frac{3x^5}{40a^4} \times \frac{y^2\dot{y}}{c} & -\frac{3x^5}{32a^4} \times \frac{y^4\dot{y}}{c^3} & -\frac{7x^5\dot{y}}{64a^4} \\
-\frac{cx^7}{56a^6} \times \dot{y} & -\frac{5x^7}{112a^6} & -\frac{5x^7}{64a^6} & -, \\
-\frac{5cx^9}{576a^8} & -\frac{35x^9}{1152a^8} & -, & \\
-\frac{7c.x^{11}}{1408a^{10}} & -, & &
\end{array}$$

Cujus

vel extractionem radicum ſimplicium affectarumve, pervenire. Sed quomodo in iſtis caſibus procedendum ſit, jam non vacat dicere; ut neque alia quædam tradere, quæ circa reductionem infinitarum ſerierum in finitas, ubi rei natura tulerit, excogitavi. Nam parciùs ſcribo, quòd hæ ſpeculationes diu mihi faſtidio eſſe cœperint, adeo ut ab iiſdem jam per quinque ferè annos abſtinuerim. CONSTRUCT. MECHANIC.

13. UNUM tamen addam: quòd poſtquam Problema aliquod ad infinitam æquationem deducitur, poſſint inde variæ Approximationes, in uſum Mechanicæ, nullo ferè negotio formari; quæ, per alias methodos quæſitæ, multo labore temporiſque diſpendio conſtare ſolent (t).

14. Cujus rei exemplo eſſe poſſunt tractatus *Hugenii* aliorumque de Quadraturâ circuli. Nam, ut ex datâ arcûs chordâ, A, & demidii arcûs chordâ, B, arcum illum proximè aſſequaris; finge arcum illum eſſe z, & circuli radium r; juxtaque ſuperiora erit A (nempe duplum ſinûs dimidii z) $= z - \frac{z^3}{4 \times 6r^2} + \frac{z^5}{4 \times 4 \times 120r^4} -$ &c (u). Et $B = \frac{1}{2}z - \frac{z^3}{2 \times 16 \times 6r^2} + \frac{z^5}{2 \times 16 \times 16 \times 120r^4} -$ &c (x). Duc jam B in numerum fictitium n, & à producto aufer A, & reſidui ſecundum terminum (nempe $- \frac{nz^3}{2 \times 16 \times 6r^2} + \frac{z^3}{4 \times 6r^2}$,) eo ut evaneſcat, pone $= 0$; indeque emerget $n = 8$, et erit $8B - A = 3z * - \frac{3z^5}{64 \times 120r^4} +$ &c. hoc eſt $\frac{8B - A}{3} = z$; errore tantùm exiſtente $\frac{z^5}{7680r^4} -$ &c. in exceſſu (y). Quod eſt Theorema *Hugenianum* (z).

Inſuper,

Cujus fluens per formulam exponetur ex ſeriebus infinitis, numero infinitis, conflatam, qualis à Newtono allata eſt.

(t) Confer Geom. Analyt. C. XI. Sect. 1.

(u) In formulam utique generalem ſinûs ex arcu ſuprà § 2. traditam ſi pro illis z, z^3, z^5, z^7, hæc $\frac{1}{2}z$, $\frac{z^3}{8}$, $\frac{z^5}{32}$, $\frac{z^7}{128}$ ordine ſubſtituantur, veniet ſinûs dimidii arcûs formula generalis; $\frac{1}{2}z - \frac{z^3}{8 \times 6r^2} + \frac{z^5}{2 \times 16 \times 120r^4} -$; cujus membra omnia cum binario ſi multiplicaveris, veniet $z - \frac{z^3}{4 \times 6r^2} + \frac{z^5}{4 \times 4 \times 120r^4} -$, $=$ A, atque hæc Chordæ quidem ex arcu Formula generalis erit.

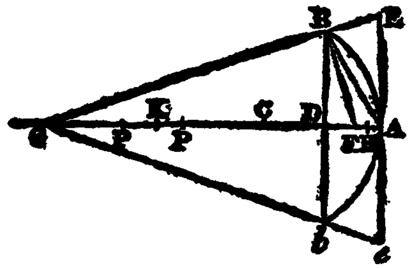
(x) Hanc æquationem ita obtinebis, ſi in formulam generalem chordæ ex arcu modò inventam, pro illis z, z^3, z^5, z^7, hæc $\frac{z}{2}$, $\frac{z^3}{8}$, $\frac{z^5}{32}$, $\frac{z^7}{128}$ ordine ſubſtitueris.

(y) In defectu potiùs; nempe rectæ eo modo inventæ arcubus minores ſunt: ſicut rectè ſtatuit Hugenius.

(z) Hugen. de Circuli Magnitudine inventâ, Prop. XII.

 (aa) Per

EXCERPTUM QUARTUM.



15. Inſuper, ſi in arcûs, Bb, ſagittâ AD indefinitè productâ, quæratur punctum G, à quo actæ rectæ, GB, Gb, abſcindant tangentem Ee quàm proximè æqualem arcui iſti: eſto circuli centrum C, diameter AK$=d$, & ſagitta AD$=x$: et erit DB $(=\sqrt{dx-x^2}) = d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16d^{\frac{5}{2}}}$ − &c. Et AE (=AB)$=d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}}$ + &c [aa]. Et AE − DB : AD :: AE : AG. Quare AG $= \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d}$ − vel + &c [bb]. Finge ergo AG$=\frac{3}{2}d-\frac{1}{5}x$; et viciſſim erit DG $(\frac{3}{2}d-\frac{6}{5}x)$: DB :: DA : AE − DB. Quare AE − DB $= \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{23x^{\frac{7}{2}}}{300d^{\frac{5}{2}}}$ + &c. Adde DB; & prodit AE$=d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{17x^{\frac{7}{2}}}{1200d^{\frac{5}{2}}}$ + &c. Hoc aufer de valore ipſius AE ſuprà habito, & reſtabit error $\frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}}$ + vel − &c. Quare in AG, cape AH quintam partem DA, et KG=HC, et actæ GBE, Gbe abſcindent tangentem Ee quàm proximè æqualem arçui BAb; errore tantùm exiſtente

(aa) Per formulam ſuprà traditam § 1. arcûs ex ſinu verſo. Quòd ſi huic æquationi prior dematur, pars ſcilicet parti, hæc tertia veniet AE − DB $= \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{3x^{\frac{7}{2}}}{28d^{\frac{5}{2}}} +$,

(bb) NEMPE ex hâc analogiâ AE − DB:AD = AE:AG, efficitur AG $= \frac{\text{AD} \times \text{AE}}{\text{AE} - \text{DB}}$.

Sed AD × AE = x × AE $= x \times d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} +$,

Et AE − DB $= \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{3x^{\frac{7}{2}}}{28d^{\frac{5}{2}}} +$,

Qua-e AG $= \left\{ \frac{x \times d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} +,}{\frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} +,} \right. = \frac{d + \frac{x}{6} + \frac{3x^2}{40d} + \frac{5x^3}{112d^2} +,}{\frac{2}{3} + \frac{x}{5d} + \frac{3x^2}{28d} +,}$

Unde diviſione peractâ veniet AG $= \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d}$ − vel +,

(cc) Nimirum ſi capiatur AG $= \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d}$, ablatâ AK $= d$, relinquetur KG $= \frac{1}{2}d - \frac{1}{5}x - \frac{12x^2}{175d} =$ CH $- \frac{12x^2}{175d}$. Sed $\frac{12x^2}{175d} = \frac{3}{5}x \times \frac{4}{5}x \times \frac{1}{7d} = \frac{3\text{AH} \times \text{DH}}{7\text{AK}}$.

(dd) Analyſ. per Æquationes numero terminorum Infinitas. Cap. III. § 7.

(ee) Immò errore hujus ferè duplo, id quod ex ſequenti computatione manifeſtum fit.

$\frac{2}{3}$ AB

iſtente $\frac{16x^3}{525d^3}\sqrt{dx}$ + vel – &c. multo minore ſcilicet quàm in Theoremate *Hugenii*. Quòd ſi fiat 7AK : 3AH :: DH : n; et capiatur KG = CH – n, erit error adhuc multo minor (cc). CONSTRUCT. MECHANIC.

16. Atque ita, ſi Circuli ſegmentum aliquod BAb per Mechanicam deſignandum eſſet: primò reducerem aream iſtam in infinitam ſeriem, puta hanc; BbA $=\frac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36d^{\frac{5}{2}}}$ – &c (dd). Dein quærerem conſtructiones Mechanicas, quibus hanc ſeriem proximè aſſequerer: cujuſmodi ſunt hæ. Age rectam AB, et erit ſegmentum BbA $= \overline{\frac{2}{3}\text{AB}+\text{BD}} \times \frac{4}{5}$AD proximè; exiſtente ſcilicet errore tantùm $\frac{x^3}{70d^2}\sqrt{dx}$ + &c. in defectu (ee): vel proximiùs, erit ſegmentum illud (biſecto AD in F, & rectâ actâ BF) $= \frac{4\text{BF}+\text{AB}}{15} \times 4\text{AD}$; exiſtente errore ſolummodo $\frac{x^3}{560d^2}\sqrt{dx}$ + (ff) &c. qui ſemper minor erit quàm $\frac{1}{1500}$ totius ſegmenti, etiamſi ſegmentum illud ad uſque ſemicirculum augeatur.

17. Sic et in Ellipſi BAb, [*vid. fig. præcedent.*] cujus vertex A, axis alteruter AK, & latus rectum AP; cape PG $= \frac{1}{2}$AP $+ \frac{19\text{AK}-21\text{AP}}{10\text{AK}} \times$ AD. In Hyperbolâ verò, cape PG $= \frac{1}{2}$AP $+ \frac{19\text{AK}+21\text{AP}}{10\text{AK}} \times$ AD (gg). Et acta.

$$\tfrac{2}{3}\text{AB} = \tfrac{2}{3}d^{\frac{1}{2}}x^{\frac{1}{2}}$$
$$\text{BD} = d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} -,$$
$$\tfrac{2}{3}\text{AB}+\text{BD} = \tfrac{5}{3}d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} -,$$
$$\tfrac{4}{5}\text{AD} = \tfrac{4}{5}x$$
$$\overline{\tfrac{2}{3}\text{AB}+\text{BD}} \times \tfrac{4}{5}\text{AD} = \tfrac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{10d^{\frac{3}{2}}} -,$$
$$\text{Area B}b\text{A} = \tfrac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}}$$

Differentia $= \left(\frac{1}{10} - \frac{1}{14}\right)\frac{x^{\frac{7}{2}}}{d^{\frac{3}{2}}} = \frac{2}{70}\frac{x^{\frac{7}{2}}}{d^{\frac{3}{2}}}$

(ff) Errore rurſum duplo

Nempe $\text{BF}^2 = \text{AB}^2 - \text{AF}^2 - \overline{2\text{DF}\times\text{FA}} = \text{AB}^2 - 3\text{AF}^2 = dx - \frac{3}{4}x^2$

$$\text{BF} = \sqrt{dx - \tfrac{3}{4}x^2} = d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{3x^{\frac{3}{2}}}{8d^{\frac{1}{2}}} - \frac{9x^{\frac{5}{2}}}{128d^{\frac{3}{2}}} - \frac{27x^{\frac{7}{2}}}{1024d^{\frac{5}{2}}} -,$$
$$4\text{BF} = 4d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{3x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{9x^{\frac{5}{2}}}{32d^{\frac{3}{2}}} - \frac{27x^{\frac{7}{2}}}{256d^{\frac{5}{2}}} -,$$
$$\text{AB} = d^{\frac{1}{2}}x^{\frac{1}{2}}$$
$$4\text{BF}+\text{AB} = 5d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{3x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{9x^{\frac{5}{2}}}{32d^{\frac{3}{2}}} - \frac{27x^{\frac{7}{2}}}{256d^{\frac{5}{2}}} -,$$
$$\frac{4\text{AD}}{15} = \frac{4x}{15}$$
$$\frac{4\text{BF}+\text{AB}}{15} \times 4\text{AD} = \tfrac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{3x^{\frac{7}{2}}}{40d^{\frac{3}{2}}} - \frac{9x^{\frac{9}{2}}}{64\times 5d^{\frac{5}{2}}}$$

4

Area

EXCERPTUM QUARTUM. acta recta GBE abſcindet tangentem AE quàm proximè æqualem Arcui Elliptico vel Hyperbolico AB, dummodo arcus ille non ſit nimis magnus (hh).

Et

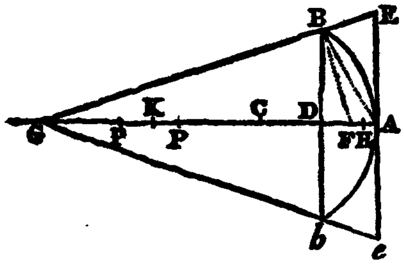
$$\text{Area } BbA = \tfrac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} -$$

$$\text{Error} = \overline{\tfrac{3}{40} - \tfrac{1}{14}}\,\frac{x^{\frac{7}{2}}}{d^{\frac{3}{2}}} = \frac{x^3}{280d^2}\sqrt{dx}$$

2. CÆTERUM viam inveſtigandi generalem, quâ hæc et hiſce ſimilia aſſequi liceat, et errores etiam, quantùm velis, imminuere, Newtonus ipſe indicavit Geometriæ Analyticæ Cap. XI. § 1.

(gg) In Parabolâ capiatur $PG = \frac{1}{2}AP + \frac{19}{12}AD$.

(hh) NEMPE poſito Problemate quàm maximè generali, ut Curvâ quâlibet AB poſitione datâ, inveniatur in axe ejus punctum G, unde ſi recta GB ad punctum quodlibet in curvâ B educatur, producta illa à rectâ AE, quæ curvam in axis vertice contingat, auferat partem AE arcui AB quàm proximè æqualem; hoc inquam Problemate ita generaliter propoſito, quæcunque demum ſit Curva AB, modò talis ſit ut arcûs ejus AB longitudo ex ſinu ejus, vel recto quem vocant vel verſo, per ſeriem infinitam exponi poſſit, eadem ſervanda erit inveſtigandi ratio, quam Newtonus uſurpabat, cùm de arcu Circulari ageretur.



Nimirum factum puta id quod efficiendum eſt. Sit G punctum illud quod quærimus; junctaque GB tangenti AE in E occurrat, et abſcindat AE arcui AB æqualem. A puncto B in axem AG ad perpendiculum deducatur recta BD. Jam cùm arcûs AB longitudo ex utrâque rectarum DB, vel DA, puta ex DA, per ſeriem infinitam data ſit, id enim ponimus; per eandem ſeriem infinitam data erit rectæ AE longitudo. Sed ex curvæ naturâ data erit recta DB ex illâ DA. Symbolum igitur rectæ DB in ſeriem infinitam, ſi opus ſit, reſolutum auferatur ſeriei, quâ arcus AB exponitur. Relinquetur nova ſeries; cujus, ſi ea infinitè producatur, ſumma arcûs AB rectæque DB, hoc eſt rectarum AE, DB, differentiæ ultimò erit æqualis.

Jam ex analogiâ illâ AE — DB : AD = DB : GD, quæ Curvas pariter apud omnes viget, utpote quæ ex triangulorum GDB, GAE inter ipſa ſimilitudine, non ex peculiari curvæ alicujus naturâ ortum ducat; ex hâc igitur analogiâ eliciatur ſeries, quæ rectæ DG longitudinem exponat. Nempe hoc modo. Quantitas Algebraica, quâ rectæ DB longitudo exhibetur, dividatur à ſerie illâ cujus inventionem modò expoſuimus, quæ rectarum AE, DB differentiam exhibeat. In ſerie per diviſionem factâ, indices poteſtatum ejus literæ, quâ recta AD deſignata fuerit, unitate augeantur. Ita veniet nova ſeries; cujus, ſi ea infinitè producatur, ſumma rectæ DG ultimò fit æqualis. Huic ſi auferatur ſymbolum rectæ DP, ſive $p - x$, veniet tandem ſeries, cujus, infinitè utique productæ, ſumma rectæ PG ultimò æqualis erit. Hujus demum ope Problema abſolvendum erit. Capienda nimirum eſt PG ejus longitudinis, quam membra aliquot ſeriei noviſſimæ, inde ab initio ordine ſumenda, algebraicè conſummata præfinierint. Duo plerumque ſuffecerint, niſi nimia ſit arcûs AB longitudo. Sed quo plura fuerint adhibita, eò veritatem propiùs acceſſeris.

2. HÆC etſi ſatis explicaté dixiſſe mihi videor, majoris tamen evidentiæ gratiâ, poſito curvam Parabolam eſſe primùm, tum Ellipſin, calculos ſubducam.

3. SIT igitur curva AB primùm Parabola, cujus parametro recta AP ſit æqualis. Rectam AP litera p deſignet, rectam verò AD, arcûs AB ſinum verſum, litera x.

Arcus igitur AB, ſive recta AE, $= p^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{2x^{\frac{3}{2}}}{3p^{\frac{1}{2}}} - \frac{2x^{\frac{5}{2}}}{5p^{\frac{3}{2}}} + \frac{4x^{\frac{7}{2}}}{7p^{\frac{5}{2}}} - \frac{10x^{\frac{9}{2}}}{p^{\frac{7}{2}}} -$,

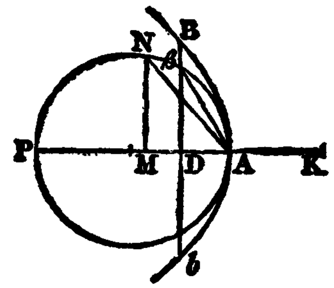
ſecundum formulam ſuprà traditam: vide not. q. § 1.

Et è naturâ parabolæ, $DB = p^{\frac{1}{2}}x^{\frac{1}{2}}$

$$\text{Ergo } AE - DB = \frac{2x^{\frac{3}{2}}}{3p^{\frac{1}{2}}} - \frac{2x^{\frac{5}{2}}}{5p^{\frac{3}{2}}} + \frac{4x^{\frac{7}{2}}}{7p^{\frac{5}{2}}} - \frac{10x^{\frac{9}{2}}}{p^{\frac{7}{2}}}$$

$$\text{Ergo } \frac{DB}{AE - DB} = \frac{p^{\frac{1}{2}}x^{\frac{1}{2}}}{\frac{2x^{\frac{3}{2}}}{3p^{\frac{1}{2}}} - \frac{2x^{\frac{5}{2}}}{5p^{\frac{3}{2}}} + \frac{4x^{\frac{7}{2}}}{7p^{\frac{5}{2}}} -}, = \frac{p}{\frac{2}{3}x - \frac{2}{5p}x^2 + \frac{4}{7p^2}x^3 -} = \frac{3p}{2x} + \frac{9}{10} - \frac{261}{350p}x$$

Hinc



18. Et pro areâ ſegmenti Hyperbolici $B b A$; in DP cape $MD = \frac{3AD^2}{4AK}$, & ad D & M erige perpendicula $D\beta$, MN occurrentia ſemicirculo ſuper diametro AP deſcripto: eritque $\frac{4AN + A\beta}{15} \times 4AD = B b A$ proximè: vel proximiùs, erit $\frac{21AN + 4A\beta}{75} \times 4AD = B b A$; ſi modò capiatur $DM = \frac{5AD^2}{7AK}$ (ii). CONSTRUCT. MECHANIC.

TRAC-

Hinc $\frac{DB \times AD}{AE - DB}$ $(= DG) = \frac{3p}{2} + \frac{9x}{10} - \frac{261}{350p}x^2 +$,

Sed $DP = \qquad p - x$

Ergo $PG = (DG - DP =)\ \frac{1}{2}p + \frac{19}{10}x - \frac{261}{350p}x^2$

Hinc ſi modicus ſit arcus AB, ſatis erit ſumpſiſſe PG æqualem rectæ $\frac{1}{2}AP + \frac{19}{10}AD$.

4. SIT jam curva AB Ellipſis, cujus axis AD literâ d deſignetur, parametur AP literâ p, literâ x ut priùs arcûs AB ſinum verſum deſignante.

Arcus igitur AB, ſive recta AE, $= p^{\frac{1}{2}} x^{\frac{1}{2}} +$

$$\begin{array}{lll} \frac{2}{3p^{\frac{1}{2}}} & -\frac{2}{5p^{\frac{3}{2}}} & +\frac{4}{7p^{\frac{5}{2}}} \\ -\frac{p^{\frac{1}{2}}}{2d}\,x^{\frac{3}{2}} & +\frac{3}{5p^{\frac{1}{2}}d}\,x^{\frac{5}{2}} & -\frac{9}{7p^{\frac{3}{2}}d}\,x^{\frac{7}{2}} \\ & -\frac{p^{\frac{1}{2}}}{8d^2} & +\frac{23}{28p^{\frac{1}{2}}d^2} \\ & & -\frac{p^{\frac{1}{2}}}{16d^3} \end{array}$$

(per formulam ſuprà traditam §. 5.)

Sed è naturâ Ellipſeos, $DB = \sqrt{\frac{p}{d}dx - x^2} = p^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{p^{\frac{1}{2}}}{2d}x^{\frac{3}{2}} - \frac{p^{\frac{1}{2}}}{8d^2}x^{\frac{5}{2}} - \frac{p^{\frac{1}{2}}}{16d^3}\,x^{\frac{7}{2}}$.

Ergo $AE - DB =$

$$\begin{array}{lll} \frac{2}{3p^{\frac{1}{2}}}x^{\frac{3}{2}} & -\frac{2}{5p^{\frac{3}{2}}} & +\frac{4}{7p^{\frac{5}{2}}} \\ & +\frac{3}{5p^{\frac{1}{2}}d}\;x^{\frac{5}{2}} & -\frac{9}{7p^{\frac{3}{2}}d}\,x^{\frac{7}{2}} \\ & & +\frac{23}{28p^{\frac{1}{2}}d^2} \end{array}$$

$$\text{Et } \frac{DB}{AE - DB} = \frac{p^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{p^{\frac{1}{2}}}{2d}x^{\frac{3}{2}} - \frac{p^{\frac{1}{2}}}{8d^2}x^{\frac{5}{2}} - \frac{p^{\frac{1}{2}}}{16d^3}x^{\frac{7}{2}}}{\begin{array}{lll} \frac{2}{3p^{\frac{1}{2}}}x^{\frac{3}{2}} - \frac{2}{5p^{\frac{3}{2}}} & & -\frac{4}{7p^{\frac{5}{2}}} \\ +\frac{3}{5p^{\frac{1}{2}}d}\,x^{\frac{5}{2}} & & -\frac{9}{7p^{5}d}\,x^{\frac{7}{2}} \\ & & +\frac{23}{28p^{\frac{1}{2}}d^2} \end{array}}$$

Hoc

Hoc eſt $\frac{BD}{AE-DB} = \dfrac{p - \frac{px}{2d} - \frac{px^2}{8d^2} -,}{\begin{array}{l} \frac{2}{3}x - \frac{2}{5p}x^2 + \frac{4}{7p^2}x^3 -, \\ \quad + \frac{3}{5d}x^2 - \frac{9}{7pd}x^3 +, \\ \quad\quad + \frac{23}{28d^2}x^3 -, \end{array}} = \begin{array}{l} \frac{3p}{2x} + \frac{9}{10} - \frac{261}{350p}x \\ \quad - \frac{21p}{10d} + \frac{144}{175d}x \\ \quad\quad - \frac{51p}{350d^2}x \\ \quad\quad +, \end{array}$

$$\text{Ergo } \frac{DB \times AD}{AE-DB} (= GD) = \begin{array}{l} \frac{3p}{2} + \frac{9}{10}x - \frac{261}{350p}x^2 \pm \\ \quad - \frac{21p}{10d}x + \frac{144}{175d}x^2 \pm \\ \quad\quad - \frac{51p}{350d^2}x^2 \pm \end{array}$$

$$DP = p - x \pm$$

$$\text{Ergo } PG = \begin{array}{l} \frac{1}{2}p + \frac{19}{10}x - \frac{261}{350p}x^2 \pm \\ \quad - \frac{21p}{10d}x + \frac{144}{175d}x^2 \pm \\ \quad\quad - \frac{51p}{350d^2}x^2 \pm \\ \quad\quad\quad \pm \end{array}$$

Hinc, ſi modicus ſit arcus AB, ſatis erit ſumpſiſſe $PG = \frac{1}{2}AP + \frac{19AK - 21AP}{10AK}AD$.

(ii) $A\beta = \sqrt{px}$, è naturâ Circuli. $MD = \frac{3x^2}{4d}$ (ex hypotheſi) Ergo $AM = x + \frac{3x^2}{4d}$. $AN^2 = AM \times AP = px + \frac{3px^2}{4d}$. Ergo $AN = \sqrt{px + \frac{3px^2}{4d}} = \frac{1}{2}\sqrt{\frac{p}{d}} \times \sqrt{4dx + 3x^2}$.

Ergo $4AN = 2\sqrt{\frac{p}{d}} \times \sqrt{4dx + 3x^2}$.

Sed $\sqrt{4dx + 3x^2} = 2 d^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{3x^{\frac{3}{2}}}{4d^{\frac{1}{2}}} - \frac{9x^{\frac{5}{2}}}{64d^{\frac{3}{2}}} + \frac{27x^{\frac{7}{2}}}{512d^{\frac{5}{2}}} -,$

Ergo $4AN = 4p^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{3p^{\frac{1}{2}}x^{\frac{3}{2}}}{2d} - \frac{9p^{\frac{1}{2}}x^{\frac{5}{2}}}{32d^2} + \frac{27x^{\frac{7}{2}}}{256d^3} -,$

Et $4AN + A\beta = 5p^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{3p^{\frac{1}{2}}x^{\frac{3}{2}}}{2d} - \frac{9p^{\frac{1}{2}}x^{\frac{5}{2}}}{32d^2} + \frac{27x^{\frac{7}{2}}}{256d^3} -,$

Et $\frac{4.AN + AB}{15} = \frac{1}{3}p^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{p^{\frac{1}{2}}x^{\frac{3}{2}}}{10d} - \frac{3p^{\frac{1}{2}}x^{\frac{5}{2}}}{160d^2} + \frac{9p^{\frac{1}{2}}x^{\frac{7}{2}}}{1280d^3} -,$

Ergo $\frac{4AN + AB}{15} \times 4AD = \frac{4p^{\frac{1}{2}}x^{\frac{3}{2}}}{3} + \frac{2p^{\frac{1}{2}}x^{\frac{5}{2}}}{5d} - \frac{3p^{\frac{1}{2}}x^{\frac{7}{2}}}{40d^2} + \frac{9p^{\frac{1}{2}}x^{\frac{9}{2}}}{320d^3} -,$

Sed area hyperbolica $BbA = \frac{4p^{\frac{1}{2}}x^{\frac{3}{2}}}{3} + \frac{2p^{\frac{1}{2}}x^{\frac{5}{2}}}{5d} - \frac{p^{\frac{1}{2}}x^{\frac{7}{2}}}{14d^2} +,$ (Geometr. Analyt.)

Error $\overline{\frac{3}{40} - \frac{1}{14}}\,\frac{p^{\frac{1}{2}}x^{\frac{7}{2}}}{d^2} = \frac{p^{\frac{1}{2}}x^{\frac{7}{2}}}{280d^2}$

2. Ponatur autem $MD = \frac{5AD^2}{7AK} = \frac{5x^2}{7d}$. Tum $AM = x + \frac{5x^2}{7d}$,

Et $AN^2 = AM \times AP = px + \frac{5px^2}{7d}$. Ergo $AN = \sqrt{px + \frac{5px^2}{7d}} = \sqrt{\frac{p}{7d}} \times \sqrt{7dx + 5x^2}$

Et

Et 21.AN $= 3\sqrt{\frac{7p}{d}} \times \sqrt{7dx+5x^2}$

Sed $\sqrt{7dx+5x^2} = 7^{\frac{1}{2}} d^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{5.x^{\frac{3}{2}}}{2.d^{\frac{1}{2}}7^{\frac{1}{2}}} - \frac{25.x^{\frac{5}{2}}}{56.d^{\frac{3}{2}}7^{\frac{1}{2}}} + \frac{125x^{\frac{7}{2}}}{784.d^{\frac{5}{2}}7^{\frac{1}{2}}} -,$

Ergo 21AN $= 21.p^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{15.p^{\frac{1}{2}} x^{\frac{3}{2}}}{2.d} - \frac{75.p^{\frac{1}{2}} x^{\frac{5}{2}}}{56.d^2} + \frac{375.p^{\frac{1}{2}} x^{\frac{7}{2}}}{784.d^3} -,$

Sed 4Aβ $= 4.p^{\frac{1}{2}} x^{\frac{1}{2}}$

Ergo 21AN+4Aβ $= 25.p^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{15.p^{\frac{1}{2}} x^{\frac{3}{2}}}{2.d} - \frac{75.p^{\frac{1}{2}} x^{\frac{5}{2}}}{56.d^2} + \frac{375\, p^{\frac{1}{2}} x^{\frac{7}{2}}}{784.d^3} -$

Et $\frac{21AN+4A\beta}{75} = \frac{1}{3} p^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{p^{\frac{1}{2}} x^{\frac{3}{2}}}{10.d} - \frac{p^{\frac{1}{2}} x^{\frac{5}{2}}}{56.d^2} + \frac{5.p^{\frac{1}{2}} x^{\frac{7}{2}}}{784.d^3} -,$

Et $\frac{21AN+4A\beta}{75} \times 4AD = \frac{4}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} + \frac{2p^{\frac{1}{2}} x^{\frac{5}{2}}}{5.d} - \frac{p^{\frac{1}{2}} x^{\frac{7}{2}}}{14.d^2} + \frac{5p^{\frac{1}{2}} x^{\frac{9}{2}}}{196d^3} -,$

Sed area hyperbolica BbA $= \frac{4}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} + \frac{2p^{\frac{1}{2}} x^{\frac{5}{2}}}{5d} - \frac{p^{\frac{1}{2}} x^{\frac{7}{2}}}{14d^2} + \frac{p^{\frac{1}{2}} x^{\frac{9}{2}}}{36d^3} -,$

Error $\overline{\frac{1}{36} - \frac{5}{196}}\, \frac{p^{\frac{1}{2}} x^{\frac{9}{2}}}{d^3} = \frac{p^{\frac{1}{2}} x^{\frac{9}{2}}}{441d^3}.$

TRACTATUS

DE

QUADRATURA CURVARUM.

INTRODUCTIO

AD

QUADRATURAM CURVARUM.

QUANTITATES Mathematicas, non ut ex partibus quàm minimis conſtantes, ſed ut motu continuo deſcriptas, hîc conſidero. Lineæ deſcribuntur, ac deſcribendo generantur, non per appoſitionem partium, ſed per motum continuum punctorum; ſuperficies per motum linearum; ſolida per motum ſuperficierum; anguli per rotationem laterum; tempora per fluxum continuum; & ſic in cæteris. Hæ Geneſes in rerum naturâ locum verè habent, & in motu corporum quotidie cernuntur. Et ad hunc modum Veteres, ducendo rectas mobiles in longitudinem rectarum immobilium, geneſin docuerunt rectangulorum.

2. Conſiderando igitur, quòd quantitates æqualibus temporibus creſcentes & creſcendo genitæ, pro velocitate majori vel minori quâ creſcunt ac generantur, evadunt majores vel minores; methodum quærebam, determinandi quantitates, ex velocitatibus motuum vel incrementorum, quibus generantur; & has motuum vel incrementorum velocitates nominando *Fluxiones*, & quantitates genitas nominando *Fluentes*, incidi paulatim *annis* 1665 & 1666 in Methodum Fluxionum, quâ hic uſus ſum in Quadraturâ Curvarum.

3. Fluxiones ſunt, quàm proximè, ut fluentium augmenta æqualibus temporis particulis quàm minimis genita, &, ut accurate loquar, ſunt in primâ ratione augmentorum naſcentium; exponi autem poſſunt per lineas quaſcunque, quæ ſunt ipſis proportionales ([a]).

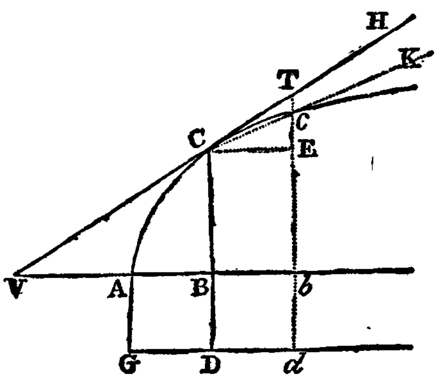
4. Ut ſi areæ ABC, ABDG ordinatis BC, BD, ſuper baſi AB uniformi cum motu progredientibus, deſcribantur; harum arearum

([a]) Vide Noſtram Fluxionum Geometriam Def. 4 & 5. & Prop. 1.

fluxiones

INTRODUCTIO.

fluxiones erunt inter ſe ut ordinatæ deſcribentes BC & BD, & per ordinatas illas exponi poſſunt, propterea quòd ordinatæ illæ ſunt ut arearum augmenta naſcentia.



Progrediatur ordinata BC de loco ſuo BC in locum quemvis novum *bc*. Compleatur parallelogrammum BCE*b*, ac ducatur recta VTH, quæ curvam tangat in C, ipſiſque *bc* & BA productis occurrat in T & V: & abſciſſæ AB, ordinatæ BC, & lineæ curvæ AC*c* augmenta modò genita, erunt B*b*, E*c* & C*c*; & in horum augmentorum naſcentium ratione primâ ſunt latera trianguli CET, ideoque fluxiones ipſarum AB, BC & AC ſunt ut trianguli illius CET latera CE, ET & CT, & per eadem latera exponi poſſunt; vel, quòd perinde eſt, per latera trianguli conſimilis VBC.

5. Eodem recidit, ſi ſumantur fluxiones in ultimâ ratione partium evaneſcentium. Agatur recta C*c*, & producatur eadem ad K. Redeat ordinata *bc* in locum ſuum priorem BC, & coeuntibus punctis C & *c*, recta CK coincidet cum tangente CH ([b]), & triangulum evaneſcens CE*c*, in ultimâ ſuâ forma, evadet ſimile triangulo CET, & ejus latera evaneſcentia CE, E*c* & C*c* erunt ultimò inter ſe, ut ſunt trianguli alterius CET latera CE, ET & CT; & propterea in hâc ratione ſunt fluxiones linearum AB, BC & AC. Si puncta C & *c* parvo quovis intervallo ab invicem diſtant, recta CK parvo intervallo à tangente CH diſtabit. Ut recta CK cum tangente CH coincidat, & rationes ultimæ linearum CE, E*c* & C*c* inveniantur, debent puncta C & *c* coire, & omninò coincidere. Errores quàm minimi in rebus mathematicis non ſunt contemnendi.

6. Simili argumento ſi Circulus, centro B radio BC deſcriptus, in longitudinem abſciſſæ AB ad angulos rectos, uniformi cum motu, ducatur; fluxio ſolidi geniti ABC erit ut circulus ille generans, & fluxio ſuperficiei ejus erit ut perimeter circuli illius & fluxio lineæ curvæ AC conjunctim. Nam quo tempore ſolidum

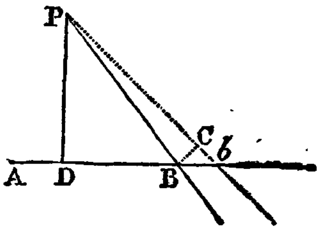
([b]) Per Lemma 6m de Method. Rat. Prim. et Ult.

ABC

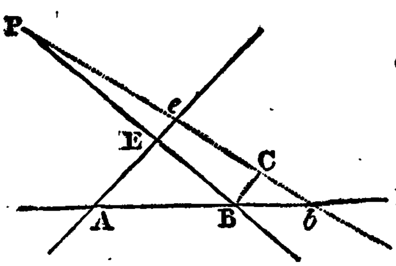
ABC generatur, ducendo circulum illum in longitudinem abſciſſæ AB, eodem ſuperficies ejus generatur, ducendo perimetrum circuli illius in longitudinem curvæ AC. Hujus methodi accipe etiam exempla quæ ſequuntur.

7. *Recta* PB, *circa polum datum* P *revolvens, ſecet aliam poſitione datam rectam* AB : *quæritur proportio fluxionum rectarum illarum* AB & PB.

Progrediatur recta PB de loco ſuo PB in locum novum P*b*. In P*b* capiatur PC ipſi PB æqualis, & ad AB ducatur PD ſic, ut angulus *b*PD æqualis ſit angulo *b*BC ; & ob ſimilitudinem triangulorum *b*BC, *b*PD erit augmentum B*b* ad augmentum C*b*, ut P*b* ad D*b*. Redeat jam P*b* in locum ſuum priorem PB, ut augmenta illa evaneſcant, & evaneſcentium ratio ultima, id eſt ratio ultima P*b* ad D*b*, ea erit, quæ eſt PB ad DB, exiſtente angulo PDB recto ; & propterea in hâc ratione eſt fluxio ipſius AB ad fluxionem ipſius PB.



8. *Recta* PB, *circa datum polum* P *revolvens, ſecet alias duas poſitione datas rectas* AB & AE *in* B & E : *quæritur proportio fluxionum rectarum illarum* AB & AE.



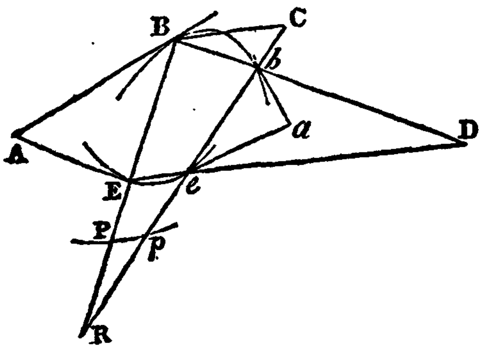
Progrediatur recta revolvens PB de loco ſuo PB in locum novum P*b*, rectas AB, AE in punctis *b* & *e* ſecantem, & rectæ AE parallela BC ducatur, ipſi P*b* occurrens in C ; & erit B*b* ad BC ut A*b* ad A*e*, & BC ad E*e* ut PB ad PE ; &, conjunctis rationibus, B*b* ad E*e* ut A*b*×PB ad A*e*×PE. Redeat jam linea P*b* in locum ſuum priorem PB, & augmentum evaneſcens B*b* erit ad augmentum evaneſcens E*e*, ut AB×PB ad AE×PE ; ideoque in hâc ratione eſt fluxio rectæ AB ad fluxionem rectæ AE.

9. Hinc ſi recta revolvens PB lineas quaſvis curvas, poſitione datas, ſecet in punctis B & E, & rectæ jam mobiles, AB, AE, curvas illas tangant in ſectionum punctis B & E : erit fluxio curvæ, quam recta AB tangit, ad fluxionem curvæ, quam recta AE tangit, ut AB×PB ad AD×PE. Id quod etiam eveniet, ſi recta

PB

PB curvam aliquam, poſitione datam, perpetuò tangat in puncto mobili P ([c]).

FORMULA 10. *Fluat quantitas x uniformiter, & invenienda ſit fluxio quantitatis* x^n. Quo tempore quantitas x fluendo evadit $x+o$, quantitas x^n evadet $\overline{x+o}|^n$; id eſt per methodum ſerierum infinitarum, $x^n + nox^{n-1} + \frac{nn-n}{2} oox^{n-2} +$ &c. Et augmenta, o, & $nox^{n-1} +$ $nn-n$



([c]) IN curvas duas Bb, Ee recta PEB incidat, quæ dum quolibet motu transferatur, extremitate ſuâ P curvam, poſitione datam, Pp perpetuò tangat. Per puncta B, E, ubi recta mobilis PB curvis Bb, Ee, occurrit, ducantur rectæ BA, EA quæ curvas illas contingant. Rectæ BA, EA in A concurrant. Dico fluxionem curvæ Bb eſſe ad fluxionem curvæ Ee, ut rectangulum AB × PB ad rectangulum AE × PE. Rectæ enim PB, ex ſitu PB egreſſæ, ſit alſius quivis ſitus *peb*. Junctæ Bb, Ee in D concurrant; necnon rectæ PB, *pb* in R. Ducatur per B recta BC cum rectâ Ee parallela. Erit igitur Bb ad BC ut Db ad De: et BC ad Ee ut BR ad RE. Ergo Bb erit ad Ee ut rectangulum Db × BR ad rectangulum De × RE. Redeunte autem rectâ *pb* in locum ſuum priorem, PB, anguli DBA, DEA vel imminuti uſque vel uſque creſcentes, ultimò quidem, punctis *b*, B necnon illis *e*, E coeuntibus, aut in nihilum erunt redacti, aut in duos rectos uterque creverint; aut altero denique in nihilum redacto in duos rectos alter creverit. Unde punctum D cum ipſo A ultimò jungetur. Jungetur etiam ultimò punctum R cum ipſo P, lineâ PR infinitè imminutâ. Quare ſubtenſæ evaneſcentis Bb ad evaneſcentem Ee ea erit ratio ultima, quæ rectanguli AB × PB ad rectangulum AE × PE eſt ratio. Sed ſubtenſæ arcubus ſuis evaneſcentibus ultimò ſunt æquales. (Per Lemma VII. De Rat Prim. et Ult.) Arcuum igitur et ſubtenſarum rationes ultimæ eædem erunt. Quare curvarum Bb, Ee incrementa evaneſcentia erunt ultimò inter ſe ut rectangula AB × PB, AE × PE. Ergo et curvarum illarum fluxiones eorundem rectangulorum inter ſe proportionem gerent. Q. E. D.

([d]) POST multas jam olim evulgatas hujus Formulæ demonſtrationes, quarum duas potiſſimùm plurimi facio, Robineſii alteram, alteram ipſius Newtoni, quam, in ſecundo Principiorum Libro Lemmate 2°. à Newtono adumbratim traditam, nos in Prop. VIII. Geometriæ noſtræ Fluxionum ejuſque Corollariis accuratiùs expoſuimus, poſt has inquam aliaſque plurimas jam pridem pervagatas, alia nuper in lucem prodiit, quæ auctorem habuit Simſonum celebrem illum Glaſguenſium Geometram; quæ neſcio ſanè an omnes hactenus excogitatas ſimplicitate vincat. Certè dignam eſſe judico, quæ inter pretioſiſſima Geometriæ hîc reponatur, tanquam in fiſco Newtoni; ita tamen ſi ſermone reddita ſit Newtoniano, quem Simſonus ſine cauſâ, me judice, magnoperè refugiebat. Ex eo autem quòd fluxio poteſtatis quadraticæ, ſive x^2, ſit $2x\dot{x}$, et quòd rectanguli xy fluxio ſit $x\dot{y}+y\dot{x}$, ad hunc modum ratiocinando Formulæ generalis veritatem Vir ſummus evincit.

THEOREMA.

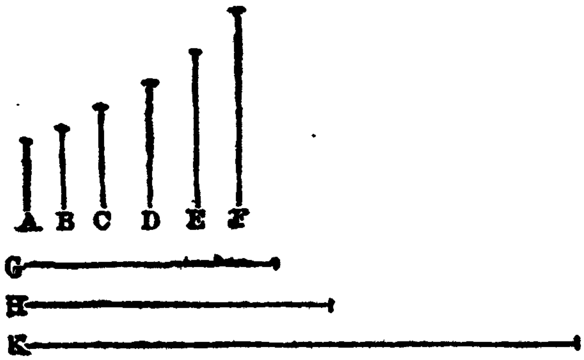
Si ſint rectæ quovis numero proportione convenientes, quarum primâ permanente reliquæ fluant; tum ſi fluxio penultimæ ſit ad fluxionem ſecundæ, ſicut antepenultimæ multiplex aliqua ad primam, et capiatur æquè multiplex penultimæ; fluxio ultimæ erit ad fluxionem ſecundæ ſicut ſumma penultimæ et multiplicis ipſius ad primam.

Sint rectæ quovis numero proportione convenientes A, B, C, D, E, F, quarum primâ A permanente, reliquæ fluant. Harum antepenultimæ D ſit recta G multiplex, et penultimæ E ſit H æquè multiplex; et fluxio rectæ E ad fluxionem ſecundæ B eam rationem habeat quam G ad A. Dico fluxionem ultimæ F ad fluxionem ſecundæ B eam rationem habituram quam ſumma duarum E, H, ad A. Sit enim recta K ad A ut fluxio rectæ F ad fluxionem rectæ B. Cùm igitur $\dot{F}$ ſit ad $\dot{B}$ ut K ad A, rectangula A × $\dot{F}$, K × $\dot{B}$ [*notis* $\dot{F}$, $\dot{B}$ *rectæ intelligantur deſignari, nempe rectarum* F, B *fluxiones*

$\frac{nn-n}{2}oox^{n-2}$ + &c. ſunt ad invicem, ut 1 & $nx^{n-1} + \frac{nn-n}{2}ox^{n-2}$ + &c. Evaneſcant jam augmenta illa, & eorum ratio ultima erit 1 ad nx^{n-1}: ideoque fluxio quantitatis x eſt ad fluxionem quantitas x^n, ut 1 ad nx^{n-1} (d). FLUXIONIS QUANTITATIS x^n.

II. Similibus argumentis, per methodum Rationum Primarum & Ultimarum, colligi poſſunt fluxiones linearum, ſeu rectarum,

ſenſu Def. 5. Geomet. Flux.] erunt inter ſe æqualia. Rectangulum autem $A \times \dot{F}$ fluxio eſt rectanguli $A \times F$, cujus latus F fluit, altero A manente. Et, propter continuam rectarum A, B, C, D, E, F, proportionis convenientiam, rectangula $A \times F$, $B \times E$ ſunt inter ſe æqualia. Horum igitur fluxiones erunt inter ſe æquales. (Geom. Flux. Prop. 11.) Fluxio autem rectanguli BE rectangulorum $B \times \dot{E}$, $E \times \dot{B}$ ſumma erit. Et fluxio rectanguli $A \times F$ eſt ipſum rectangulum $A \times \dot{F}$. Rectangulum igitur $A \times \dot{F}$ duobus $B \times \dot{E}$, $E \times \dot{B}$ ſimul ſumptis eſt æquale. Iiſdem igitur rectangulum $K \times \dot{B}$ æquale erit. Atqui $\dot{B} : \dot{E} = A : G$. (ſic enim poſuimus.) Permutando $\dot{B} : A = \dot{E} : G$. Rectangulum igitur $K \times \dot{B} : K \times A = B \times \dot{E} : B \times G$. Et permutando $K \times \dot{B} : B \times \dot{E} = K \times A : B \times G$. Rurſum $K \times \dot{B} : E \times \dot{B} = K : E = K \times A : A \times E$. Quare rectangulum $K \times \dot{B}$ erit ad rectangula duo $B \times \dot{E}$, $E \times \dot{B}$ ſimul ſumpta, ſicut rectangulum $K \times A$ ad rectangula duo $B \times G$, $A \times E$ ſimul ſumpta (24. 5. Elem.) Sed ex priùs oſtenſis rectangulum $K \times \dot{B}$ duobus $B \times \dot{E}$, $E \times \dot{B}$ ſimul ſumptis eſt æquale. Ergo duobus $B \times G$, $A \times E$ ſimul ſumptis rectangulum $K \times A$ æquale erit. Sed cùm rectæ G, H rectarum D, E æquè ſint multiplices, erit G ad H ut D ad E. Sed D ad E ut A ad B. Ergo G ad H ut A ad B. Rectangula igitur $G \times B$, $H \times A$ erunt inter ſe æqualia. Et rectangulo $A \times E$ communiter addito, duo $G \times B$, $A \times E$ ſimul ſumpta duobus $A \times H$, $A \times E$ ſimul ſumptis æqualia erunt. Sed duobus $G \times B$, $A \times E$ ſimul ſumptis rectangulum $A \times K$ modò oſtenſum eſt æquale. Duobus igitur $A \times H$, $A \times E$ ſimul ſumptis idem $A \times K$ æquale. Ac propterea recta K æqualis erit rectis H, E ſimul ſumptis. Sed recta K ad A ut $\dot{F}$ ad $\dot{B}$. Id enim poſuimus. Quare $\dot{F}$ ad $\dot{B}$ ut ſumma duarum H, E ad A. Q. E. D. DEMONSTRATIO FORMULÆ FLUX. x^n.



HINC ſi litera n numerum deſignet, cujus tot ſunt unitates quot ſunt rectæ præter primam A proportione convenientes, ſi ſit F proportione convenientium ultima, E penultima, fluxio ultimæ F erit ad fluxionem ſecundæ B, ſive $\dot{F}$ ad $\dot{B}$, ut $n.E$ ad A. Nam ob continuam rectarum A, B, C analogiam, $A \times C = B^2$. Ergo $A \times \dot{C} = 2B\dot{B}$. Ergo $\dot{C} : \dot{B} = 2B : A$. Ac proinde, propter ea quæ modò ſunt oſtenſa; erit $\dot{D} : \dot{B} = 2C + C : A$, hoc eſt $\dot{D} : \dot{B} = 3C : A$. Unde rurſum concludetur eſſe $\dot{E} : \dot{B} = 3D + D : A$. Hoc eſt $\dot{E} : \dot{B} = 4D : A$. Eodemque modo progrediendo obtinebitur tandem $\dot{F} : \dot{B} = n.E : A$.

Jam ſi proportione convenientium primam, quam datam ponimus, litera a deſignet, ſecundam x; tertia hoc ſymbolo ſignificanda erit, $\frac{x^2}{a}$; quarta hoc alio, $\frac{x^3}{a^2}$; penultimam ultimamque ſymbola $\frac{x^{n-1}}{a^{n-2}}$, $\frac{x^n}{a^{n-1}}$ deſignabunt. Et fluxio ultimæ erit ad fluxionem ſecundæ, ut $n\frac{x^{n-1}}{a^{n-2}}$ ad a. Hoc eſt ut $n.x^{n-1}$ ad a^{n-1}. Vel ſi arithmeticè omnia æſtimare libeat, ponatur $a = 1$. Et ſi ſecunda proportione convenientium ſit x, erit tertia x^2, quarta x^3 penultima x^{n-1}, ultima x^n. Et fluxio ultimæ x^n erit ad fluxionem ſecundæ x ſicut $n.x^{n-1}$ ad 1. Ac proinde ſi fluxio quantitatis x deſignetur hâc notâ $\dot{x}$, fluxio quantitatis x^n erit $nx^{n-1}\dot{x}$. Q. E. D.

Vide Librum Roberti Simſoni De Limitibus Quantitatum et Rationum Prop. x.

INTRODUCTIO

ſeu curvarum, in caſibus quibuſcunque, ut & fluxiones ſuperficierum, angulorum & aliarum quantitatum. In finitis autem quantitatibus analyſin ſic inſtituere, & finitarum, naſcentium vel evaneſcentium, rationes primas vel ultimas inveſtigare, conſonum eſt Geometriæ Veterum: & volui oſtendere, quòd in methodo Fluxionum non opus ſit figuras infinitè parvas in Geometriam introducere. Peragi tamen poteſt analyſis, in figuris quibuſcunque, ſeu finitis, ſeu infinitè parvis, quæ figuris evaneſcentibus finguntur ſimiles; ut & in figuris quæ per methodos Indiviſibilium pro infinitè parvis haberi ſolent, modò cautè procedas.

12. Ex Fluxionibus invenire Fluentes, Problema difficilius eſt; & ſolutionis primus gradus æquipollet Quadraturæ Curvarum; de quâ ſequentia olim ſcripſi.

DE

QUADRATURA CURVARUM.

NOTATIO FLUXIONALIS.

QUANTITATES indeterminatas ut motu perpetuo creſcentes vel decreſcentes, id eſt, ut fluentes vel defluentes, in ſequentibus conſidero; deſignoque literis x, y, z, v, & earum fluxiones, ſeu celeritates creſcendi, noto iiſdem literis punctatis $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$. Sunt & harum fluxionum fluxiones, ſeu mutationes, magis aut minùs celeres; quas ipſarum z, y, x, v, fluxiones ſecundas nominare licet, & ſic deſignare $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$; & harum fluxiones primas, ſeu ipſarum z, y, x, v fluxiones tertias, ſic $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$; & quartas ſic $\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$, $\ddddot{v}$. Et quemadmodum $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$ ſunt fluxiones quantitatum $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, & hæ ſunt fluxiones quantitatum $\dot{z}$, $\dot{y}$, $\dot{x}$, v, & hæ ſunt fluxiones quantitatum primarum z, y, x, v: ſic hæ quantitates conſiderari poſſunt ut fluxiones aliarum, quas ſic deſignabo, $\overset{\prime}{z}$, $\overset{\prime}{y}$, $\overset{\prime}{x}$, $\overset{\prime}{v}$; & hæ ut fluxiones aliarum $\overset{\prime\prime}{z}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime}{x}$, $\overset{\prime\prime}{v}$; & hæ ut fluxiones aliarum $\overset{\prime\prime\prime}{z}$, $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime\prime}{x}$, $\overset{\prime\prime\prime}{v}$. Deſignant igitur $\overset{\prime\prime}{z}$, $\overset{\prime}{z}$, z, $\dot{z}$, $\ddot{z}$, $\dddot{z}$, $\ddddot{z}$, $\ddddot{z}$, &c. ſeriem quantitatum, quarum quælibet poſterior eſt fluxio præcedentis, & quælibet prior eſt fluens

fluens quantitas fluxionem habens ſubſequentem. Similis eſt ſeries $\overset{||}{\sqrt{az-zz}}$, $\overset{|}{\sqrt{az-zz}}$, $\sqrt{az-zz}$, $\overset{\cdot}{\sqrt{az-zz}}$, $\overset{\cdot\cdot}{\sqrt{az-zz}}$, $\overset{\cdot\cdot\cdot}{\sqrt{az-zz}}$, ut & ſeries $\overset{||}{\frac{az+zz}{a-z}}$, $\overset{|}{\frac{az+zz}{a-z}}$, $\frac{az+zz}{a-z}$, $\overset{\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot\cdot}{\frac{az+zz}{a-z}}$ (c).

Et notandum eſt, quòd quantitas quælibet prior, in his ſeriebus, eſt ut Area figuræ curvilineæ, cujus ordinatim applicata rectangula eſt quantitas poſterior, & abſciſſa eſt z: uti $\overset{|}{\sqrt{az-zz}}$ area curvæ cujus ordinata eſt $\sqrt{az-zz}$ & abſciſſa z. Quò autem ſpectant hæc omnia patebit in propoſitionibus quæ ſequuntur.

PROP. I. PROB. I.

Datâ æquatione quotcunque fluentes quantitates involvente, invenire fluxiones. Prop. I.

Solutio.

Multiplicetur omnis æquationis terminus per indicem dignitatis quantitatis cujuſque fluentis, quam involvit; & in ſingulis multiplicationibus mutetur dignitatis latus in fluxionem ſuam, & aggregatum factorum omnium, ſub propriis ſignis, erit æquatio nova.

Explicatio.

Sunto a, b, c, d, &c. quantitates determinatæ & immutabiles, & proponatur æquatio quævis quantitates fluentes z, y, x, &c. involvens, uti $x^3-xy^2+a^2z-b^3=0$. Multiplicentur termini primò per indices dignitatum x, &c, in ſingulis multiplicationibus, pro dignitatis latere, ſeu x unius dimenſionis, ſcribatur $\dot{x}$, & ſumma factorum erit $3\dot{x}x^2-\dot{x}y^2$. Idem fiat in y, & prodibit $-2xy\dot{y}$. Idem fiat in z & prodibit $aa\dot{z}$. Ponatur ſumma factorum æqualis nihilo, & habebitur æquatio $3\dot{x}x^2-\dot{x}y^2-2xy\dot{y}+a^2\dot{z}=0$. Dico quòd hâc æquatione definitur relatio fluxionum.

Demonſtratio.

Nam ſit o quantitas admodum parva & ſunto $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, quantitatum z, y, x momenta; id eſt, incrementa momentanea ſyn-

(c) Vide noſtram Fluxionum Geometriam Def. 5.

 chrona.

Prop. I.

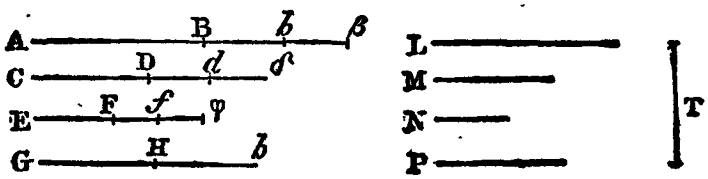
chrona. Et ſi quantitates fluentes jam ſunt z, y & x, hæ, poſt momentum temporis, incrementis ſuis $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ auctæ, evadent $z+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$, quæ in æquatione primâ pro z, y, & x ſcriptæ dant æquationem $x^3+3x^2o\dot{x}+3xo^2\dot{x}\dot{x}+o^3\dot{x}^3-xy^2-o\dot{x}y^2-2xo\dot{y}y-2\dot{x}o^2\dot{y}y-xo^2\dot{y}\dot{y}-\dot{x}o^3\dot{y}\dot{y}+a^2z+a^2o\dot{z}-b^3=0$.

Subducatur æquatio prior, & reſiduum diviſum per o erit $3\dot{x}x^2+3\dot{x}\dot{x}ox+\dot{x}^3o^2-\dot{x}y^2-2x\dot{y}y-2\dot{x}o\dot{y}y-xo\dot{y}\dot{y}-\dot{x}o^2\dot{y}\dot{y}+a^2\dot{z}=0$. Minuatur quantitas o in infinitum, & neglectis terminis evaneſcentibus reſtabit $3\dot{x}x^2-\dot{x}y^2-2x\dot{y}y+a^2\dot{z}=0$. Q. E. D. (f)

Expli-

(f) EXPLICATIO DEMONSTRATIONIS NEWTONIANÆ.

Demonſtratio



Puta rectas, AB, CD, EF, motu punctorum, B, D, F, genitas magnitudines eſſe mutabiles, quas Newtonus literis z, y, x deſignari voluit, literâ a deſignante aliam rectam datæ longitudinis. Hæ rectæ, tempore quovis T, longitudines novas Ab, Cd, Ef ſint adeptæ; ut ſint, Bb, Dd, Ff incrementa rectarum ſimul genita. Sit GH alia recta, æquabili velocitate mutabilis, cui, tempore T, incrementum Hb acceſſerit. Exponatur recta P datæ cujuſvis longitudinis. Sint L, M, N aliæ rectæ, ad quas P rationes habeat, quas velocitas æquabilis puncti H, rectam GH generantis, ad velocitates quibus puncta B, D, F, rectas AB, CD, EF generantia, ex ſuo unumquodque loco B, vel D, vel F diſcedunt. Erunt L, M, N magnitudines, quæ Newtono notis $\dot{z}$, $\dot{y}$, $\dot{x}$ deſignantur (Geom. Flux. Def. v.) Sint Bβ, Dδ, Fφ incrementa quæ rectis, AB, CD, EF, tempore T, acceſſiſſent, ſi puncta B, D, F velocitatibus quibuſcum è locis B, D, F diſcedunt, totum illud tempus T æquabiliter lata eſſent. Jam cùm æquatio propoſita, $x^3-xy^2+a^2z-b^3=0$, quantitatum mutabilium x, y, z relationes mutuas generaliter exprimat, ut quibuſlibet harum magnitudinibus, quæ fluendo ſimul fuerint factæ, pro literis, x, y, z ſubſtitutis, æqualitas maneat; cùm præterea ſint AB, CD, EF magnitudines fluendo ſimul factæ, et Ab, Cd, Ef aliæ ſimul factæ, literis z, y, x ſignificatæ, ſi illas primò, tum has, pro literis z, y, x ſubſtitueris, duæ orientur æquationes

$$\mathrm{EF}^3-\mathrm{EF}\times\mathrm{CD}^2+a^2\mathrm{AB}-b^3=0$$

$$\overline{\mathrm{EF}+\mathrm{F}f}|^3-\overline{\mathrm{EF}+\mathrm{F}f}\times\overline{\mathrm{CD}+\mathrm{D}d}|^2+a^2\times\overline{\mathrm{AB}+\mathrm{B}b}-b^3=0.$$

Harum poſteriori priore detractâ, tertia exſiſtet,

$$3\overline{\mathrm{EF}}|^2.\mathrm{F}f+3\mathrm{EF}.\overline{\mathrm{F}f}|^2+\overline{\mathrm{F}f}|^3-2\mathrm{EF.CD.D}d-\mathrm{EF}.\overline{\mathrm{D}d}|^2-\mathrm{F}f.\overline{\mathrm{CD}}|^2-2\mathrm{CD.F}f.\mathrm{D}d-\mathrm{F}f.\overline{\mathrm{D}d}|^2+a^2\mathrm{B}b=0.$$

Quæ incrementorum ſimul factorum, Bb, Dd, Ff, mutuas relationes ſignificabit; idque generaliter, qualeſcunque fuerint illorum magnitudines. Si igitur tempore T infinitè decreſcente, incrementa illa evaneſcant, evaneſcentium relationes ultimas eadem æquatio ſignificabit. Sed cùm fluentium GH, AB. quarum fluxiones ſunt P, L, incrementa ſimul genita ſint Hb, Bb, evaneſcentium Hb, Bb ultima inter ipſas ratio rectarum P, L ratio erit. (Geometr. Flux. Prop. I.) Sunt autem Hb, Bβ ſpatia eodem tempore T æquabiliter confecta velocitatibus, quæ ſunt inter ſe ut rectæ P, L. Neque huic rationi obſtant ſpatiorum Hb, Bβ longitudines magnæ parvæve fuerint. Quare tempore T infinitè decreſcente, evaneſcentium Hb, Bβ ratio inter ipſas ultima rectarum P, L ratio erit. Evaneſcentis igitur Hb ad evaneſcentes Bb, Bβ eadem erunt rationes ultimæ. Evaneſcentes igitur Bb, Bβ erunt ultimò inter ſe æquales. Simili modo oſtendetur evaneſcentes Dd, Dδ, necnon Ff, Fφ eſſe ultimò inter ſe æquales. Ac propterea evaneſcentium, Bβ, Dδ, Fφ, eædem erunt ac illarum Bb, Dd, Ff relationes ultimæ; et ſi in æquationem noviſſimam pro his Bb, Dd, Ff, illæ Bβ, Dδ, Fφ ſubſtituantur, tranſmutabitur illa in aliam, quæ evaneſcentium Bβ, Dδ, Fφ ultimas inter ipſas relationes exprimet. Nempe hoc modo. $3\overline{\mathrm{EF}}|^2\mathrm{F}\varphi+3.\mathrm{EF}.\overline{\mathrm{F}\varphi}|^2+\mathrm{F}\varphi^3-2\mathrm{EF.CD.D}\delta-\mathrm{EF}.\overline{\mathrm{D}\delta}|^2-\mathrm{F}\varphi.\overline{\mathrm{CD}}|^2-2\mathrm{CD.F}\varphi.\mathrm{D}\delta-\mathrm{F}\varphi.\overline{\mathrm{D}\delta}|^2+a^2\mathrm{B}\beta=0$, ultimò

Jam.

Explicatio plenior.

Ad eundem modum ſi æquatio eſſet $x^3 - xy^2 + a^2\sqrt{ax-yy} - b^3 = 0$, produceretur $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{\overline{\sqrt{ax-yy}}} = 0$. Ubi ſi fluxionem $\dot{\overline{\sqrt{ax-yy}}}$ tollere velis, pone $\sqrt{ax-yy} = z$, & erit $ax - y^2 = z^2$, & (per hanc propoſitionem) $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, ſeu $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$; hoc eſt $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-yy}} = \dot{\overline{\sqrt{ax-yy}}}$. Et inde $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + \frac{a^3\dot{x} - 2a^2\dot{y}y}{2\sqrt{ax-yy}} = 0$.

Et per operationem repetitam pergitur ad fluxiones ſecundas, tertias & ſequentes. Sit æquatio $zy^3 - z^4 + a^4 = 0$; & fiet per operationem primam, $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$: per ſecundam $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2$

Jam cùm $\mathrm{H}b$ ſit ad $\mathrm{B}\beta$ ut P ad L, erit $\mathrm{B}\beta = \frac{\mathrm{H}b \times \mathrm{L}}{\mathrm{P}}$; vel ſi recta $\mathrm{H}b$ per literam o déſignetur, $\mathrm{B}\beta = \frac{o \times \mathrm{L}}{\mathrm{P}}$. Similiter oſtendetur $\mathrm{D}\delta = \frac{o \times \mathrm{M}}{\mathrm{P}}$, necnon $\mathrm{F}\varphi = \frac{o \times \mathrm{N}}{\mathrm{P}}$. Et in omnibus rectarum, $\mathrm{B}\beta$, $\mathrm{D}\delta$, $\mathrm{E}\varphi$, magnitudinibus, dummodo illæ conferantur quæ ſimul genitæ ſint, hæ æqualitates manent. Quare ſi in æquationem relationum ultimarum pro illis $\mathrm{B}\beta$, $\mathrm{D}\delta$, $\mathrm{F}\varphi$, hæ $\frac{o\mathrm{L}}{\mathrm{P}}$, $\frac{o\mathrm{M}}{\mathrm{P}}$, $\frac{o\mathrm{N}}{\mathrm{P}}$ ſubſtituantur, nova exſiſtet hujuſcemodi; $\frac{3\mathrm{EF}^2.\mathrm{N}o}{\mathrm{P}} + \frac{3\mathrm{EF}.\mathrm{N}^2o^2}{\mathrm{P}^2} \pm \frac{\mathrm{N}^3o^3}{\mathrm{P}^3} - 2\mathrm{EF}.\mathrm{CD}.\frac{o.\mathrm{M}}{\mathrm{P}} - \frac{\mathrm{EF}.o^2.\mathrm{M}^2}{\mathrm{P}^2} - \frac{\overline{\mathrm{CD}}^2.o.\mathrm{N}}{\mathrm{P}} - \frac{2\mathrm{CD}.\mathrm{N}.\mathrm{M}.o^2}{\mathrm{P}^2} - \frac{\mathrm{N}.\mathrm{M}^2.o^3}{\mathrm{P}^3} + \frac{a^2\mathrm{L}.o}{\mathrm{P}} = 0$; ultimò ſcilicet. Omnibuſque à quantitate $\frac{o}{\mathrm{P}}$ diviſis, factâque tranſlatione, ut quantitates illæ omnes, quas, peractâ diviſione, litera o non erit ingreſſa, ſeorſum conſiſtant, $3\mathrm{EF}^2.\mathrm{N} - 2\mathrm{EF}.\mathrm{CD}.\mathrm{M} - \overline{\mathrm{CD}}^2.\mathrm{N} + a^2\mathrm{L} = \frac{\mathrm{N}.\mathrm{M}^2.o^2}{\mathrm{P}^2} + \frac{2\mathrm{CD}.\mathrm{N}.\mathrm{M}.o}{\mathrm{P}} + \frac{\mathrm{EF}.\mathrm{M}^2.o}{\mathrm{P}} - \frac{\mathrm{N}^3.o^2}{\mathrm{P}^2} - \frac{3\mathrm{EF}.\mathrm{N}^2.o}{\mathrm{P}}$ ultimò. Hoc eſt, duo illa multinomia hinc illinc conſtituta, ut ſint inæqualia ſi tempus T datæ capiatur magnitudinis, tempore tamen T infinitè decreſcente, ad æqualitatem uſque tendant; quam propiùs tandem acceſſerint quàm pro datâ quâvis differentiâ. Sed tempore T infinitè decreſcente recta o evaneſcit, et unà cum tempore T ad nihilum redigetur. Quare è duobus multinomiis, quæ conſtituunt æquationem noviſſimam, alterum illud, cujus membra ſingula litera o ingreſſa eſt, evaneſcet; ut unà cum tempore T rectâque o in nihilum abierit. Quare et alterum $3\mathrm{EF}^2.\mathrm{N} - 2\mathrm{EF}.\mathrm{CD}.\mathrm{M} - \mathrm{CD}^2.\mathrm{N} + a^2\mathrm{L}$ nihilo æquale erit. Non enim. Sed ſi fieri poſſit, æquale ſit finitæ alicui magnitudini, quam litera Y deſignet. Multinomium igitur alterum, cujus membra ſingula litera o ingreſſa eſt, magnitudinem Y propiùs ultimò acceſſerit quàm pro datâ quâvis differentiâ; ac propterea hanc ipſam Y magnitudinem ſuam habebit ultimam; nec unquam propiùs quàm pro datâ illâ differentiâ Y nihilum acceſſerit. Non igitur unà cum tempore T rectâque o multinomium illud in nihilum abierit. Quod eſt abſurdum. Abierit enim ex priùs demonſtratis.

Multinomium igitur $3\overline{\mathrm{EF}}^2.\mathrm{N} - 2\mathrm{EF}.\mathrm{CD}.\mathrm{M} - \overline{\mathrm{CD}}^2.\mathrm{N} + a^2\mathrm{L}$ nihilo eſt æquale. Hoc eſt, pro rectis CD, EF, literis y, x; et pro L, M, N notis $\dot{z}$, $\dot{y}$, $\dot{x}$, ſubſtitutis, $3x^2\dot{x} - 2xy\dot{y} - y^2\dot{x} + a^2\dot{z} = 0$. Q. E. D.

Ita ni fallor abſolutam dedimus ac perfectam, quam Newtonus adumbrare voluit, demonſtrationem; ab omni certè individuorum vel infinitè parvorum notione liberam.

Cæterùm veritas regulæ de quâ agitur è Corollariis Prop. VIII. Libri noſtri de Geometriâ Fluxionum ejuſdemque Libri Prop. V. ſatis elucefcit.

Prop. I. & II. $+3z\ddot{y}y^2+6z\dot{y}^2y-4\ddot{z}z^3-12\dot{z}^2z^2=0$: per tertiam $\dddot{z}y^3+9\ddot{z}\dot{y}y^2+9\dot{z}\ddot{y}y^2+18\dot{z}\dot{y}^2y+3z\dddot{y}y^2+18z\ddot{y}\dot{y}y+6z\dot{y}^3-4\dddot{z}z^3-36\ddot{z}\dot{z}z^2-24\dot{z}^3z=0$.

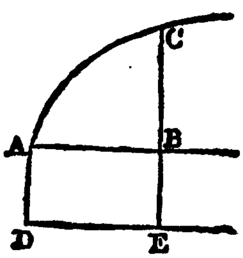
Ubi verò ſic pergitur ad fluxiones ſecundas, tertias & ſequentes, convenit quantitatem aliquam ut uniformiter fluentem conſiderare; & pro ejus fluxione primâ unitatem ſcribere, pro ſecundâ verò & ſequentibus nihil. Sit æquatio $zy^3-z^4+a^4=0$, ut ſuprà; & fluat z uniformiter, ſitque ejus fluxio unitas, & fiet per operationem primam $y^3+3z\dot{y}y^2-4z^3=0$: per ſecundam $6\dot{y}y^2+3z\ddot{y}y^2+6z\dot{y}^2y-12z^2=0$: per tertiam $9\ddot{y}y^2+18\dot{y}^2y+3z\dddot{y}y^2+18z\ddot{y}\dot{y}y+6z\dot{y}^3-24z=0$.

In hujus autem generis æquationibus concipiendum eſt, quòd fluxiones in ſingulis terminis ſint ejuſdem ordinis; id eſt, vel omnes primi ordinis $\dot{y}$, $\dot{z}$; vel omnes ſecundi $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$; vel omnes tertii $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. Et ubi res aliter ſe habet, complendus eſt ordo per ſubintellectas fluxiones quantitatis uniformiter fluentis. Sic æquatio noviſſima complendo ordinem tertium fit $9\dot{z}\ddot{y}y^2+18\dot{z}\dot{y}^2y+3z\dddot{y}y^2+18z\ddot{y}\dot{y}y+6z\dot{y}^3-24z\dot{z}^3=0$ (g).

PROP. II. PROB. II.

Invenire Curvas quæ Quadrari poſſunt.

Sit ABC figura invenienda, BC ordinatim applicata rectangula, & AB abſciſſa. Producatur CB ad E, ut ſit BE = 1, & compleatur parallelogrammum ABED: & arearum ABC, ABED fluxiones erunt ut BC & BE. Aſſumatur igitur æquatio quævis quâ relatio arearum definiatur, & inde dabitur relatio ordinatarum BC & BE per Prop. 1. Q. E. I. (h).



Hujus rei exempla habentur in propoſitionibus duabus ſequentibus (i).

(g) Vide Geomet. Analytic. Cap. IV. § 21. not. *d*.

(h) Hoc Problema Septimum eſt Geometriæ Analyticæ, ubi reſolutionem ejus, explicatiùs longè quàm hic fecit, Newtonus tradidit. Confer etiam Geometr. Analyt. C. IV. § 13.

(i) Et Tironum captui magis accommodata in Geometriâ Analyticâ, poſt Problema ſeptimum.

PROP.

PROP. III. THEOR. I.

Prop. III. & IV.

Si pro abſciſſâ AB, & areâ AE, ſeu AB × I, promiſcuè ſcribatur z; & ſi pro $e + fz^n + gz^{2n} + hz^{3n} +$ &c. ſcribatur R: ſit autem area curvæ $z^\theta R^\lambda$, erit ordinatim applicata BC æqualis

$$\theta e + {\theta \atop +\lambda\eta} \times fz^n + {\theta \atop +2\lambda\eta} \times gz^{2n} + {\theta \atop +3\lambda\eta} \times hz^{3n} + \&c. \times z^{\theta-1} R^{\lambda-1}.$$

Demonſtratio.

Nam ſi ſit $z^\theta R^\lambda = v$, erit (per Prop. I.) $\theta \dot{z} z^{\theta-1} R^\lambda + \lambda z^\theta \dot{R} R^{\lambda-1} = \dot{v}$. Pro R^λ, in primo æquationis termino, & z^θ, in ſecundo, ſcribe $RR^{\lambda-1}$ & $zz^{\theta-1}$; & fiet $\theta \dot{z} R + \lambda z \dot{R}$ in $z^{\theta-1} R^{\lambda-1} = \dot{v}$. Erat autem $R = e + fz^n + gz^{2n} + hz^{3n} +$ &c. et inde (per Prop. I.) fit $\dot{R} = \eta f \dot{z} z^{n-1} + 2\eta g \dot{z} z^{2n-1} + 3\eta h \dot{z} z^{3n-1} +$ &c. quibus ſubſtitutis, & ſcriptâ BE, ſeu I, pro $\dot{z}$, fiet

$$\theta e + {\theta \atop +\lambda\eta} \times fz^n + {\theta \atop +2\lambda\eta} \times gz^{2n} + {\theta \atop +3\lambda\eta} \times hz^{3n} + \&c. \times z^{\theta-1} R^{\lambda-1} = \dot{v} = BC.$$

Q. E. D.

PROP. IV. THEOR. II.

Si Curvæ abſciſſa AB ſit z, & ſi pro $e + fz^n + gz^{2n} +$ &c. ſcribatur R, & pro $k + lz^n + mz^{2n} +$ &c. ſcribatur S; ſit autem Area Curvæ $z^\theta R^\lambda S^\mu$: erit ordinatim applicata BC æqualis

$$\left.\begin{array}{l} \theta ek + {\theta \atop +\lambda\eta} \times fkz^n + {\theta \atop +2\lambda\eta} \times gkz^{2n} \ldots * \ldots\ldots * \ldots \\ + {\theta \atop +\mu\eta} \times elz^n + {\theta \atop {+\lambda\eta \atop +\mu\eta}} \times flz^{2n} + {\theta \atop {+2\lambda\eta \atop +\mu\eta}} \times glz^{3n} \ldots * \ldots \\ + {\theta \atop +2\mu\eta} \times emz^{2n} + {\theta \atop {+\lambda\eta \atop +2\mu\eta}} \times fmz^{3n} + {\theta \atop {+2\lambda\eta \atop +2\mu\eta}} \times gmz^{4n} \end{array}\right\} \times z^{\theta-1} R^{\lambda-1} S^{\mu-1}$$

Demonſtratur ad modum propoſitionis ſuperioris (k).

PROP.

(k) PROP. IV. EXPLICATIUS ENUNTIATA ET DEMONSTRATA.

Si curvæ abſciſſæ AB ſit z, et ſi pro $e + fz^n + gz^{2n} + lz^{3n} + iz^{4n} +$, ſcribatur R, et pro $k + lz^n + mz^{2n} + nz^{3n} + pz^{4n} +$, ſcribatur S, ſitque Area curvæ $z^\theta . R^\lambda . S^\mu$: erit Ordinata BC æqualis rectæ hoc ſymbolo deſignatæ

(ek +

Prop. V.

PROP. V. THEOR. III.

Si Curvæ abſciſſa AB ſit z, & pro $e+fz^n+gz^{2n}+hz^{3n}+$ &c. ſcribatur R: ſit autem ordinatim applicata $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^n+cz^{2n}+dz^{3n}+\text{\&c.}}$ & ponatur $\frac{\theta}{n}=r$, $r+\lambda=s$, $s+\lambda=t$, $t+\lambda=v$, &c.

$$\text{Erit area}=z^\theta R^\lambda \text{ in } +\frac{\frac{1}{n}a}{re}$$

$$+\frac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}z^n$$

$$+\frac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}z^{2n}$$

$$+\frac{\frac{1}{n}d-\overline{s+2}\times fC-\overline{t+1}\times gB-vhA}{\overline{r+3}\times e}z^{3n}$$

$$+\frac{-\overline{s+3}\times fD-\overline{t+2}\times gC-\overline{v+1}\times hB}{\overline{r+4}\times e}z^{4n}$$

$$+\ \text{\&c.}$$

Ubi

Demonſtratio.

$$\left.\begin{array}{lllll}
\theta ek & +\theta\times fkz^n & +\theta\times gkz^{2n} & +\theta\times hkz^{3n} & +\theta\times ikz^{4n}\\
 & +\lambda n & +2\lambda n & +3\lambda n & +4\lambda n\\
 & +\theta\times elz^n & +\theta\times flz^{2n} & +\theta\times glz^{3n} & +\theta\times hlz^{4n}\\
 & +\mu n & +\lambda n & +2\lambda n & +3\lambda n\\
 & & +\mu n & +\mu n & +\mu n\\
 & & +\theta\times emz^{2n} & +\theta\times fmz^{3n} & +\theta\times gmz^{4n}\\
 & & +2\mu n & +\lambda n & +2\lambda n\\
 & & & +2\mu n & +2\mu n\\
 & & & +\theta\times enz^{3n} & +\theta\times fnz^{4n}\\
 & & & +3\mu n & +\lambda n\\
 & & & & +3\mu n\\
 & & & & +\theta\times epz^{4n}\\
 & & & & +4\mu n
\end{array}\right\}\times Z^{\theta-1}.R^{\lambda-1}.S^{\mu-1}.$$

DEMONSTRATIO.

Ponatur $z^\theta.R^\lambda.S^\mu=u$. Erit igitur (Per Prop. I.)

$$\theta z^{\theta-1}\dot{z}R^\lambda S^\mu+\lambda z^\theta.R^{\lambda-1}\dot{R}S^\mu+\mu z^\theta R^\lambda S^{\mu-1}\dot{S}=\dot{u}.$$

Pro R^λ in membro æquationis primo atque tertio, pro z^θ in ſecundo et tertio, pro S^μ in primo et ſecundo, ſcribantur $R.R^{\lambda-1}$, $z.z^{\theta-1}$, $S.S^{\mu-1}$. Ita fiet $\overline{\theta\dot{z}RS+\lambda zS\dot{R}+\mu zR\dot{S}}\times z^{\theta-1}.R^{\lambda-1}.S^{\mu-1}.=\dot{u}$. Erant autem $R=e+fz^n+gz^{2n}+hz^{3n}+iz^{4n}+$, et $S=k+lz^n+mz^{2n}+nz^{3n}+pz^{4n}+$. Quare (per Prop. I.) $\dot{R}=nfz^{n-1}\dot{z}+2ngz^{2n-1}\dot{z}+3nhz^{3n-1}\dot{z}+4niz^{4n-1}\dot{z}+$; et $\dot{S}=nlz^{n-1}\dot{z}+2nmz^{2n-1}\dot{z}+3nnz^{3n-1}\dot{z}+4npz^{4n-1}\dot{z}+$. Ex hiſce verò per formulam generalem noſtram multiplicationis primam, in Logiſticâ Infinitorum traditam, efficietur.

RS =

Ubi A, B, C, D, &c. denotant totas coefficientes datas terminorum ſingulorum in ſerie cum ſignis ſuis + & −, nempe PROP. V.

A primi termini coefficientem $\dfrac{\frac{1}{\eta}a}{re}$

B ſecundi termini coefficientem $\dfrac{\frac{1}{\eta}b - sfA}{\overline{r+1}\times e}$

C tertii termini coefficientem $\dfrac{\frac{1}{\eta}c - \overline{s+1}\times fB - tgA}{\overline{r+2}\times e}$

Et ſic deinceps.

Demonſtratio.

Sunto juxta propoſitionem tertiam.

CURVARUM ORDINATÆ		AREÆ
1. $\theta eA + \overline{\theta+\lambda\eta}\times fAz^{\eta} + \overline{\theta+2\lambda\eta}\times gAz^{2\eta} + \overline{\theta+3\lambda\eta}\times hAz^{3\eta}$ &c.	$\times z^{\theta-1}R^{\lambda-1}$	$Az^{\theta}R^{\lambda}$
2. . . $+\overline{\theta+\eta}\times eBz^{\eta} + \overline{\theta+\eta+\lambda\eta}\times fBz^{2\eta} + \overline{\theta+\eta+2\lambda\eta}\times gBz^{3\eta}$ &c.		$Bz^{\theta+\eta}R^{\lambda}$
3. . . . . $+\overline{\theta+2\eta}\times eCz^{2\eta} + \overline{\theta+2\eta+\lambda\eta}\times fCz^{3\eta}$ &c.		$Cz^{\theta+2\eta}R^{\lambda}$
4. . . . . $+\overline{\theta+3\eta}\times eDz^{3\eta}$ &c.		$Dz^{\theta+3\eta}R^{\lambda}$

Et ſi ſumma ordinatarum ponatur æqualis ordinatæ $a + bz^{\eta} + cz^{2\eta}$

PROP. IVª.

$$RS = ek + \begin{matrix}+el\\+kf\end{matrix}\, z^{\eta} \begin{matrix}+em\\+fl\\+gk\end{matrix}\, z^{2\eta} \begin{matrix}+en\\+fm\\+gl\\+hk\end{matrix}\, z^{3\eta} \begin{matrix}+ep\\+fn\\+gm\\+hl\\+ik\end{matrix}\, z^{4\eta} +,$$

$$\text{Et } \dot{R}S = \eta fk.z^{\eta-1} \begin{matrix}+\eta fl\\+2\eta gk\end{matrix}\, z^{2\eta-1} \begin{matrix}+\eta fm\\+2\eta gl\\+3\eta hk\end{matrix}\, z^{3\eta-1} \begin{matrix}+\eta fn\\+2\eta gm\\+3\eta hl\\+4\eta ik\end{matrix}\, z^{4\eta-1} +, \times \dot{z}$$

$$\text{Et } \dot{S}R = \eta lez^{\eta-1} \begin{matrix}+\eta lf\\+2\eta me\end{matrix}\, z^{2\eta-1} \begin{matrix}+\eta lg\\+2\eta mf\\+3\eta ne\end{matrix}\, z^{3\eta-1} \begin{matrix}+\eta lh\\+2\eta mg\\+3\eta nf\\+4\eta pe\end{matrix}\, z^{4\eta-1} +, \times \dot{z}$$

Et hiſce ſeriebus pro tribus illis RS, ṘS, ṠR, in æquationem modò expoſitam $\overline{\theta\dot{z}RS + \lambda z\dot{R}S + \mu z\dot{S}R} \times z^{\theta-1}.R^{\lambda-1}.S^{\mu-1} = \dot{u}$ ritè ſubſtitutis, efficietur

PROP. V. $cz^{2\eta}+dz^{3\eta}+$ &c. in $z^{\theta-1}R^{\lambda-1}$, ſumma arearum $z^{\theta}R^{\lambda}$ in $A+Bz^{\eta}+Cz^{2\eta}+Dz^{3\eta}+$ &c. æqualis erit areæ curvæ cujus iſta eſt ordinata. Æquentur igitur ordinatarum termini correſpondentes,

et fiet $a=\theta eA$,

$$b=\overline{\theta+\lambda\eta}\times fA+\overline{\theta+\eta}\times eB,$$

$$c=\overline{\theta+2\lambda\eta}\times gA+\overline{\theta+\eta+\lambda\eta}\times fB+\overline{\theta+2\eta}\times eC,$$

&c ([l]).

et inde $A=\frac{a}{\theta e}$

$$B=\frac{b-\overline{\theta+\lambda\eta}\times fA}{\overline{\theta+\eta}\times e}$$

$$C=\frac{c-\overline{\theta+2\lambda\eta}\times gA-\overline{\theta+\eta+\lambda\eta}\times fB}{\overline{\theta+2\eta}\times e}$$

Et ſic deinceps in infinitum ([m]).

Pone jam $\frac{\theta}{\eta}=r$, $r+\lambda=s$, $s+\lambda=t$, &c. et in areâ $z^{\theta}R^{\lambda}$ in $A+Bz^{\eta}+Cz^{2\eta}+Dz^{3\eta}+$ &c. ſcribe ipſorum A, B, C, &c. valores inventos, & prodibit ſeries propoſita. Q. E. D.

1. Et notandum eſt, quòd Ordinata omnis duobus modis in ſeriem reſolvitur. Nam index η vel affirmativus eſſe poteſt vel negativus.

Proponatur

$$\theta\times ek+\left.\begin{array}{l}\theta\times el\\+\theta\times fk\\+\lambda n\times fk\\+\mu n\times le\end{array}\right\} z^{n}+\left.\begin{array}{l}\theta\times em\\+\theta\times fl\\+\theta\times gk\\+\lambda n\times fl\\+2\lambda n\times gk\\+\mu n\times lf\\+2\mu n\times me\end{array}\right\} z^{2n}+\left.\begin{array}{l}\theta\times en\\+\theta\times fm\\+\theta\times gl\\+\theta\times hk\\+\lambda n\times fm\\+2\lambda n\times gl\\+3\lambda n\times hk\\+\mu n\times lg\\+2\mu n\times mf\\+3\mu n\times ne\end{array}\right\} z^{3n}+\left.\begin{array}{l}\theta\times ep\\+\theta\times fn\\+\theta\times gm\\+\theta\times hl\\+\theta\times ik\\+\lambda n\times fn\\+2\lambda n\times gm\\+3\lambda n\times hl\\+4\lambda n\times ik\\+\mu n\times lh\\+2\mu n\times mg\\+3\mu n\times nf\\+4\mu n\times pe\end{array}\right\} z^{4n}+,\times z^{\theta-1}.R^{\lambda-1}.S^{\mu-1}\times\dot{z}$$

$$=\dot{u}=BC\times\dot{z}.$$

Unde omnibus ab illâ $\dot{z}$ diviſis, et ritè ordinatis, veniet ſymbolum rectæ BC, quod initio poſuimus. Q. E. D.

([l]) $d=\overline{\theta+3\lambda n}\times hA+\overline{\theta+n+2\lambda n}\times gB+\overline{\theta+2n+\lambda n}\times fC+\overline{\theta+3n}\times eD.$

([m]) Nimirum $D=\dfrac{\frac{1}{n}d-\overline{s+2}fC-\overline{t+1}gB-uhA}{\overline{r+3}e}.$

Et

Proponatur Ordinata $\frac{3k-lzz}{zz\sqrt{kz-lz^3+mz^4}}$: hæc vel ſic ſcribi poteſt PROP. V.
$z^{-\frac{5}{2}} \times \overline{3k-lz^2} \times \overline{k-lz^2+mz^3}|^{-\frac{1}{2}}$; vel ſic $z^{-2} \times \overline{-l+3kz^{-2}} \times \overline{m-lz^{-1}+kz^{-3}}|^{-\frac{1}{2}}$

In caſu priore eſt $a=3k$, $b=0$, $c=-l$, $e=k$, $f=0$, $g=-l$, $h=m$, $\lambda=\frac{1}{2}$, $\eta=1$, $\theta-1=-\frac{5}{2}$, $\theta=-\frac{3}{2}=r$, $s=-1$, $t=-\frac{1}{2}$, $v=0$.

In poſteriore eſt $a=-l$, $b=0$, $c=3k$, $e=m$, $f=-l$, $g=0$, $h=k$, $\lambda=\frac{1}{2}$, $\eta=-1$, $\theta-1=-2$, $\theta=-1$, $r=1$, $s=1\frac{1}{2}$, $t=2$, $v=2\frac{1}{2}$. Tentandus eſt caſus uterque. Et ſi ſerierum alterutra, ob terminos tandem deficientes, abrumpitur ac terminatur, habebitur area curvæ in terminis finitis. Sic in exempli hujus priore caſu, ſcribendo in ſerie valores ipſorum a, b, c, e, f, g, h, λ, θ, r, s, t, v, termini omnes poſt primum evaneſcunt in infinitum, & area curvæ prodit $-2\sqrt{\frac{k-lz^2+mz^3}{z^3}}$. Et hæc area ob ſignum negativum adjacet abſciſſæ ultra ordinatam productæ. Nam area omnis affirmativa adjacet tam abſciſſæ quàm ordinatæ; negativa verò cadit ad contrarias partes ordinatæ, & adjacet abſciſſæ productæ, manente ſcilicet ſigno ordinatæ. Hoc modo ſeries alterutra, & nonnunquam utraque, ſemper terminatur, & finita evadit, ſi curva

Et ſi multinomio literâ R deſignato accedat novum nomen $iz^{4\eta}$, et multinomio $a+bz^{\eta}+$, novum etiam nomen $\varepsilon z^{4\eta}$, erit

$$\varepsilon = \overline{\theta+4\lambda\eta} \times iA + \overline{\theta+\eta+3\lambda\eta}|hB + \overline{\theta+2\eta+2\lambda\eta}|gC + \overline{\theta+3\eta+\lambda\eta}|fD + \overline{\theta+4\eta}|eE.$$

Et $E = \frac{\frac{1}{\eta}\varepsilon - \overline{s+3}|fD - \overline{t+2}|gC - \overline{u+1}|hB - wiA}{\overline{r+4}|e}$, ſicut in ſuis etiam ad hunc locum commentariis rectè admonuit J. Stewart Abredonenſis.

Newtonus autem in numeratore coefficientis E membrum primum, $\frac{1}{\eta}\varepsilon$, et ultimum, wiA, omiſit.

Quod eo conſilio eum feciſſe arbitror, ut inſinuaret, formulis illis, $e+fz^{\eta}+gz^{2\eta}+$, $a+bz^{\eta}+cz^{2\eta}+$, non ſeries neceſſariò infinitas deſignari, ſed multinomia Algebraica, ſive infinita ſive finita quotvis nominum: et quando illa finita ſint, in areæ formulâ literas illas quæ nominum, illis in quibus formulæ $e+fz^{\eta}+$, $a+bz^{\eta}+$, ſubſiſtant, elatiorum coefficientes deſignarent, nihilo ſingulatim æquandas eſſe. Et cujuſque coefficientis eam quantitatem ſumendam, quæ ex membris reliquis conflatur, quæ literis illis, nihilo æquatis, non ſunt implicata. Ut ſi formulæ illæ $e+fz^{\eta}+$, $a+bz^{\eta}+$, in quarto quæque nomine ſubſiſtant; minimè quidem ex eo conſequetur, Areæ quoque formulam in nomine quarto neceſſariò terminatum iri. Qui etiam evenire poteſt, ut illa uſque proſerpat. Literæ vero ε, i, et aliæ omnes, quibus formulæ utriuſque, $e+fz^{\eta}+$, $a+bz^{\eta}+$, ultra quartum nomen productæ, coefficientes indicarentur, nihilo ſingulatim ſunt æquandæ. Unde coefficiens E formam induet, quali eam Newtonus protulit, membris $\frac{1}{\eta}\varepsilon$, wiA è medio ſublatis. Et nominis proximi coefficiens F erit $\frac{-\overline{s+4}fE - \overline{t+3}gD - \overline{u+2}hC}{\overline{r+5}|e}$.

 geometricè

PROP. V. geometricè quadrari poteſt. At ſi curva talem quadraturam non admittit, ſeries utraque continuabitur in infinitum, & earum altera converget, & aream dabit approximando; præterquam ubi r (propter aream infinitam) vel nihil eſt, vel numerus integer & negativus, vel ubi $\frac{z^\eta}{e}$ æqualis eſt unitati. Si $\frac{z^\eta}{e}$ minor eſt unitate, converget ſeries in quâ index η affirmativus eſt: ſin $\frac{z^\eta}{e}$ unitate major eſt, converget ſeries altera. Si in uno caſu area adjacet abſciſſæ ad uſque ordinatam ductæ, in altero adjacet abſciſſæ ultra ordinatam productæ.

2. Nota inſuper, quòd ſi Ordinata contentum eſt ſub factore rationali

(n) NIMIRUM calculi ita fient ſimpliciſſimi, ſi Ordinata QR^π ad eam formam revocetur, quâ latus partis irrationalis R^π non metiatur partem rationalem. Si igitur R metiatur Q, metiendo faciat A. Erit igitur $Q = AR$; et ordinata QR^π in $AR^{\pi+1}$ migraverit. Si R^2 metiatur Q, metiendo faciat B. Erit $Q = B.R^2$ et ordinata $Q.R^\pi$ in $B.R^{\pi+2}$ migraverit. Denique ſi R^n metiatur Q, metiendo faciat P. Erit $Q = PR^n$ et ordinata $Q.R^\pi$ in $P.R^{\pi+n}$ migraverit.

(o) Viâ in Arithmeticâ Univerſali traditâ. Cap. VII. Sect. 2.

(p) HUJUS demonſtratio hâc nititur propoſitione.

THEOREMA.

DE FLUXIONIBUS FLUENTIUM

Si quantitas fracta rationalis fluat, ejus fluxio quantitas erit fracta; cujus, minimis ſcilicet quantitatibus expoſitæ, denominator vel quadratus erit, vel ex poteſtatibus cubicis, vel ultra-cubicis etiam, cum quadraticis compoſitus.

FLUAT quantitas fracta algebraicè rationalis $\frac{X}{Y}$, hiſce quidem, X, Y, minimis expoſita. Hujus fluxio quantitas erit fracta $\frac{Y\dot{X}-X\dot{Y}}{Y^2}$: quæ, ſi quantitates $Y\dot{X}-X\dot{Y}$, Y^2 inter ſe ſint primæ, his quidem minimis expoſita erit, et denominatorem habet Y^2 quadratum. Quantitatum autem, $Y\dot{X}-X\dot{Y}$ et Y^2, ſi non ſint inter ſe primæ, inveniatur maximus diviſor communis algebraicus, qui literâ Z deſignetur. Quantitas Z quantitates $Y\dot{X}-X\dot{Y}$, Y^2 dividens faciat U, S. Hiſce igitur U, S fluxio quantitatis $\frac{X}{Y}$ minimis exponetur. Nempe hoc modo $\frac{U}{S}$. Dico denominatorem S ex poteſtatibus cubicis, vel etiam ultra-cubicis cum aliis quadraticis, compoſitum eſſe.

Quantitas Z vel prima erit, vel ex primis compoſita. Sit primùm prima. Cùm igitur quantitas prima Z quantitatem quadraticam Y^2 metiatur, radicem ejus Y eadem Z metietur (Elem. VII. 32.) Quare Z metietur $Y\dot{X}$. Sed metitur quoque hujus $Y\dot{X}$, aliuſque $X\dot{Y}$ differentiam. Quare et aliam illam $X\dot{Y}$ metitur. Sed non metietur X. Nam ſi metiatur illam, cùm oſtenſum ſit, eandem Z quantitatem etiam Y metiri, duæ X, Y non erunt inter ſe primæ. Sed primas inter ſe poſuimus. Quantitas igitur prima Z quæ non metitur X, cùm tamen $X\dot{Y}$ metiatur, neceſſariò metietur $\dot{Y}$ (Elem. VII. 32.) Metitur igitur Z illam $\dot{Y}$. Et oſtenſum eſt metiri Y. Ponatur igitur Z^n poteſtas illius Z, elatiſſima earum quæ fluentem Y metiantur. Jam

rationali Q & factore ſurdo irreducibili R^{π}, & factoris ſurdi latus Prop. V.
R non dividit factorem rationalem Q; erit $\lambda - 1 = \pi$, & $R^{\lambda-1} = R^{\pi}$. Sin factoris ſurdi latus R dividit factorem rationalem ſemel, erit $\lambda - 1 = \pi + 1$, & $R^{\lambda-1} = R^{\pi+1}$: ſi dividit bis, erit $\lambda - 1 = \pi + 2$, & $R^{\lambda-1} = R^{\pi+2}$: ſi ter, erit $\lambda - 1 = \pi + 3$, & $R^{\lambda-1} = R^{\pi+3}$: & ſic deinceps (n).

3. Si Ordinata eſt fractio rationalis irreducibilis cum denominatore ex duobus vel pluribus terminis compoſito: reſolvendus eſt denominator in diviſores ſuos omnes primos (o). Et ſi diviſor ſit aliquis cui nullus alius eſt æqualis, curva quadrari nequit (p): ſin

cùm Z^n metiatur fluentem Y, quantitatem quidem quadraticam Y^2 metietur quadratica Z^{2n}, et fluxionem $\dot{Y}$ quantitas Z^{n-1} metietur. Sed oſtenſum eſt Z metiri fluxionem $\dot{Y}$. Quare index ille $n - 1$ non minor erit unitate, neque ſanè major. Non minor: quoniam ſi minor eſſet, quantitas Z^{n-1} non eſſet elatiſſima in poteſtatibus quantitatis Z quæ fluxionem $\dot{Y}$ metirentur: neque igitur Z^n elatiſſima quæ metiretur fluentem Y: quam tamen elatiſſimam poſuimus. Non major: quoniam ſi major eſſet unitate index ille $n - 1$, quantitas prima Z non eſſet duarum $Y\dot{X} - X\dot{Y}$, et Y^2 maximus diviſor communis. Sed maximum poſuimus. Quare $n - 1 = 1$. Ergo $n = 2$. Et $2n = 4$. Quantitas igitur Z^4 metietur quantitatem Y^2. Metiatur, et faciat T. Ut ſit $TZ^4 = Y^2$. Sed $S \times Z = Y^2$. Quare $TZ^4 = S \times Z$, et $TZ^3 = S$. Quantitatis igitur fractæ, $\frac{U}{S}$, denominator, S, compoſitus eſt ex poteſtate cubicâ primæ quantitatis Z, cum aliâ quantitate T. Fractarum Rationalium.

Sed non ſit Z quantitas prima. Ex primis igitur erit compoſita. Reſolvatur igitur Z in diviſores ſui primos α, β, γ. Jam cùm α metiatur Z, et Z metiatur Y^2, idcirco illa α eandem Y^2 metietur. Ex eo autem quòd quantitas prima α quadraticam Y^2 metiatur, efficietur eandem α tum ſimplicem Y tum fluxionem $\dot{Y}$ metiri. Id enim iiſdem planè argumentis obtinendum erit, quibus priùs idem de quantitate Z evicimus, cùm illam primam poſuiſſemus. Et pari ratione alii quantitatis Z diviſores primi, β, γ, fluentem Y et fluxionem $\dot{Y}$ ſingulatim metientur. Quantitatum α, β, γ ponantur α^n, β^p, γ^q poteſtates elatiſſimæ, quæ quantitatem Y ſingulatim metiantur. Fluxionem igitur $\dot{Y}$ poteſtates, α^{n-1}, β^{p-1}, γ^{q-1}, ſingulatim metientur, neque illis ullæ quidem elatiores: quantitatem verò quadraticam Y^2 poteſtates α^{2n}, β^{2p}, γ^{2q} ſingulatim metientur. Jam cùm quantitatem Y metiantur primarum, α, β, γ, poteſtates illæ α^n, β^p, γ^q, ſingulatim, factum quidem ex poteſtatum illarum omnium mutuâ in ſe multiplicatione, $\alpha^n \times \beta^p \times \gamma^q$, eandem Y metietur. (Hoc autem demonſtratione mox firmabimus.) Et factum $\alpha^{n-1} \times \beta^{p-1} \times \gamma^{q-1}$, fluxionem $\dot{Y}$ metietur, nimirum, quam illæ α^{n-1}, β^{p-1}, γ^{q-1} primarum poteſtates ſingulatim. Quantitatem denique quadraticam Y^2 factum $\alpha^{2n} \times \beta^{2p} \times \gamma^{2q}$ metietur. Hinc efficitur quantitatem $\alpha^{n-1} \times \beta^{p-1} \times \gamma^{q-1}$ duarum $Y\dot{X} - X\dot{Y}$ et Y^2 communem eſſe diviſorem; et maximum quidem illum, qui ex poteſtatibus primarum α, β, γ compoſitus communis eſſe poteſt. Major enim ſi quis eſſet, eum ex elatioribus poteſtatibus componi oporteret. Ita verò fluxionem $\dot{Y}$ poteſtates quantitatum α, β, γ, elatiores quàm ſunt illæ α^{n-1}, β^{p-1}, γ^{q-1} ſingulatim metirentur. Cujus contrarium oſtendimus. Quantitatum igitur $\dot{Y}X - X\dot{Y}$, Y^2 erit illa $\alpha^{n-1} \times \beta^{p-1} \times \gamma^{q-1}$ diviſor communis maximus qui ex poteſtatibus illarum α, β, γ componi poterit. Eſt verò Z duarum $\dot{Y}X - X\dot{Y}$, Y^2 diviſor maximus communis, et ex illarum α, β, γ poteſtatibus totus eſt compoſitus.

Quare Z facto illi $\alpha^{n-1} \times \beta^{p-1} \times \gamma^{q-1}$ æqualis eſt. Sed cùm Z dividens Y^2 fecerit S, erit

$SZ =$

PROP. V. ſin duo vel plures ſint diviſores æquales, rejiciendus eſt eorum unus; & ſi adhuc alii duo vel plures ſint ſibi mutuò æquales, & prioribus

DE FLUXIONIBUS FLUENTIUM

$sz = Y^2$, hoc eſt $s \times \alpha^{n-1} \times \beta^{p-1} \times \gamma^{q-1} = Y^2$. Sed illam Y^2 hæc $\alpha^{2n} \times \beta^{2p} \times \gamma^{2q}$ metitur (oſtenſum enim.) Metiendo faciat T. Ut ſit $T \times \alpha^{2n} \times \beta^{2p} \times \gamma^{2q} = Y^2$. Ergo $T \times \alpha^{2n} \times \beta^{2p} \times \gamma^{2q} = s \times \alpha^{n-1} \times \beta^{p-1} \times \gamma^{q-1}$. Ac propterea $T \times \alpha^{n+1} \times \beta^{p+1} \times \gamma^{q+1} = s$. Quantitatis igitur fractæ $\frac{\dot{U}}{S}$, quæ fluentis fractæ, $\frac{X}{Y}$, fluxio eſt, minimis expoſita; hujus denominator s compoſitus eſt ex quantitâte quâdam per literam T deſignata, et primarum α, β, γ poteſtatibus $\alpha^{n+1}, \beta^{p+1}, \gamma^{q+1}$; quæ ſanè cubicis haud erunt depreſſiores. Nam cùm illæ $\alpha^{n-1}, \beta^{p-1}, \gamma^{q-1}$ ſint primarum α, β, γ poteſtates elatiſſimæ, quæ fluxionem $\dot{Y}$ metiantur, et oſtenſum ſit ipſas α, β, γ illam $\dot{Y}$ metiri, indicum illorum, $n-1$, $p-1$, $q-1$, nullus quidem unitate minor erit. Illorum igitur $n+1$, $p+1$, $q+1$, nullus ternario minor. Poteſtates igitur $\alpha^{n+1}, \beta^{p+1}, \gamma^{q+1}$ cubicis non erunt depreſſiores.

Quantitatis igitur fractæ $\frac{U}{S}$, quæ facta eſt ex illâ $\frac{Y\dot{X} - X\dot{Y}}{Y^2}$, illas $Y\dot{X} - X\dot{Y}$, Y^2 dividendo à z, quem maximum illæ communem diviſorem habent; hujus inquam quantitatis fractæ, ſic minimis expoſitæ, denominator s, in caſu utroque, ſive z prima fuerit ſive compoſita, totus erit compoſitus ex quantitate quâdam per literam T deſignatâ, et poteſtatibus quibuſdam, ſive ipſius z, ſive primorum ipſius z diviſorum, quæ cubicis non erunt depreſſiores.

JAM verò dico illam T ex poteſtatibus quadraticis totam eſſe compoſitam. Primùm enim, ſi z prima ſit, erit z^2 quantitatis z poteſtas, earum quæ Y metiantur elatiſſima (per ea quæ in caſu primo jam antè oſtenſa ſunt.) Metiatur igitur z^2 illam Y, et faciat t. Ut ſit $z^2 \times t = Y$. Erit igitur $z^4 \times t^2 = Y^2$. Sed poſitum eſt $z^4 \times T = Y^2$. Quare $T = t^2$.

In altero caſu, quando z eſt quantitas compoſita, metiatur $\alpha^n \times \beta^p \times \gamma^q$ fluentem Y et t faciat, Ut ſit $\alpha^n \beta^p \gamma^q \times t = Y$; erit igitur $\alpha^{2n} \beta^{2p} \gamma^{2q} \times t^2 = Y^2$. Sed poſitum erat $T \times \alpha^{2n} \beta^{2p} \gamma^{2q} = Y^2$. Unde $T = t^2$.

In caſu igitur utroque quantitas T quadraticæ t^2 æqualis erit. Unde ſi t prima ſit nihilominus T quadratica erit. Si t non ſit prima, reſolvatur ea in diviſores ſui primos, π, ρ, σ; ut ſit $t = \pi \times \rho \times \sigma$. Erit igitur t^2 ſive T, huic facto $\pi^2 \times \rho^2 \times \sigma^2$ æqualis. Quantitas igitur T ex quadraticis tota componetur; pluribus quidem eis, ſi t ſit compoſita; ſi prima ſit t, ex unâ illâ t^2.

Quantitatis igitur fractæ $\frac{U}{S}$ denominator s, quem oſtendimus ex quantitate T & poteſtatibus cubicis, vel ultra-cubicis, totum eſſe compoſitum, is quidem ex cubicis quibuſdam, vel ultrà-cubicis, et aliis quadraticis totus componetur. Q. E. D.

2. ILLUD modò demonſtratione nobis firmandum eſt, quod aſſumpſimus, factum ex quantitatum primarum poteſtatibus, quæ aliam aliquam ſingulatim metiantur, id eandem metiri.

Sint igitur quantitates duæ primæ α, β quæ aliam A metiantur. Et ſint α^n, β^p illarum α, β poteſtates quæ illam A ſingulatim metiantur. Dico $\alpha^n \times \beta^p$ eandem A metiri. Metiatur enim α^n illam A, et faciat a. Ut ſit $\alpha^n \times a = A$. Jam cùm β^p metiatur A, metitur certè $\alpha^n \times a$. Sed β^p non metitur α^n, ad quam utique prima eſt (Elem. IX 13.) Metitur igitur illam a. Nam ſi duæ quantitates inter ſe multiplicatæ aliam fecerint, factam verò alia aliqua metiatur, ea, ſi ad alteram initio poſitarum prima ſit, alteram metietur. Hoc enim iiſdem planè argumentis oſtendi poſſit, quibus evincitur ſecunda à triceſimâ libri ſeptimi Elementorum. Igitur β^p, cùm illam A metiatur è duabus, α^n, a, factam, et ad α^n prima ſit, metietur alteram a. Sed $\beta^p : a = \alpha^n \beta^p : \alpha^n a$. Quare $\alpha^n \beta^p$ metietur $\alpha^n a$; hoc eſt ipſam A metietur. Q. E. D.

A

α β

a

Sint jam tres primæ quantitates α, β, γ, quarum poteſtates α^n, β^p, γ^q quantitatem A ſingulatim

prioribus inæquales, rejiciendus eſt etiam eorum unus; & ſic in aliis omnibus æqualibus, ſi adhuc plures ſint: deinde diviſor qui relinquitur, vel contentum ſub diviſoribus omnibus qui relinquuntur, PROP. V.

ſingulatim metiantur. Dico $\alpha^n \times \beta^p \times \gamma^q$ eandem metiri. Metiatur α^n quantitatem A et faciat a. Jam β^p metietur a, id enim ut priùs oſtendetur. Sed pari ratione γ^q eandem a metietur. Quare factum etiam $\beta^p \times \gamma^q$ metietur eandem a, per caſum priorem. Sed $\beta^p \times \gamma^q : \text{a} = \alpha^n \times \beta^p \times \gamma^q : \alpha^n \text{a}$. Quare $\alpha^n \times \beta^p \times \gamma^q$ metietur α^na; hoc eſt ipſam A metietur. FRACTARUM RATIONALIUM.

A

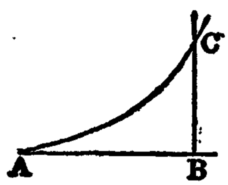
$\alpha \quad \beta \quad \gamma$

a

Et ſimili argumentatione, quotcunque poſitæ fuerint quantitates primæ, veritatem aſſumpti uſque probabimus. Q. E. D.

3. PROPOSITIONE noſtrâ ſic tandem ex omni parte firmatâ, illius ope, veritas ejus, quod à Newtono dictum eſt, manifeſta fiet: Curvam utique, cujus ordinata ſit fracta rationalis, ſi fractæ illius, minimis expoſitæ, denominator in diviſoribus ſui primis ullum habeat, cui alius non inveniatur æqualis, eam curvam non poſſe quadrari: non poſſe utique definitâ quadraturâ, per æquationem Algebraicam, nominibus finitam, exponendâ. Nam per ſeries quidem infinitas nihil obſtat, quo minùs talis etiam curvæ area, propiùs ſaltem quàm pro differentiâ datâ, æſtimari poſſit.

4. SIT enim quantitas fracta rationalis $\frac{\text{v}}{\text{s}}$, his v, s minimis expoſita: ſignificantibus utique literis v, s, multinomia Algebraica rationalia, ex poteſtatibus ſimplicis alicujus quantitatis z compoſita. Denominator autem s in diviſoribus ſui primis habeat aliquem α, cui alius non inveniatur æqualis. Sit curva ABC, cujus abſciſſâ AB per literam z ſignificatâ, ordinata BC quantitati $\frac{\text{v}}{\text{s}}$ ſemper ſit æqualis. Dico hanc curvam, aliter ac per ſeriem infinitam non poſſe quadrari. Quadretur enim, et area ejus dicatur Q. Jam cùm Q area ſit curvæ, cujus ordinata eſt $\frac{\text{v}}{\text{s}}$, abſciſſa z, erit fluxio areæ Q æqualis huic $\frac{\text{v}}{\text{s}}\dot{z}$. Hoc eſt ſi ponatur $\dot{z} = 1$, erit $\dot{Q} = \frac{\text{v}}{\text{s}}$. Quantitas igitur Q aut fracta rationalis erit, aut ſeries infinita, ſiquidem fluxionem fractam rationalem habeat. Nam fluxio rationalis à fluente irrationali, ab unâ fluente ſimplici factâ, haud venerit: neque fracta à non fractâ rationali; niſi forte ex ſerie infinitâ. Sit igitur Q, modò eſſe poſſit, quantitas fracta rationalis. Quantitatis igitur fractæ rationalis Q, hujus eſt illa $\frac{\text{v}}{\text{s}}$, fracta rationalis, fluxio; cujus tamen, minimis quidem illis, v, s, expoſitæ, denominator s, cùm in diviſoribus ſui primis habeat quendam α, cui alius non inveniatur æqualis, certè neque quadratus ille erit, neque ex poteſtatibus cubicis, vel ultra-cubicis aliiſque quadraticis totus quidem compoſitus. Hoc autem abſurdum eſt: pugnat enim cum propoſitione modò oſtenſâ. Non igitur fracta rationalis erit quantitas illa Q. Reſtat ut ſeries infinita ſit Q. Curva igitur ACB, niſi per ſeriem infinitam, non poteſt quadrari. Q. E. D.



5. CÆTERUM fluxiones fractas quaſdam rationales ex ſeriebus infinitis nonnunquam emanare, quod tacitè quidem aſſumpſimus; illud et poſſe fieri, et reverâ ſolere, ſatis ut opinor intelligatur vel hujus exemplo, $\frac{\dot{z}}{1+z}$, quæ ex hac ſerie infinitâ, $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 -$, emanârit. Hujus enim ſeriei fluxio erit, $\dot{z} - z\dot{z} + z^2\dot{z} - z^3\dot{z} + z^4\dot{z} -, = \frac{\dot{z}}{1+z}$. Hanc autem æquationem verè poni, ſi aliunde non ſatis notum ſit, ex eo intelligatur, quòd ſeriem $\dot{z} - z\dot{z} + z^2\dot{z} - z^3\dot{z} + z^4\dot{z} -$, in binomium $1 + z$ multiplicando, illa $\dot{z}$ effecta fuerit; nominibus aliis omnibus ipſâ multiplicationis operâ deletis, ut videre eſt:

PROP. V. linquuntur, ſi plures ſunt, ponendum eſt pro R; & ejus quadrati reciprocum R^{-2} pro $R^{\lambda-1}$, (præterquam ubi contentum illud eſt quadratum vel cubus vel quadrato-quadratum, &c. quo caſu ejus latus ponendum eſt pro R, & poteſtatis index, 2 vel 3 vel 4, negativè ſumptus pro λ,) et Ordinata ad denominatorem R^2 vel R^3 vel R^4 vel R^5 &c. reducenda [q].

4. Ut ſi Ordinata ſit $\frac{z^5+z^4-8z^3}{z^5+z^4-5z^3-z^2+8z-4}$; quoniam hæc fractio irreducibilis eſt, & denominatoris diviſores ſunt pares, nempe $z-1$, $z-1$, $z-1$, & $z+2$, $z+2$; rejicio magnitudinis utriuſque diviſorem unum, & reliquorum $z-1$, $z-1$, $z+2$ contentum, z^3-3z+2, pono pro R; & ejus quadrati reciprocum, $\frac{1}{R^2}$ ſeu R^{-2}, pro $R^{\lambda-1}$. Dein Ordinatam ad denominatorem, R^2 ſeu $R^{1-\lambda}$, reduco, & fit $\frac{z^6-9z^4+8z^3}{\overline{z^3-3z+2}|^2}$, i. e. $z^3\times\overline{8-9z+z^3}\times\overline{2-3z+z^3}|^{-2}$. Et inde eſt $a=8$, $b=-9$, $c=0$, $d=1$, &c. $e=2$, $f=-3$, $g=0$, $h=1$, $\lambda-1=-2$, $\lambda=-1$, $\eta=1$, $\theta-1=3$, $\theta=4=r$, $s=3$, $t=2$, $v=1$. Et his in ſerie ſcriptis prodit area $\frac{z^4}{z^3-3z+2}$, terminis omnibus in totâ ſerie poſt primum evaneſcentibus.

5. Si denique Ordinata eſt fractio irreducibilis, & ejus denominator contentum eſt ſub factore rationali Q & factore ſurdo irreducibili

$$\begin{array}{l} \dot z - z\dot z + z^2\dot z - z^3\dot z + z^4\dot z - \\ \qquad\qquad 1+z \\ \hline \quad z\dot z - z^2\dot z + z\dot z - z^4 z + z^5\dot z -, \\ \dot z - z\dot z + z^2\dot z - z^3\dot z + z^4\dot z - z^5\dot z +, \\ \hline \dot z \quad * \quad * \quad * \quad * \quad * \end{array}$$

[r] NIMIRUM ſi factum è diviſoribus relictis neque quadratum fuerit, neque cubus neque ulla denique poteſtas numero quopiam indicata. Sin fuerit, ſit quidem poteſtas numero n indicata. Tum, facto illo dicto F, jubet Newtonus quantitatem R huic $F^{\frac{1}{n}}$ æqualem ſumi. Et ordinatam ad denominatorem R^{n+1} vult reduci. Cùm enim indices λ, $-n$ æquandos dixit, perinde eſt ac ſi hos $\lambda-1$, $-n-1$ æquandos dixiſſet.

[s] NEMPE harum præceptionum, ſicut earum quæ ſuprà traduntur (§ 2 & 3.) ea mens eſt: ut Ordinata, ſive illa rationalis ſive irrationalis ſit, ad denominatorem quâm maximè ſimplicem reducatur. Quod ita plerumque ſiet, ſi quibus modis Newtonus præſcripſit, ea tractetur.

[t] RAPHSONI in hoc ſymbolo ſecutus ſum emendationes, quas Stewarto quoque ſequi placuit, ſed celato Raphſoni nomine. (Vide *the Hiſtory of Fluxions, &c. by the late Mr. Joſeph Raphſon, A. M. and R. S. S.* 4to. *London*, 1715, *pag.* 42: et confer *Sir Iſaac Newton's two Treatiſes of the Quadrature of Curves and Analyſis by Equations, &c. explained by John Stewart, A. M.*

reducibili R*, inveniendi ſunt lateris R diviſores omnes primi, & rejiciendus eſt diviſor unus magnitudinis cujuſque, & per diviſores qui reſtant, ſiqui ſint, multiplicandus eſt factor rationalis Q: & ſi factum æquale eſt lateri R, vel lateris illius poteſtati alicui cujus index eſt numerus integer, eſto index ille m, & erit $\lambda - 1 = -\pi - m$, & $R^{\lambda-1} = R^{-\pi-m}$ (r). PROP. V.

Ut ſi ordinata ſit $\frac{3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{\overline{q^2-x^2} \times \overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$ (ſ); quoniam factoris ſurdi latus, R, ſeu $q^3+q^2x-qx^2-x^3$, diviſores habet $q+x, q+x, q-x$, qui duarum ſunt magnitudinum; rejicio diviſorem unum magnitudinis utriuſque: & per diviſorem, $q+x$, qui relinquitur, multiplico factorem rationalem q^2-x^2. Et quoniam factum, $q^3+q^2x-qx^2-x^3$, æquale eſt lateri R, pono $m=1$, & inde, cùm π ſit $\frac{1}{3}$, fit $\lambda - 1 = -\frac{4}{3}$. Ordinatam igitur reduco ad denominatorem (t) $R^{\frac{4}{3}}$, & fit $x^0 \times \overline{3q^6 + 2q^5x + 8q^4x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5} \times \overline{q^3 + q^2x - qx^2 - x^3}|^{-\frac{4}{3}}$. Unde eſt $a = 3q^6$; $b = 2q^5$, &c. $e = q^3$; $f = q^2$, &c. $\theta - 1 = 0$; $\theta = 1 = \eta$; $\lambda = -\frac{1}{3}$; $r = 1$; $s = \frac{2}{3}$; $t = \frac{1}{3}$; $v = 0$. Et his in ſerie ſcriptis prodit area $\frac{3q^3x + 3qx^3}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, terminis omnibus in ſerie totâ poſt tertium evaneſcentibus (u).

PROP.

A. M. Profeſſor of Math. in the Mariſhal College, Aberdeen, 4to. London, 1745, *page* 13.) Editio Princeps habuit $\frac{3q^5 - q4x + 9q^1xx - qqx^3 - 6x^3}{qq - xx \sqrt{\text{cub.}\ q^3 + qqx - qxx - x^3}}$. Quod in pejus etiam mutatum Joneſius dedit, hoc modo, $\frac{3q^5 - q^4x^4 + 9q^2x^2 - q^2x^3 - 6x^3}{\overline{q^2-x^2} \times \overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$. Horum enim utrumque vitioſum eſſe, ex eo intelligere eſt, quòd neutrum quidem cum alio congruit, inferiùs expoſito, ejuſdem ordinatæ ſymbolo ad denominatorem $R^{\frac{4}{3}}$ reductæ; quod utraque editio, princeps illa, et Joneſii, uno modo protulerunt; quodque eo modo prolatum rectè ſe habere, ſigno eſt indicum ſeries ordine progrediens. Cæterùm vel ante Raphſonum, multo certè ante Stewartum, Harriſius ediderat, $\frac{3q^5 - q^4x + 9q^2xx - qqx^3 - 6qx^4}{qq - xx \sqrt[3]{q^3 + qqx - qxx - x^3}}$. Rectè quidem omnia, niſi quòd indicem literæ q, in nomine numeratoris tertio, non ſatis emendaverat. (Vide *Lexicon Technicum, &c. by John Harris, D. D. &c. vol. 2. fol. London*, 1710, ſub voce *Quadrature*.).

(t) Sic emendavi, pro $R^{-\frac{4}{3}}$, quod editiones habuere ante hanc noſtram omnes, et quod ipſe quidem Raphſonus habuit.

(u) STEWARTI in hoc ſymbolo emendationes ſequimur. Editio Joneſii, aliæque numeratorem habuere $3q^2x + 3x^3$. Vitioſè. Nam ſi areæ exquirendæ ponatur hoc ſymbolum generale, $\frac{x}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}} \times A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$, coefficientes hiſce æquationibus definientur. $A = 3q^3$. $B = 0$. $C = 3q$. $D = 0$. $E = 0$. $F = 0$.

Prop. VI.

PROP. VI. THEOR. IV.

Si Curvæ abſciſſa AB ſit z, & ſcribantur R pro $e+fz^{\eta}+gz^{2\eta}+hz^{3\eta}+$ &c. & S pro $k+lz^{\eta}+mz^{2\eta}+nz^{3\eta}+$ &c. ſit autem ordinatim applicata $z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ in $a+bz^{\eta}+cz^{2\eta}+dz^{3\eta}+$ &c. et ſi terminorum, e, f, g, h, &c. et k, l, m, n, &c. rectangula ſint

$$\begin{array}{lllll} ek & fk & gk & hk & \&c. \\ el & fl & gl & hl & \&c. \\ em & fm & gm & hm & \&c. \\ en & fn & gn & hn & \&c. \end{array}$$

Et ſi rectangulorum illorum coefficientes numerales ſint reſpectivè

$$\frac{\theta}{\eta}=r.\ r+\lambda=s.\ s+\lambda=t\ \ t+\lambda=v.\ \&c.$$

$$r+\mu=s'.\ s+\mu=t'.\ t+\mu=v'.\ v+\mu=w'.\ \&c.$$

$$s'+\mu=t''.\ t'+\mu=v''.\ v'+\mu=w''.\ w'+\mu=x''.\ \&c.$$

$$t''+\mu=v'''.\ v''+\mu=w'''.\ w''+\mu=x'''.\ x''+\mu=y'''.\ \&c.$$

Area

Hæ autem æquationes è formulâ generali Propoſitionis V. Newtonianæ veniunt, pro literis, a, b, c, &c. e, f, g, &c. η, θ, λ, r, s, t, &c. eis earum æſtimationibus in formulam illam ſubſtitutis, quas ſymbolum ordinatæ promit.

Itaque coefficiente ſecundâ, B, omnibuſque poſt tertiam C in nihilum abeuntibus, Area erit $\frac{x}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}} \times 3q^3+3qx^2 = \frac{3q^3x+3qx^3}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$.

(x) DEMONSTRATIO PROPOSITIONIS SEXTÆ.

Sunto juxta Propoſitionem quartam.

CURVARUM ORDINATÆ — AREÆ.

$$1^{\circ}.\ \left.\begin{array}{llll} \theta ekA & \begin{array}{l}+\theta\\+\lambda\eta\end{array} fkAz^{\eta} & \begin{array}{l}+\theta\\+2\lambda\eta\end{array} gkAz^{2\eta} & \begin{array}{l}+\theta\\+3\lambda\eta\end{array} hkAz^{3\eta} \\ & \begin{array}{l}+\theta\\+\mu\eta\end{array} elAz^{\eta} & \begin{array}{l}+\theta\\+\lambda\eta\\+\mu\eta\end{array} flAz^{2\eta} & \begin{array}{l}+\theta\\+2\lambda\eta\\+\mu\eta\end{array} glAz^{3\eta}\ \&c. \\ & & \begin{array}{l}+\theta\\+2\mu\eta\end{array} emAz^{2\eta} & \begin{array}{l}+\theta\\+\lambda\eta\\+2\mu\eta\end{array} fmAz^{3\eta} \\ & & & \begin{array}{l}+\theta\\+3\mu\eta\end{array} enAz^{3\eta} \end{array}\right\} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} \ \Big|\ Az^{\theta}R^{\lambda}S^{\mu}$$

Area curvæ erit hæc

$$z^{\theta}R^{\lambda}S^{\mu} \text{ in } + \frac{\frac{1}{n}a}{rek}$$

$$+ \frac{\frac{1}{n}b \begin{matrix} -sfk \\ -s'el \end{matrix} A}{\overline{r+1} \times ek} z^{n}$$

$$+ \frac{\frac{1}{n}c \begin{matrix} -\overline{s+1} \times fk \\ -\overline{s'+1} \times el \end{matrix} B \begin{matrix} -tgk \\ -t'fl \\ -t''em \end{matrix} A}{\overline{r+2} \times ek} z^{2n}$$

$$+ \frac{\frac{1}{n}d \begin{matrix} -\overline{s+2} \times fk \\ -\overline{s'+2} \times el \end{matrix} C \begin{matrix} -\overline{t+1} \times gk \\ -\overline{t'+1} \times fl \\ -\overline{t''+1} \times em \end{matrix} B \begin{matrix} -vhk \\ -v'gl \\ -v''fm \\ -v'''en \end{matrix} A}{\overline{r+3} \times ek} z^{3n}$$

$+$ &c.

Ubi A denotat termini primi coefficientem datam, $\frac{\frac{1}{n}a}{rek}$, cum ſigno ſuo + vel −; B coefficientem datam ſecundi; C coefficientem datam tertii; & ſic deinceps. Terminorum vero, a, b, c, &c. e, f, g, &c. k, l, m, &c. unus vel plures deeſſe poſſunt.

Demonſtratur Propoſitio ad modum præcedentis, & quæ ibi notantur hîc obtinent. Pergit autem ſeries talium Propoſitionum in infinitum, & progreſſio ſeriei manifeſta eſt (x).

Demonstratio

2°. $$\left.\begin{matrix} \overline{\theta+n}ekBz^{n} \; \begin{matrix}+\overline{\theta+n}\\+\lambda n\end{matrix} fkBz^{2n} \; \begin{matrix}+\overline{\theta+n}\\+2\lambda n\end{matrix} gkBz^{3n} \\ \begin{matrix}+\overline{\theta+n}\\+\mu n\end{matrix} elBz^{2n} \; \begin{matrix}+\overline{\theta+n}\\+\lambda n\\+\mu n\end{matrix} flBz^{3n} \;\&c. \\ \begin{matrix}+\overline{\theta+n}\\+2\mu n\end{matrix} emBz^{3n} \end{matrix}\right\} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} \;\Big|\; Bz^{\theta+n}R^{\lambda}S^{\mu}$$

3°. $$\left.\begin{matrix} \overline{\theta+2n}ekCz^{2n} \; \begin{matrix}+\overline{\theta+2n}\\+\lambda n\end{matrix} fkCz^{3n} \\ \begin{matrix}+\overline{\theta+2n}\\+\mu n\end{matrix} elCz^{3n} \end{matrix} \;\&c.\right\} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} \;\Big|\; Cz^{\theta+2n}R^{\lambda}S^{\mu}$$

4°. $$\overline{\theta+3n}ekDz^{3n} \;\&c. \; \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} \quad Dz^{\theta+3n}R^{\lambda}S^{\mu}$$

Et ſi ſumma Ordinatarum ponatur æqualis ordinatæ $a + bz^{n} + cz^{2n} + dz^{3n} +, \times z^{\theta-1}.R^{\lambda-1}.S^{\mu-1}$, ſumma arearum $z^{\theta}.R^{\lambda}.S^{\mu} \times A + Bz^{n} + Cz^{2n} + Dz^{3n} +$, æqualis erit areæ curvæ, cujus iſta eſt ordinata. Æquentur igitur ordinatarum membra homologa,

Prop. VII.

PROP. VII. THEOR. V.

Si pro $e+fz^{\eta}+gz^{2\eta}+$ &c. ſcribatur R ut ſuprà, & in Curvæ alicujus Ordinatâ, $z^{\theta\pm\eta\sigma}R^{\lambda\pm\tau}$, maneant quantitates datæ θ, η, λ, e, f, g, &c. et pro σ ac τ ſcribantur ſucceſſivè numeri quicunque integri: & ſi detur Area unius ex curvis, quæ per ordinatas innumeras ſic prodeuntes deſignantur, ſi Ordinatæ ſunt duorum nominum in vinculo radicis; vel ſi dentur Areæ duarum ex curvis, ſi Ordinatæ ſunt trium nominum in vinculo radicis; vel Areæ trium ex curvis, ſi Ordinatæ ſunt quatuor nominum in vinculo radicis; & ſic deinceps in infinitum: dico quòd dabuntur Areæ curvarum omnium.

Pro nominibus hîc habeo terminos omnes in vinculo radicis, tam deficientes quàm plenos, quorum indices dignitatum ſunt in progreſſione arithmeticâ. Sic Ordinata $\sqrt{a^4-ax^3+x^4}$, ob terminos duos inter a^4 & $-ax^3$ deficientes, pro quinquinomio haberi debet. At $\sqrt{a^4+x^4}$ binomium eſt; & $\sqrt{a^4+x^4-\frac{x^8}{a^4}}$ trinomium;

Prop. VI.

et fiet $a=\theta ek\text{A}$.

$$b=\overline{\theta+\lambda\eta}\times fk\text{A}+\overline{\theta+\mu\eta}\times el\text{A}+\overline{\theta+\eta}\times ek\text{B}$$

$$c=\overline{\theta+2\lambda\eta}\times gk\text{A}+\overline{\theta+\lambda\eta+\mu\eta}\times fl\text{A}+\overline{\theta+2\mu\eta}\times em\text{A}+\overline{\theta+\eta+\lambda\eta}\times fk\text{B}+\overline{\theta+\eta+\mu\eta}\times el\text{B}+\overline{\theta+2\eta}\times ek\text{C}.$$

$$d=\overline{\theta+3\lambda\eta}\times hk\text{A}+\overline{\theta+2\lambda\eta+\mu\eta}\times gl\text{A}+\overline{\theta+\lambda\eta+2\mu\eta}\times fm\text{A}+\overline{\theta+3\mu\eta}\times en\text{A}+\overline{\theta+\eta+2\lambda\eta}\times gk\text{B}+\overline{\theta+\eta+\lambda\eta+\mu\eta}\times fl\text{B}+\overline{\theta+\eta+2\mu\eta}\times em\text{B}+\overline{\theta+2\eta+\lambda\eta}\times fk\text{C}+\overline{\theta+2\eta+\mu\eta}\times el\text{C}+\overline{\theta+3\eta}\times ek\text{D}.$$

Ex quibus venient

$$\text{A}=\frac{a}{\theta ek}.\quad \text{B}=\frac{b-\overline{\theta+\lambda\eta}\times fk\text{A}-\overline{\theta+\mu\eta}\times el\text{A}}{\overline{\theta+\eta}\times ek}.$$

$$\text{C}=\frac{c-\overline{\theta+2\lambda\eta}\,gk\text{A}-\overline{\theta+\lambda\eta+\mu\eta}\,fl\text{A}-\overline{\theta+2\mu\eta}\,em\text{A}-\overline{\theta+\eta+\lambda\eta}\,fk\text{B}-\overline{\theta+\eta+\mu\eta}\,el\text{B}}{\overline{\theta+2\eta}\times ek}.$$

$$\text{D}=\frac{d-\overline{\theta+3\lambda\eta}\,hk\text{A}-\overline{\theta+2\lambda\eta+\mu\eta}\,gl\text{A}-\overline{\theta+\lambda\eta+2\mu\eta}\,fm\text{A}-\overline{\theta+3\mu\eta}\,en\text{A}-\overline{\theta+\eta+2\lambda\eta}\,gk\text{B}-\overline{\theta+\eta+\lambda\eta+\mu\eta}\,fl\text{B}-\overline{\theta+\eta+2\mu\eta}\,em\text{B}-\overline{\theta+2\eta+\lambda\eta}\,fk\text{C}-\overline{\theta+2\eta+\mu\eta}\,el\text{C}}{\overline{\theta+3\eta}\,ek}$$

Jam

mium; cùm progreſſio jam per majores differentias procedat. Propoſitio verò ſic demonſtratur. PROP. VII.

CAS. I.

Sunto Curvarum duarum Ordinatæ $pz^{\theta-1}R^{\lambda-1}$ (y) & $qz^{\theta+\eta-1}R^{\lambda-1}$, & Areæ pA & qB, exiſtente R quantitate trium nominum $e+fz^{\eta}+gz^{2\eta}$. Et cùm per Prop. III. ſit $z^{\theta}R^{\lambda}$ Area curvæ, cujus Ordinata eſt $\theta e+\overline{\theta+\lambda\eta}\times fz^{\eta}+\overline{\theta+2\lambda\eta}\times gz^{2\eta}$ in $z^{\theta-1}R^{\lambda-1}$, ſubduc ordinatas & areas priores de areâ & ordinatâ poſteriori, & manebit $\overline{\theta e-p+\overline{\theta+\lambda\eta}\times f-q\times z^{\eta}+\overline{\theta+2\lambda\eta}\times gz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}$ Ordinata nova curvæ, et $z^{\theta}R^{\lambda}-pA-qB$ ejuſdem Area. Pone $\theta e=p$, & $\theta f+\lambda\eta f=q$ (z), & Ordinata evadet $\overline{\theta+2\lambda\eta}\times gz^{2\eta}\times z^{\theta-1}R^{\lambda-1}$, & Area $z^{\theta}R^{\lambda}-\theta eA-\theta fB-\lambda\eta fB$. Divide utramque per $\theta g+2\lambda\eta g$, & aream prodeuntem dic C; & aſſumptâ utcunque r, erit rC Area curvæ, cujus Ordinata eſt $rz^{\theta+2\eta-1}R^{\lambda-1}$. Et quâ ratione ex areis pA & qB aream rC, ordinatæ $rz^{\theta+2\eta-1}R^{\lambda-1}$ congruentem, invenimus, licebit ex areis qB & rC Aream quartam, puta sD, Ordinatæ $sz^{\theta+3\eta-1}R^{\lambda-1}$ congruentem, invenire; & ſic deinceps in infinitum. Et par eſt ratio progreſſionis

Jam pone $\frac{\theta}{n}=r$, $r+\lambda=s$, $s+\lambda=t$, $t+\lambda=u$, $u+\lambda=w$

$r+\mu={}'s$, $s+\mu={}'t$, $t+\mu={}'u$, $u+\mu={}'w$

${}'s+\mu={}''t$, ${}'t+\mu={}''u$, ${}'u+\mu={}''w$

${}''t+\mu={}'''u$, ${}''u+\mu={}'''w$

Et in areâ $z^{\theta}R^{\lambda}S^{\mu}\times A+Bz^{n}+Cz^{2n}+Dz^{3n}+$, in membro primo pro A, in ſecundo pro B, in tertio pro C, in quarto pro D, ſcribantur literarum A, B, C, D, æſtimationes illæ, quas æquationes jam ſuprà poſitæ, ſi noviſſimarum ope transformentur, promant; ita exſiſtet Areæ formula qualis à Newtono poſita eſt. Q. E. D.

Atque hæc eſt ſextæ propoſitionis demonſtratio, ad ſuperioris exemplar confecta, quam et ante nos Stewartus, in ſuis ad hunc locum commentariis, eodem ferè modo concinnatam dedit.

(y) In hâc demonſtratione ſymbola, $\theta-1$, $\lambda-1$, ponuntur pro θ, et λ enunciationis.

(z) NIMIRUM cùm id ſemper obtineat, ſecundum omnem literarum p, q æſtimationem, ut arearum differentia area ſit curvæ, cujus ordinata ordinatarum differentiæ æqualis ſit, obtinebit etiam, ſi p, q, earum ſumantur magnitudinum, quæ efficiant, ut duo prima ordinatarum differentiæ membra in nihilum ſingulatim abeant. Hoc autem efficietur ſi ſumatur $p=\theta e$, & $q=\theta f+\lambda\eta f$. Ita enim fiet $\theta e-p=0$, & $\theta f+\lambda\eta f-q=0$. Ponatur igitur $\theta e=p$, & $\theta f+\lambda\eta f=q$, atque ea conſequentur quæ Newtonus dicit.

(aa) Hæc

PROP. VII. progreſſionis ab areis B & A in partem contrariam pergentis. Si terminorum θ, $\theta+\lambda\eta$, & $\theta+2\lambda\eta$ aliquis deficit, & ſeriem abrumpit, aſſumatur area pA, in principio progreſſionis unius, & area qB, in principio alterius, & ex his duabus areis dabuntur areæ omnes in progreſſione utrâque ([aa]). Et contrà, ex aliis duabus areis aſſumptis fit regreſſus, per analyſin, ad areas A & B; adeo ut ex duabus datis cæteræ omnes dentur. Q. E. O.

Hic eſt caſus Curvarum ubi ipſius z index θ augetur vel diminuitur perpetuâ additione vel ſubductione quantitatis η. Caſus alter eſt curvarum ubi index λ augetur vel diminuitur unitatibus.

C A S. II.

Ordinatæ $pz^{\theta-1}R^{\lambda}$ ([bb]) et $qz^{\theta+\eta-1}R^{\lambda}$, quibus areæ pA et qB jam reſpondeant, ſi in R, ſeu $\overline{e+fz^{\eta}+gz^{2\eta}}$, ducantur, ac deinde ad R viciſſim applicentur, evadunt $\overline{pe+pfz^{\eta}+pgz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}$, et $\overline{qez^{\eta}+qfz^{2\eta}+qgz^{3\eta}}\times z^{\theta-1}R^{\lambda-1}$.

Et (per Prop. III.) eſt $az^{\theta}R^{\lambda}$ Area curvæ, cujus Ordinata eſt $\overline{\theta ae+\overline{\theta+\lambda\eta}\times afz^{\eta}+\overline{\theta+2\lambda\eta}\times agz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}$; et $bz^{\theta+\eta}R^{\lambda}$ Area curvæ, cujus Ordinata eſt $\overline{\overline{\theta+\eta}\times bez^{\eta}+\overline{\theta+\eta+\lambda\eta}\times bfz^{2\eta}+\overline{\theta+\eta+2\lambda\eta}\times bgz^{3\eta}}\times z^{\theta-1}R^{\lambda-1}$.

Et

([aa]) HÆC præciſè nimis et obſcurè quidem dicta ſunt, credo tamen hâc mente dici. "Si forte eveniat ut vel θ, vel $\theta+\lambda\theta$, vel $\theta+2\lambda\eta$, hoc eſt, ut in ſymbolo illo generali $z^{\theta}R^{\lambda}-\theta:A-\overline{\theta+\lambda\eta}\,fB=\overline{\theta+2\lambda\eta}\,gC$, coefficiens cujuſlibet arearum A, B, C nihilo æqualis ſit, hoc fortè ſi eveniat, nihilo minus, per eandem inveſtigationis viam dabuntur areæ omnes utriuſque progreſſionis; ſive illius quæ, initio ab illâ pA ſumpto, infinitè uſque aſcendit, ſive alterius, quæ, ab alterâ pB incipiens, infinitè uſque deſcendit; ſive utriuſque. Hæc autem explicatiùs mox dicam, ac verè dicta eſſe oſtendam.

([bb]) Rurſum ponitur $\theta-1$, pro θ enunciationis.

([cc]) NEMPE hoc modo.
Propter coefficientem membri ſecundi nihilo æqualem, fiet
$$p=-\frac{\theta af+\lambda\eta af+\theta be+\eta be+qe}{f}.\quad \text{Rurſum,}\quad p=-\frac{\theta ag+2\lambda\eta ag+\theta bf+\eta bf+\lambda\eta bf+qf}{g};$$
propter coefficientem membri tertii nihilo æqualem. Rurſum, propter coefficientem membri quarti nihilo æqualem, $q=-\theta b-\eta b-2\lambda\eta b$.

Ex his autem efficietur $b=\frac{afg}{f^{2}-2ge}$; et $p=\frac{2\lambda\eta age}{f^{2}-2ge}-\theta a-\lambda\eta a$; et $q=\frac{\theta+\eta+2\lambda\eta}{2ge-f^{2}}afg$.

Jam verò cùm id ſemper obtineat, ut Arearum ſumma $p\text{A}+q\text{B}+az^{\theta}R^{\lambda}+bz^{\theta+\eta}R^{\lambda}$ curvæ ſit Area,

cujus

Et harum quatuor Arearum ſumma eſt $pA+qB+az^{\theta}R^{\lambda}+bz^{\theta+\eta}R^{\lambda}$; PROP. VII.
& ſumma reſpondentium Ordinatarum

$$\theta ae+\overline{\theta+\lambda\eta}\times afz^{\eta}+\overline{\theta+2\lambda\eta}\times agz^{2\eta}+\overline{\theta+\eta+2\lambda\eta}\times bgz^{3\eta}\times z^{\theta-1}R^{\lambda-1}.$$
$$\begin{array}{llll} +pe & +\overline{\theta+\eta}\times be & +\overline{\theta+\eta+\lambda\eta}\times bf & +\ 1\times qg \\ & +1\times pf & +1\times pg & \\ & +1\times qe & +1\times qf & \end{array}$$

Si terminus primus, tertius, & quartus, ponatur ſeorſim æquales nihilo, per primum fiet $\theta ae+pe=0$, ſeu $-\theta a=p$; per quartum $-\theta b-\eta b-2\lambda\eta b=q$, & per tertium (eliminando p & q) $\frac{2ag}{f}=b$. Unde ſecundus fit $\frac{\lambda\eta aff-4\lambda\eta age}{f}$; adeoque ſumma quatuor Ordinatarum eſt $\frac{\lambda\eta aff-4\lambda\eta age}{f}z^{\theta+\eta-1}R^{\lambda-1}$, & ſumma totidem reſpondentium Arearum eſt $az^{\theta}R^{\lambda}+\frac{2ag}{f}z^{\theta+\eta}R^{\lambda}-\theta aA-\frac{\overline{2\theta+2\eta+4\lambda\eta}}{f}agB$. Dividantur hæ ſummæ per $\frac{\lambda\eta aff-4\lambda\eta age}{f}$; & ſi quotum poſterius dicatur D, erit D Area curvæ, cujus Ordinata eſt quotum prius, $z^{\theta+\eta-1}R^{\lambda-1}$. Et eâdem ratione, ponendo omnes ordinatæ terminos, præter primum, æquales nihilo, poteſt Area curvæ inveniri, cujus Ordinata eſt $z^{\theta-1}R^{\lambda-1}$(cc). Dicatur area iſta C; & quâ ratione ex areis A & B inventæ ſunt areæ C ac D, ex his areis, C ac D, inveniri poſſunt aliæ duæ, E & F, ordinatis $z^{\theta-1}R^{\lambda-2}$ et $z^{\theta+\eta-1}R^{\lambda-2}$ congruentes, & ſic deinceps in infinitum. Et, per analyſin contrariam, regredi licet ab areis E & F ad areas C ac D, & inde ad

cujus Ordinata ſummæ Ordinatarum eſt æqualis, ſi ſumantur b, p, q, earum magnitudinum, quæ efficiant, ut membra omnia ſummæ ordinatatarum poſt primum in nihilum ſingulatim abeant, erit $pA+qB+az^{\theta}R^{\lambda}+bz^{\theta+\eta}R^{\lambda}$ Area curvæ, cujus Ordinata erit $\overline{\theta a+p}\times ez^{\theta-1}R^{\lambda-1}$.

Quare $\frac{pA+qB+az^{\theta}.R^{\lambda}+bz^{\theta+\eta}R^{\lambda}}{\overline{\theta a+p}\times e}$ erit Area curvæ, cujus Ordinata $z^{\theta-1}R^{\lambda-1}$. Hoc eſt, ſi pro p, q, b ſcribantur earum æſtimationes illæ, quas è membris Ordinatæ ſecundo, tertio atque quarto, ſingulatim nihilo æquatis elicuimus, et pro Areâ curvæ cujus Ordinata eſt $z^{\theta-1}R^{\lambda-1}$, ſcribatur C, erit

$$\overline{az^{\theta}.R^{\lambda}+\frac{afg}{f^2-2ge}z^{\theta+\eta}R^{\lambda}+\overline{\frac{2\lambda\eta ge}{f^2-2ge}-\theta-\lambda\eta}\,aA+\overline{\frac{\theta+\eta+2\lambda\eta}{2ge-f^2}}\,afgB}\times\frac{f^2-2ge}{4\lambda\eta age^2-f^2\lambda\eta ae}=C.$$

Hoc eſt $\frac{\overline{f^2-2ge}\,z^{\theta}.R^{\lambda}+fgz^{\theta+\eta}R^{\lambda}+\overline{\overline{\theta+2\lambda\eta}\,2ge-\overline{\theta+\lambda\eta}\,f^2}\,A-\overline{\theta+\eta+2\lambda\eta}\,fgB}{\overline{4ge-f^2}\times\lambda\eta e}=C$. Quali etiam formâ ſymbolum areæ C, ſimili computandi ratione, Stewartus protulit. Symbolum, quod in Hiſtoriâ ſuâ, capite undecimo, (p. 85.) Raphſonus poſuit, vitioſum eſt.

4 areas

PROP. VII. areas A & B, aliaſque quæ in progreſſione ſequuntur. Igitur ſi index λ, perpetuâ unitatum additione vel ſubductione, augeatur vel minuatur, & ex Areis, quæ Ordinatis ſic prodeuntibus reſpondent, duæ ſimpliciſſimæ habentur; dantur aliæ omnes in infinitum. Q. E. O.

CAS. III.

Et per caſus hoſce duos conjunctos, ſi tam index θ perpetuâ additione vel ſubductione ipſius η, quàm index λ perpetuâ additione

PROP. VII. ÆQUAT. CANON.

(d.) APERTIORA quidem ſunt hæc, quàm ut explicatione egeant. Illud modò nobis agendum eſt, ut hujus Propoſitionis ſummam formulis quibuſdam exponamus ad uſum computandi accommodatis.

Hunc igitur in finem, Areæ curvarum quarum Ordinatæ ſunt $pz^{\theta-1}R^{\lambda-1}$, $qz^{\theta+\eta-1}R^{\lambda-1}$, $rz^{\theta+2\eta-1}R^{\lambda-1}$, $sz^{\theta+3\eta-1}R^{\lambda-1}$, $tz^{\theta+4\eta-1}$, $R^{\lambda-1}$, $*$, $*$, $*$, $\pi z^{\theta+\sigma\eta-1}R^{\lambda-1}$, harum inquam Areæ ſignificentur notis pA, qB, rC, sD, tE, $*$, $*$, $*$, πZ. Tum, ope Casûs Primi hujus Propoſitionis, hæ elicientur Æquationes, quas Canonicas vocare liceat.

Primùm quidem, ſi quantitas R è duobus conſtet nominibus

$$1.\quad B = \frac{z^{\theta}R^{\lambda} - \theta eA}{\overline{\theta+\lambda\eta}\times f}.\qquad 2.\quad A = \frac{z^{\theta}R^{\lambda} - \overline{\theta+\lambda\eta}\times fB}{\theta e}.$$

II. SI R ex tribus conſtet nominibus.

$$1.\quad C = \frac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+\lambda\eta}\,fB}{\overline{\theta+2\lambda\eta}\times g}.$$

$$2.\quad B = \frac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+2\lambda\eta}\times gC}{\overline{\theta+\lambda\eta}\times f}.$$

$$3.\quad A = \frac{z^{\theta}R^{\lambda} - \overline{\theta+\lambda\eta}\,fB - \overline{\theta+2\lambda\eta}\times gC}{\theta e}.$$

III. SI R ex quatuor conſtet nominibus.

$$1.\quad D = \frac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+\lambda\eta}\,fB - \overline{\theta+2\lambda\eta}\times gC}{\overline{\theta+3\lambda\eta}\times h}.$$

$$2.\quad C = \frac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+\lambda\eta}\,fB - \overline{\theta+3\lambda\eta}\,hD}{\overline{\theta+2\lambda\eta}\,g}.$$

$$3.\quad B = \frac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+2\lambda\eta}\,gC - \overline{\theta+3\lambda\eta}\,hD}{\overline{\theta+\lambda\eta}\,f}.$$

$$4.\quad A = \frac{z^{\theta}R^{\lambda} - \overline{\theta+\lambda\eta}\,fB - \overline{\theta+2\lambda\eta}\,gC - \overline{\theta+3\lambda\eta}\,hD}{\theta e}.$$

DENIQUE ſi quantitas R ex tot conſtet nominibus, quot ſint in numero $\sigma+1$ unitates

$$1.\quad Z = \frac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+\lambda\eta}\,fB - \overline{\theta+2\lambda\eta}\,gC - \overline{\theta+3\lambda\eta}\,hD - * - * - * - \overline{\theta+\overline{\sigma-1}\,\lambda\eta}\,yY}{\overline{\theta+\sigma\lambda\eta}\,z}.$$

deſignante

ditione vel ſubductione unitatis, utcunque augeatur vel minuatur, dabuntur areæ ſingulis prodeuntibus ordinatis reſpondentes. Q. E. O. PROP. VII.

CAS. IV.

Et ſimili argumento, ſi Ordinata conſtat ex quatuor nominibus in vinculo radicali, & dantur tres arearum, vel ſi conſtat ex quinque nominibus & dantur quatuor arearum, & ſic deinceps; dabuntur Areæ omnes, quæ, addendo vel ſubducendo numerum η indici θ, vel unitatem indici λ, generari poſſunt. Et par eſt ratio Curvarum ubi Ordinatæ ex binomiis conflantur, & Area una earum, quæ non ſunt geometricè quadrabiles, datur (dd). Q. E. O.

PROP.

deſignante ν denominatoris coefficientem nominis extremi quantitatis R.

2. $$\mathrm{Y} = \frac{z^{\theta}\mathrm{R}^{\lambda} - \theta e\mathrm{A} - \overline{\theta+\lambda\eta}\,f\mathrm{B} - \overline{\theta+2\lambda\eta}\,g\mathrm{C} - \overline{\theta+3\lambda r}\,h\mathrm{D} - * - * - * - \overline{\theta+\sigma\lambda\eta}\,\nu z}{\overline{\theta+\overline{\sigma-1}\,\lambda r}\,y}.$$

$* = *$
$* = *$
$* = *$

$$\mathrm{D} = \frac{z^{\theta}\mathrm{R}^{\lambda} - \theta e\mathrm{A} - \overline{\theta+\lambda\eta}\,f\mathrm{B} - \overline{\theta+2\lambda\eta}\,r\mathrm{C} - * - * - * - \overline{\theta+\overline{\sigma-1}\,\lambda\eta}\,y\mathrm{Y} - \overline{\theta+\sigma\lambda\eta}\,\nu z}{\overline{\theta+3\lambda\eta}\,h},$$

2. Atque harum formularum ope optimè quidem explicantur ea, quæ de alicujus è coefficientibus hiſce θ, $\theta+\lambda\eta$, $\theta+2\lambda\eta$ defectu Newtonus dixit; ſaltem quæ Newtonum dicentem mihi videor audiviſſe. Primùm enim exiſtente R quantitate trium nominum, θ deficiat. Hoc eſt ponatur $\theta = 0$. Nihilominus ſi detur B, dabitur quidem C per æquationem ſecundæ claſſis primam, quæ, poſito $\theta = 0$, in hanc migraverit: $\mathrm{C} = \frac{\mathrm{R}^{\lambda} - \lambda\eta f\mathrm{B}}{2\lambda r g}$. Si verò detur C, dabitur B per æquationem ſecundæ claſſis ſecundam, quæ poſito $\theta = 0$, in hanc migraverit; $\mathrm{B} = \frac{\mathrm{R}^{\lambda} - 2\lambda r g\mathrm{C}}{\lambda\eta f}$. Ex datis autem duabus B, C invenietur D; et ex datis C, D, invenietur E; eodemque uſque modo progredi licebit, pro θ, illis η, 2η, 3η, 4η, pro C, illis D, E, F, pro B illis, C, D, E, pro A illis, B, C, D, in primam ſecundæ claſſis æquationem ordine ſubſtitutis. Nempe hoc modo, ut exemplis res magis eluceſcat;

$$\mathrm{D} = \frac{z^{\eta}\mathrm{R}^{\lambda} + \eta e\mathrm{B} - \overline{\eta+\lambda\eta}\,f\mathrm{C}}{\overline{\eta+2\lambda\eta}\,g}.$$

$$\mathrm{E} = \frac{z^{2\eta}\mathrm{R}^{\lambda} - 2\eta e\mathrm{C} - \overline{2\eta+\lambda\eta}\,f\mathrm{D}}{\overline{2\eta+2\lambda\eta}\,g}.$$

$$\mathrm{F} = \frac{z^{3\eta}\mathrm{R}^{\lambda} - 3\eta e\mathrm{D} - \overline{3\eta+\lambda\eta}\,f\mathrm{E}}{\overline{3\eta+2\lambda\eta}\,g}.$$

In hoc igitur caſu dabuntur areæ omnes progreſſionis ejus, quæ, ab illâ A initio ſumpto, infinitè uſque aſcendit. Ipſa autem A, cujus, in æquationibus Canonicis, coefficientem nihilo æqualem poſuimus, illa quidem infinita erit; id quod æquatio ſecundæ claſſis tertia ſatis indicat; quæ, poſito $\theta = 0$, in hanc migraverit. $\mathrm{A} = \frac{\mathrm{R}^{\lambda} - \lambda r f\mathrm{B} - 2\lambda r g\mathrm{C}}{0}$.

 Progreſſionis

PROP. VIII. THEOR. VI.

Si pro $e+fz^{n}+gz^{2n}+$ &c. et $k+lz^{n}+mz^{2n}+$ &c. ſcribantur R &c. ut ſuprà, & in Curvæ alicujus Ordinatâ $z^{\theta\pm n\sigma}R^{\lambda\pm\tau}S^{\mu\pm\nu}$ maneant quanti-

Progreſſionis autem illius, quæ, ab illâ B incipiens, infinitè deſcendit, hujus quidem areæ omnes infinitæ erunt. Primùm enim eſſe A infinitam jam oſtendimus. Deſignet autem a curvam progreſſionis deſcendentis ab illâ A proximè inferiorem, ſive eam, cujus Ordinata eſt $z^{\theta-n}R^{\lambda-1}$. Area a, ope tertiæ æquationis ſecundæ claſſis, invenienda eſt; ſcribendo, $-n$ pro θ, a pro A, A pro B, B pro C; nempe hoc modo, $a = \frac{z^{-n}R^{\lambda} - \overline{\lambda n - n}\,fA - \overline{2\lambda n - n}\,gB}{-ne}$. Ubi numerator quantitatis fractæ, propter membrum, $\overline{\lambda n - n}\,fA$, infinitum, cùm reliqua finita ſint, totus quidem infinitus erit. Unde, cùm denominator ſit finitus, quantitas ipſa fracta infinita erit. Ergo area a infinita. Simili modo, ſi b deſignat aream progreſſionis deſcendentis ab illâ a proximè inferiorem, oſtendetur b infinita. Eodemque uſque modo progredi licebit. Omnes igitur hujus progreſſionis areæ poſt primam B infinitæ erunt.

3. Sed deficiat $\theta+2\lambda n$. Hoc eſt, ſit $\theta+2\lambda n=0$, exiſtente nimirum $\theta=-2\lambda n$.

Nihilominus, ſi detur A, dabitur B, per æquationem ſecundæ claſſis ſecundam; quæ, poſito $\overline{\theta+2\lambda n}=0$, in hanc migraverit $B = \frac{z^{-2\lambda n}R^{\lambda} + 2\lambda neA}{-\lambda nf}$. Vel ſi detur B, dabitur A per æquationem ſecundæ claſſis tertiam; quæ, poſito $\theta+2\lambda n=0$, in hanc migraverit: $A = \frac{z^{-2\lambda n}R^{\lambda} + \lambda nfB}{-2\lambda ne}$. E duabus autem B, A datis, dabitur a, quæ, in progreſſione deſcendente, ab illâ A proximè eſt inferior; è duabus A, a, dabitur rurſum proximè inferior b; è duabus a, b dabitur c, eodemque uſque modo progredi licebit. Omnes enim a, b, c, d ope tertiæ æquationis ſecundæ claſſis inveniendæ ſunt, pro θ, illis $-2\lambda n-n$, $-2\lambda n-2n$, $-2\lambda n-3n$, pro A, illis a, b, c, d, et pro B, illis A, a, b, c, pro C illis B, A, a, b ordine ſubſtitutis.

In hoc igitur caſu dabuntur areæ omnes progreſſionis ejus, quæ, ab illâ B initio ſumpto, infinitè deſcendit. Ipſa autem C, cujus, in æquationibus Canonicis, coefficientem nihilo æqualem poſuimus, illa quidem infinita erit. Id quod æquatio prima ſecundæ claſſis ſatis indicat; quæ, poſito $\theta+2\lambda n=0$, in hanc migraverit, $C = \frac{z^{-2\lambda n}R^{\lambda} + 2\lambda neA + \lambda nfB}{0}$.

Ac propterea progreſſionis ejus, quæ ab illâ B incipiens, infinitè aſcendit, hujus quidem, hoc in caſu, areæ omnes infinitæ erunt. Id enim eâdem plane argumentandi ratione ex æquatione primâ efficietur, quâ in priore caſu ex tertiâ efficiebatur, contrariæ progreſſionis areas omnes, poſt primam B, infinitas eſſe.

4. Denique deficiat $\theta+\lambda n$. Hoc eſt, ſit $\theta+\lambda n=0$, exiſtente utique $\theta=-\lambda n$. Jam ſi detur A, dabitur quidem C per æquationem ſecundæ claſſis primam; quæ, poſito $\theta+\lambda n=0$, in hanc migraverit: $C = \frac{z^{-\lambda n}R^{\lambda} + \lambda neA}{\lambda ng}$. Vel ſi detur C, dabitur A per æquationem ſecundæ claſſis tertiam; quæ, poſito $\theta+\lambda n=0$, in hanc migraverit: $A = \frac{z^{-\lambda n}R^{\lambda} - \lambda ngC}{-\lambda ne}$. Ipſa vero B, cujus, in æquationibus Canonicis, coefficientem nihilo æqualem poſuimus, illa quidem infinita erit. Id quod æquatio ſecunda claſſis ſecundæ ſatis indicat; quæ, poſito $\theta+\lambda n=0$, in hanc migraverit: $B = \frac{z^{-\lambda n}R^{\lambda} + \lambda neA - \lambda ngC}{0}$.

Utriuſque autem progreſſionis, tam illius quæ ab illâ B incipiens infinitè aſcendit, quàm ejus quæ ab illâ A infinitè deſcendit, areæ quidem omnes, illis, A, C exceptis, in hoc caſu infinitæ erunt. Deſcendentis enim progeſſionis infinitas eſſe areas omnes poſt A, id quidem ex tertiâ æquatione ſecundæ claſſis efficietur, Aſcendentis omnes poſt B, id verò ex primâ; ſimili prorſus argumentandi ratione, quâ in primo caſu Deſcendentis, in ſecundo Aſcendentis progreſſionis, omnes, poſt primam utriuſque, infinitæ efficiebantur.

5. In

quantitates datæ θ, η, λ, μ, e, f, g, k, l, m, &c. et pro σ, τ, & υ, ſcribantur ſucceſſivè numeri quicunque integri: & ſi dentur Areæ duarum ex curvis, quæ per Ordinatas ſic prodeuntes deſignantur, ſi quanti- PROP. VIII.

5. In ſummâ ſi ponatur $\theta = 0$, datâ ſive A, ſive B, dabuntur areæ omnes progreſſionis inde ab illâ A aſcendentis. Alterius, à B deſcendentis, omnes infinitæ erunt.

Si ponatur $\theta + 2\lambda\eta = 0$, datâ ſive A, ſive B, dabuntur areæ omnes deſcendentis inde ab illâ B progreſſionis. Alterius, à B aſcendentis, omnes infinitæ erunt.

Denique ſi ponatur $\theta + \lambda\eta = 0$, ſi duarum A, C detur altera, dabitur altera quoque. Præter illas A, C, omnes utriuſque progreſſionis infinitæ erunt.

Præciſè igitur et breviter loquenti, cujus generis Newtonus nimium amans erat, dicere quodam modo liceat, è datis illis pA, qB omnes in utrâque progreſſione dari. Si quidem unâ illâ inveſtigationis viâ, à Newtono traditâ, quæ omninò æſtimari poſſunt, harum juſtæ invenientur æſtimationes; quarum verò infinitas omnem æſtimationem fugiit, de iis id ipſum duntaxat innoteſcet, infinitas eas eſſe.

6. Cæterùm quæ de curvis trinominalibus diſſeruimus, ea, ſi operæ pretium eſſet, quod vix cenſeo, ad alias facilè transferentur. Illud verò minimè nobis reticendum eſt, Æquationes Canonicas primæ & ſecundæ claſſis, Stewartum Abredonenſem ante nos protuliſſe.

7. In ſecundo Propoſitionis hujus caſu, eam ſecutus eſt Newtonus demonſtrandi rationem, quâ nulla certè vel evidentiæ plus vel firmitatis habeat; ad æquationes tamen, ſive Canonicas, ſive pro re natâ, inferendas, minùs, ut videtur, accommodatam; tum propter creſcentem uſque, quo quantitas R plurium fuerit nominum, notarum multitudinem, tum verò maximè propter non ſatis manifeſtam, ex paucis quibuſdam formulis primis, continuationis legem. Nos igitur aliam propoſitionis in hoc caſu demonſtrationem apponemus, Newtonianâ non minùs generalem; quæ per viam planè contrariam, ex datis primùm infimi ordinis, quot opus fuerit, curvarum areis, elatiorum areas pedetentim venari docet; tum, converſis formulis, ab elatioribus ad inferiores deſcenſûs rationem indicat, et Æquationes Canonicas, è Newtonianis ægrè quidem eliciendas, tantùm non ſponte promit. Neque verò noſtra eſt hæc Demonſtratio, ſed Maclaurini. (Vide *Treatiſe of Fluxions by Colin Maclaurin, Edinburgh*, 1742, § 790) PROP. V. CAS. II. DEMON-

Sint Curvæ	A	B	C	D	E
Quarum Ordinatæ	$z^{\theta-1}R^{\lambda-1}$.	$z^{\theta+\eta-1}R^{\lambda-1}$.	$z^{\theta+2\eta-1}R^{\lambda-1}$.	$z^{\theta+3\eta-1}R^{\lambda-1}$.	$z^{\theta+4\eta-1}R^{\lambda-1}$.
Curvæ	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$
Quarum Ordinatæ	$z^{\theta-1}R^{\lambda}$.	$z^{\theta+\eta-1}R^{\lambda}$.	$z^{\theta+2\eta-1}R^{\lambda}$.	$z^{\theta+3\eta-1}R^{\lambda}$.	$z^{\theta+4\eta-1}R^{\lambda}$.
Curvæ	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$
Quarum Ordinatæ	$z^{\theta-1}R^{\lambda+1}$.	$z^{\theta+\eta-1}R^{\lambda+1}$.	$z^{\theta+2\eta-1}R^{\lambda+1}$.	$z^{\theta+3\eta-1}R^{\lambda+1}$.	$z^{\theta+4\eta-1}R^{\lambda+1}$.
Curvæ	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$
Quarum Ordinatæ	$z^{\theta-1}R^{\lambda+2}$.	$z^{\theta+\eta-1}R^{\lambda+2}$.	$z^{\theta+2\eta-1}R^{\lambda+2}$.	$z^{\theta+3\eta-1}R^{\lambda+2}$.	$z^{\theta+4\eta-1}R^{\lambda+2}$.

Dico primum datâ ſerie infimâ A, B, C, D, E, &c. ſuperiores omnes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, uſque dari.

Nam cùm $\mathfrak{A}$ area ſit curvæ cujuſdam, cujus abſciſſa eſt z, ordinata $z^{\theta-1}R^{\lambda}$; et cùm fluxio areæ æqualis ſit rectangulo ſub ordinatâ et fluxione abſciſſæ, erit $\dot{\mathfrak{A}} = z^{\theta-1}\dot{z}R^{\lambda} = z^{\theta-1}\dot{z}R\,R^{\lambda-1}$. Hoc eſt, propter $R = e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} +$, $\dot{\mathfrak{A}} = z^{\theta-1}\dot{z}R^{\lambda-1} \times \overline{e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} +}$, $= ez^{\theta-1}\dot{z}R^{\lambda-1} + fz^{\theta+\eta-1}\dot{z}R^{\lambda-1} + gz^{\theta+2\eta-1}\dot{z}R^{\lambda-1} + hz^{\theta+3\eta-1}R^{\lambda-1} +$, Jam verò $ez^{\theta-1}\dot{z}R^{\lambda-1} = e\dot{A}$. $fz^{\theta+\eta-1}\dot{z}R^{\lambda-1} = f\dot{B}$. $gz^{\theta+2\eta-1}\dot{z}R^{\lambda-1} = g\dot{C}$. $hz^{\theta+3\eta-1}\dot{z}R^{\lambda-1} = h\dot{D}$.

Quare $\dot{\mathfrak{A}} = e\dot{A} + f\dot{B} + g\dot{C} + h\dot{D} +$,

et $\mathfrak{A} = eA + fB + gC + hD +$,

Simili

Prop. VIII. quantitates R & S ſunt binomia, vel ſi dentur Areæ trium ex curvis, ſi R & S conjunctim ex quinque nominibus conſtant, et ſic deinceps

stratio Maclaurini

Simili modo oſtendetur $\mathfrak{B} = e\text{B} + f\text{C} + g\text{D} + h\text{E} +$, $\mathfrak{C} = e\text{C} + f\text{D} + g\text{E} + h\text{F} +$, $\mathfrak{D} = e\text{D} + f\text{E} + g\text{F} + h\text{G}$, eodemque uſque modo.

Datâ igitur ſerie infimâ A, B, C, D, E, dabitur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, ſeries proximè ſuperior. Eodemque modo, quo hæc ex ſerie infimâ, ex hâc invenietur hâc ipſâ proximè ſuperior. Nimirum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$. Datâque $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, rurſum ex illâ invenietur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, eodem modo quo $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. ex infimâ.

Eodemque modo pedetentim uſque progredi licebit.

Dico præterea ex unâ areâ A, ſi quantitas R duobus tantùm conſtet nominibus, è duabus A, B, ſi tribus, è tribus A, B, C, ſi quatuor; ſemper ex areis infimæ ſeriei unâ paucioribus quàm ſunt nomina quantitatis R, areas omnes progreſſionum omnium uſque dari.

Nam ſi R duorum ſit nominum, datâ A, dabitur B et ſeries tota A, B, C, D, per caſum primum; quare reliquæ omnes dabuntur. Rurſum ſi R trium ſit nominum, datis areis duabus A, B, dabitur tertia C, et ſeries tota A, B, C, D, per caſum primum. Unde ſeries reliquæ dabuntur. Et, quotvis demum nominum ſit quantitas R, viget uſque ſimilis ratiocinatio.

Atque hæc eſt demonſtratio illa cujus fundamenta Vir ſummus Maclaurinus jecit, minùs tamen accuratè eam, quàm nos fecimus, deduxit.

8. Jam verò Æquationes Canonicæ conficientur, ſi in æquationem illam $\mathfrak{A} = e\text{A} + f\text{B} + g\text{C} + h\text{D}$, illarum B, C, D, E æſtimationes ſubſtituantur, ex Canonicis primi caſûs petendæ. Nempe, ſi R duobus conſtet nominibus

Æquat. Canon Prop. VII. Cas. II.

1. $\mathfrak{A} = e\text{A} + f\text{B} = \dfrac{\lambda\eta e\text{A} + z^{\theta}\text{R}^{\lambda}}{\theta + \lambda\eta}$. (per Canonic. 1m primæ Claſſis Caſûs prioris)

2. $\text{A} = \dfrac{\overline{\theta + \lambda\eta}\,\mathfrak{A} - z^{\theta}\text{R}^{\lambda}}{\lambda\eta e}$.

Si R tribus conſtet nominibus,

1. $\mathfrak{A} = e\text{A} + f\text{B} + g\text{C} = \dfrac{2\lambda\eta e\text{A} + \lambda\eta f\text{B} + z^{\theta}\text{R}^{\lambda}}{\theta + 2\lambda\eta}$. (Caſ. 1. Claſſ. 2. Can. 1.)

2. $\text{B} = \dfrac{\overline{\theta + 2\lambda\eta}\,\mathfrak{A} - 2\lambda\eta e\text{A} - z^{\theta}\text{R}^{\lambda}}{\lambda\eta f}$.

3. $\text{A} = \dfrac{\overline{\theta + 2\lambda\eta}\,\mathfrak{A} - \lambda\eta f\text{B} - z^{\theta}\text{R}^{\lambda}}{2\lambda\eta e}$.

Si R quatuor conſtet nominibus.

1. $\mathfrak{A} = e\text{A} + f\text{B} + g\text{C} + h\text{D} = \dfrac{3\lambda\eta e\text{A} + 2\lambda\eta f\text{B} + \lambda\eta g\text{C} + z^{\theta}\text{R}^{\lambda}}{\theta + 3\lambda\eta}$. (Caſ. 1. Claſſ. 3. Can. 1.)

2. $\text{C} = \dfrac{-z^{\theta}\text{R}^{\lambda} - 3\lambda\eta e\text{A} - 2\lambda\eta f\text{B} + \overline{\theta + 3\lambda\eta}\,\mathfrak{A}}{\lambda\eta g}$.

3. $\text{B} = \dfrac{-z^{\theta}\text{R}^{\lambda} - 3\lambda\eta e\text{A} - \lambda\eta g\text{C} + \overline{\theta + 3\lambda\eta}\,\mathfrak{A}}{2\lambda\eta f}$.

4. $\text{A} = \dfrac{-z^{\theta}\text{R}^{\lambda} - 2\lambda\eta f\text{B} - \lambda\eta g\text{C} + \overline{\theta + 3\lambda\eta}\,\mathfrak{A}}{3\lambda\eta e}$.

Denique ſi R tot conſtet nominibus quot ſunt in numero $\sigma + 1$ unitates.

1. $\mathfrak{A} = \dfrac{\sigma\lambda\eta e\text{A} + \overline{\sigma - 1}\,\lambda\eta f\text{B} + \overline{\sigma - 2}\,\lambda\eta g\text{C} + \overline{\sigma - 3}\,\lambda\eta h\text{D} * * * + \lambda\eta\nu\text{Y} + z^{\theta}\text{R}^{\lambda}}{\theta + \sigma\lambda\eta}$. (per Canonicam generalem Caſus prioris)

2. Y =

deinceps in infinitum: dico, quòd dabuntur Areæ curvarum omnium. Prop. VIII.

Demonſtratur ad modum propoſitionis ſuperioris (dd).

PROP.

$$2.\ \mathrm{Y} = \frac{-z^{\theta}\mathrm{R}^{\lambda} - \sigma\lambda\nu e\mathrm{A} - \overline{\sigma-1}\,\lambda nf\mathrm{B} - \overline{\sigma-2}\,\lambda ng\mathrm{C} - \overline{\sigma-3}\,\lambda nh\mathrm{D} * * * + \overline{\theta+\sigma\lambda n}\,\mathfrak{A}}{\lambda n y}.$$

$$\mathrm{A} = \frac{-z^{\theta}\mathrm{R}^{\lambda} - \overline{\sigma-1}\,\lambda nf\mathrm{B} - \overline{\sigma-2}\,\lambda ng\mathrm{C} - \overline{\sigma-3}\,\lambda nh\mathrm{D} * * * - \lambda ny\mathrm{Y} + \overline{\theta+\sigma\lambda n}\,\mathfrak{A}}{\sigma\lambda ne}.$$

9. Per has æquationes definiuntur etiam areæ $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, pro θ, illis $\theta+n$, $\theta+2n$, $\theta+3n$, pro illis A, B, C, D, illis, B, C, D, pro λ illis $\lambda+1$, $\lambda+2$, $\lambda+3$, pro A, B, C, D, E, illis $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, vel $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, vel $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, ritè ſubſtitutis. Quæ ſi breviùs fortè dicta ſunt, ex iis tamen apertiſſimè, credo, intelligantur, quæ ad Canonicas prioris caſûs enodatè admodum diſſeruimus.

(dd) DEMONSTRATIO PROPOSITIONIS OCTAVÆ.

CAS. I.

Sunto Curvarum trium Ordinatæ $pz^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, $qz^{\theta+n-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, $rz^{\theta+2n-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, et Areæ $p\mathrm{A}$, $q\mathrm{B}$, $r\mathrm{C}$, exiſtente R quantitate trium nominum, $e+fz^{n}+gz^{2n}$, S duorum, $k+lz^{n}$. Et cùm per Prop. IV. ſit $z^{\theta}\mathrm{R}^{\lambda}\mathrm{S}^{\mu}$ area curvæ, cujus ordinata eſt $\theta ek + \overline{\overline{\theta+\lambda n}\,fk + \overline{\theta+\mu n}\,el}\,z^{n} + \overline{\overline{\theta+2\lambda n}\,gk + \overline{\theta+\lambda n+\mu n}\,fl}\,z^{2n} + \overline{\theta+2\lambda n+\mu n}\,glz^{3n}$ in $z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, ſubduc ordinatas et areas priores de areâ et ordinatâ poſteriore, et manebit $\overline{\theta ek-p} + \overline{\overline{\theta+\lambda n}\,fk + \overline{\theta+\mu n}\,el - q}\,z^{n} + \overline{\overline{\theta+2\lambda n}\,gk + \overline{\theta+\lambda n+\mu n}\,fl - r}\,z^{2n} + \overline{\theta+2\lambda n+\mu n}\,glz^{3n} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$ Ordinata novæ Curvæ, et $z^{\theta}\mathrm{R}^{\lambda}\mathrm{S}^{\mu} - p\mathrm{A} - q\mathrm{B} - r\mathrm{C}$ ejuſdem Area. Et cùm hoc ſemper obtineat, in omni literarum, p, q, r, æſtimatione, ut arearum differentia ſit Area curvæ, cujus Ordinata ordinatarum differentiæ æqualis ſit; obtinebit etiam ſi p, q, r, earum ſumantur magnitudinum, quæ efficiant, ut ordinatarum differentiæ membra tria prima in nihilum ſingulatim abeant. Hoc autem efficietur, ſi ſumatur $p = \theta ek$, $q = \overline{\theta+\lambda n}\,fk + \overline{\theta+\mu n}\,el$. $r = \overline{\theta+2\lambda n}\,gk + \overline{\theta+\lambda n+\mu n}\,fl$. Sumantur igitur p, q, r, iſtarum magnitudinum. Itaque Ordinata evadet $\overline{\theta+2\lambda n+\mu n}\,glz^{3n} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, et Area $z^{\theta}\mathrm{R}^{\lambda}\mathrm{S}^{\mu} - \theta ek\mathrm{A} \begin{array}{l} -\overline{\theta+\lambda n}\,fk \\ -\overline{\theta+\mu n}\,el \end{array} \mathrm{B} \begin{array}{l} -\overline{\theta+2\lambda n}\,gk \\ -\overline{\theta+\lambda n+\mu n}\,fl \end{array} \mathrm{C}$. Divide utramque à $\overline{\theta+2\lambda n+\mu n}\,gl$, et Aream diviſione factam dic D; et, aſſumtâ utcunque s, erit $s\mathrm{D}$ Area Curvæ, cujus Ordinata eſt $sz^{\theta+3n-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$. Et quâ ratione ex areis $p\mathrm{A}$, $q\mathrm{B}$, $r\mathrm{C}$ aream $s\mathrm{D}$ ordinatæ $sz^{\theta+3n-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$ congruentem invenimus, licebit ex areis $q\mathrm{B}$, $r\mathrm{C}$, $s\mathrm{D}$, quartam, puta $t\mathrm{E}$, ordinatæ $tz^{\theta+4n-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$ congruentem invenire; eodemque deinceps uſque modo.

Quòd ſi R, S utraque duorum tantùm nominum fuerit, vel trium utraque, pluriumve, eadem erit demonſtrandi ratio.

Hic eſt caſus Curvarum, in quibus ipſius z index θ augetur vel diminuitur perpetuâ additione, vel ſubductione, quantitatis n. Secundus caſus eſt Curvarum, in quibus index λ vel μ augetur, vel diminuitur, unitatibus.

CAS. II.

Puta indicem λ unitatibus perpetim augeri. Ordinatæ $pz^{\theta-1}\mathrm{R}^{\lambda}\mathrm{S}^{\mu-1}$, $qz^{\theta+n-1}\mathrm{R}^{\lambda}\mathrm{S}^{\mu-1}$, $rz^{\theta+2n-1}\mathrm{R}^{\lambda}\mathrm{S}^{\mu-1}$, ſi in R, ſeu $e+fz^{n}+gz^{2n}$, ducantur, ac deinde ad R viciſſim applicentur, evadunt $\overline{pe+pfz^{n}+pgz^{2n}} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, $\overline{qez^{n}+qfz^{2n}+qgz^{3n}} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, $\overline{rez^{2n}+rfz^{3n}+rgz^{4n}} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$. Et per Prop. IV. eſt $z^{\theta}\mathrm{R}^{\lambda}\mathrm{S}^{\mu}$ Area Curvæ, cujus Ordinata eſt $\mathrm{a}+\mathrm{b}z^{n}+\mathrm{c}z^{2n}+\mathrm{d}z^{3n} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$; deſignantibus a, b, c, d, coefficientes è formulâ Propoſitionis IV. deſumendas. Et $bz^{\theta+n}\mathrm{R}^{\lambda}\mathrm{S}^{\mu}$ Area erit curvæ, cujus Ordinata eſt $\grave{\mathrm{a}}bz^{n}+\grave{\mathrm{b}}bz^{2n}+\grave{\mathrm{c}}bz^{3n}+\grave{\mathrm{d}}bz^{4n} \times z^{\theta-1}\mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$, de-

ſignantibus

DEMONSTRATIO

ſignantibus notis, $\grave{a}$, $\grave{b}$, $\grave{c}$, $\grave{d}$, coefficientes è formulâ Propoſitionis IV. deſumendas. Et harum quinque arearum ſumma eſt $p\text{A} + q\text{B} + r\text{C} + z^{\theta} \text{R}^{\lambda} \text{S}^{\mu} + bz^{6+\eta} \text{R}^{\lambda} \text{S}^{\mu}$, et ſumma ordinatarum

$$\left.\begin{array}{l} \text{a} + \text{b}z^{\eta} + \text{c}z^{2\eta} + \text{d}z^{3\eta} \\ \quad + \grave{\text{a}}bz^{\eta} + \grave{\text{b}}bz^{2\eta} + \grave{\text{c}}bz^{3\eta} + \grave{\text{d}}bz^{4\eta} \\ + pe + pfz^{\eta} + pgz^{2\eta} \\ \quad + qez^{\eta} + qfz^{2\eta} + qgz^{3\eta} \\ \qquad + rez^{2\eta} + rfz^{3\eta} + rgz^{4\eta} \end{array}\right\} \times z^{\theta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}.$$

Jam ſi ſummæ ordinataram membra præter tertium omnia nihilo æquentur ſingulatim, nacti erimus quatuor æquationes, ad quatuor illas, b, q, p, r, quæ indefinitè poſitæ ſunt, definiendas. Tum ſi in ſymbolo ſummæ arearum, $p\text{A} + q\text{B} + r\text{C} + z^{\theta} \text{R}^{\lambda} \text{S}^{\mu} + bz^{\prime+\eta} \text{R}^{\lambda} \text{S}^{\mu}$, pro p, q, r, b ponantur earum æſtimationes, quæ ex æquationibus illis quatuor venerint; habebimus Aream Curvæ, cujus Ordinata erit $\overline{\text{c} + \grave{\text{b}}b + pg + qf + re}\, z^{2\eta} \times z^{\theta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. Et utramque, aream dico et ordinatam, ab illâ $\overline{\text{c} + \grave{\text{b}}b + pg + qf + re}$ dividendo, habebimus Aream curvæ, cujus Ordinata erit $z^{\theta+2\eta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. Dicatur illa area F.

Simili ratione, ſi ſummæ ordinatarum membra præter ſecundum omnia nihilo æquentur, invenietur Area Curvæ, cujus Ordinata erit $z^{\theta+\eta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. Dicatur area illa E.

Atque rurſum ſi membra omnia ſummæ ordinatarum præter primum nihilo æquentur, invenietur Area curvæ, cujus Ordinata erit $z^{\theta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. Dicatur area illa D. Et quâ ratione ex areis tribus A, B, C, inventæ ſunt areæ D, E, F, ex tribus illis D, E, F, inveniri poſſunt tres aliæ G, H, K; eodemque deinceps uſque modo.

A L I T E R.

Et ad computandi uſum magis accommodatè.

SINT Curvæ A, B, C, D,
Quarum Ordinatæ $z^{\theta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. $z^{\theta+\eta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. $z^{\theta+2\eta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$. $z^{\theta+3\eta-1} \text{R}^{\lambda-1} \text{S}^{\mu-1}$.

Curvæ $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$,
Quarum Ordinatæ $z^{\theta-1} \text{R}^{\lambda} \text{S}^{\mu-1}$. $z^{\theta+\eta-1} \text{R}^{\lambda} \text{S}^{\mu-1}$. $z^{\theta+2\eta-1} \text{R}^{\lambda} \text{S}^{\mu-1}$. $z^{\theta+3\eta-1} \text{R}^{\lambda} \text{S}^{\mu-1}$.

Curvæ $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$,
Quarum Ordinatæ $z^{\theta-1} \text{R}^{\lambda+1} \text{S}^{\mu-1}$. $z^{\theta+\eta-1} \text{R}^{\lambda+1} \text{S}^{\mu-1}$. $z^{\theta+2\eta-1} \text{R}^{\lambda+1} \text{S}^{\mu-1}$. $z^{\theta+3\eta-1} \text{R}^{\lambda+1} \text{S}^{\mu-1}$.

Curvæ $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$,
Quarum Ordinatæ $z^{\theta-1} \text{R}^{\lambda+2} \text{S}^{\mu-1}$. $z^{\theta+\eta-1} \text{R}^{\lambda+2} \text{S}^{\mu-1}$. $z^{\theta+2\eta-1} \text{R}^{\lambda+2} \text{S}^{\mu-1}$. $z^{\theta+3\eta-1} \text{R}^{\lambda+2} \text{S}^{\mu-1}$.

Et progreſſio infinita ſit.

Dico primùm datâ ſerie infinitâ A, B, C, D, E, ſuperiores omnes uſquę dati.

Erit enim $\mathfrak{A} = e\text{A} + f\text{B} + g\text{C} + h\text{D} +$,
Erit etiam $\mathfrak{B} = e\text{B} + f\text{C} + g\text{D} + h\text{E} +$,
$\mathfrak{C} = e\text{C} + f\text{D} + g\text{E} + h\text{F}$,

Et ſimili deinceps uſque modo. Hæc enim ſimili prorſus argumentatione oſtendentur, quâ in ſuperiore propoſitione oſtenſa ſunt. Datâ igitur ſerie infinitâ A, B, C, D, E, dabitur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, proximè ſuperior. Tum hâc datâ, dabitur etiam huic proxima, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, Eodemque modo pedetentim uſque progredi licebit.

Dico præterea è duabus areis A, B, ſi quantitates R, S, duorum utraque ſint nominum, è tribus A, B, C, ſi altera duorum altera trium fuerit, ſemper ex areis infimæ ſeriei binario paucioribus quàm ſunt nomina quantitatum, R, S, ſimul ſumpta, Areas omnes progreſſionum omnium uſque dari.

Nam ſi, R, S, duorum tantùm utraque ſit nominum, datis A, B, dabitur tota ſeries A, B, C, D, per Caſum primum. Quare et reliquæ dabuntur, per ea quæ modò ſunt oſtenſa. Rurſum ſi duarum R, S, altera duorum, altera trium nominum fuerit, datis tribus A, B, C, dabitur ſeries tota prima, per caſum primum. Unde rurſum reliquæ dabuntur. Et quotvis demum nominum quantitatum R, S hæc vel illa fuerit, vigebit uſque ſimilis ratiocinatio.

Eodem modo Propoſitionem approbare licebit, ſi index μ augeatur, vel diminuatur, unitatibus.

CAS.

C A S. III.

Si igitur tam index μ quàm index λ, perpetuâ additione, vel ſubductione, unitatum augeatur vel minuatur, dabuntur Areæ ordinatis ſingulatim reſpondentes.

C A S. IV.

Caſum ſecundum cum primo conjungendo, ſi tam index θ, perpetuâ additione vel ſubductione quantitatis n, quam indicum λ vel μ alteruter, perpetuâ additione vel ſubductione unitatis, augeatur vel minuatur, dabuntur Areæ ordinatis ſingulatim reſpondentes.

C A S. V.

Et caſum tertium cum primo conjungendo, ſi tam index θ, perpetuâ additione vel ſubductione quantitatis n, quam uterque λ, μ, perpetuâ unitatum additione vel ſubductione, augeatur vel minuatur, dabuntur Areæ ordinatis ſingulatim reſpondentes.

C A S. VI.

Et universè, ſi dentur Areæ duabus pauciores quàm ſunt nomina quantitatum R, S ſimul ſumpta, dabuntur omnes. Nam quotcunque nominum illæ R, S, fuerint, eadem erit demonſtrandi ratio, ſicut jam antè in caſu primo monuimus.

2. Reſtat ut æquationes Canonicas deſcribamus.

CANONICÆ PRIMI CASUS.

I. Si, quantitates R, S utraque è duobus conſtent nominibus

$$1.\ C = \frac{z^{\theta}R^{\lambda}S^{\mu} - \theta ek\,A \begin{matrix} -\overline{\theta+\quad}fk \\ -\overline{\theta+\mu n}\,el \end{matrix} B}{\theta+\lambda n+\mu n\, fl}$$

$$2.\ B = \frac{z^{\theta}R^{\lambda}S^{\mu} - \theta ek\,A - \overline{\theta+\lambda n+\mu n}\,fl\,C}{\overline{\theta+\lambda n}\,fk + \overline{\theta+\mu n}\,el}$$

$$3.\ A = \frac{z^{\theta}R^{\lambda}S^{\mu} \begin{matrix} -\overline{\theta+\lambda n}\,fk \\ -\overline{\theta+\mu n}\,l \end{matrix} B - \overline{\theta+\lambda n+\mu n}\,fl\,C}{\theta ek}$$

II. Si R ex tribus conſtet nominibus, S è duobus.

$$1.\ D = \frac{z^{\theta}R^{\lambda}S^{\mu} - \theta ck\,A \begin{matrix} -\overline{\theta+\lambda n}\,fk \\ -\overline{\theta+\mu n}\,el \end{matrix} B \begin{matrix} -\overline{\theta+2\lambda n}\,gk \\ -\overline{\theta+\lambda n+\mu n}\,fl \end{matrix} C}{\overline{\theta+2\lambda n+\mu n}\,gl}$$

$$2.\ C = \frac{z^{\theta}R^{\lambda}S^{\mu} - \theta ek\,A \begin{matrix} -\overline{\theta+\lambda n}\,fk \\ -\overline{\theta+\mu n}\,el \end{matrix} B - \overline{\theta+2\lambda n+\mu n}\,gl\,D}{\overline{\theta+2\lambda n}\,gk + \overline{\theta+\lambda n+\mu n}\,fl}$$

$$3.\ B = \frac{z^{\theta}R^{\lambda}S^{\mu} - \theta ck\,A \begin{matrix} -\overline{\theta+2\lambda\quad}\,gk \\ -\overline{\theta+\lambda n+\mu n}\,fl \end{matrix} C - \overline{\theta+2\lambda n+\mu n}\,gl\,D}{\overline{\theta+\lambda n}\,fk + \overline{\theta+\mu n}\,cl}$$

$$4.\ A = \frac{z^{\theta}R^{\lambda}S^{\mu} \begin{matrix} -\overline{\theta+\lambda n}\,fk \\ -\overline{\theta+\mu n}\,el \end{matrix} B \begin{matrix} -\overline{\theta+2\lambda n}\,gk \\ -\overline{\theta+\lambda n+\mu n}\,fl \end{matrix} C - \overline{\theta+2\lambda n+\mu n}\,gl\,D}{\theta ck}$$

III. Si R, S ex tribus utraque conſtet nominibus

Prop. IX.

PROP. IX. THEOR. VII.

Æquantur Curvarum Areæ inter ſe quarum Ordinatæ ſunt reciprocè ut fluxiones abſciſſarum(ee).

Nam contenta ſub ordinatis & fluxionibus Abſciſſarum erunt æqualia; & fluxiones arearum ſunt ut hæc contenta.

COROL. I.

Si aſſumatur relatio quævis inter abſciſſas duarum curvarum, & inde per Prop. 1. quæratur relatio fluxionum abſciſſarum, & ponantur Ordinatæ reciprocè proportionales fluxionibus, inveniri poſſunt innumeræ Curvæ, quarum Areæ ſibi mutuò æquales erunt (ff).

COROL. II.

Sic enim curva omnis, cujus hæc eſt ordinata $z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\ \&c.}\,\big|^{\lambda}$, aſſumendo quantitatem quamvis pro y, & ponendo

Æquat. Canon. Prop. VIII. Cas. I.

$$1.\ \mathrm{E}=\frac{z^{\theta}\mathrm{R}^{\lambda}s^{\mu}-\theta ek\mathrm{A}\begin{array}{l}-\overline{\theta+\lambda n}\,fk\\-\overline{\theta+\mu n}\,el\end{array}\mathrm{B}\begin{array}{l}-\overline{\theta+2\lambda n}\,gk\\-\overline{\theta+\lambda n+\mu n}\,fl\\-\overline{\theta+\mu n}\,em\end{array}\mathrm{C}\begin{array}{l}-\overline{\theta+2\lambda n+\mu n}\,gl\\-\overline{\theta+\lambda n+2\mu n}\,fm\end{array}\mathrm{D}}{\overline{\theta+2\lambda n+2\mu n}\,gm}$$

$$2.\ \mathrm{D}=\frac{z^{\theta}\mathrm{R}^{\lambda}s^{\mu}-\theta ek\,\mathrm{A}\begin{array}{l}-\overline{\theta+\lambda n}\,fk\\-\overline{\theta+\mu n}\,el\end{array}\mathrm{B}\begin{array}{l}-\overline{\theta+2\lambda n}\,gk\\-\overline{\theta+\lambda n+\mu n}\,fl\\-\overline{\theta+2\mu n}\,em\end{array}\mathrm{C}\ -\overline{\theta+2\lambda n+2\mu n}\,gm\,\mathrm{E}}{\overline{\theta+2\lambda n+\mu n}\,gl+\overline{\theta+\lambda n+2\mu n}\,fm}$$

$$3.\ \mathrm{C}=\frac{z^{\theta}\mathrm{R}^{\lambda}s^{\mu}-\theta ek\,\mathrm{A}\begin{array}{l}-\overline{\theta+\lambda n}\,fk\\-\overline{\theta+\mu n}\,el\end{array}\mathrm{B}\begin{array}{l}-\overline{\theta+2\lambda n+\mu n}\,gl\\-\overline{\theta+\lambda n+2\mu n}\,fm\end{array}\mathrm{D}\ -\overline{\theta+2\lambda n+2\mu n}\,gm\,\mathrm{E}}{\overline{\theta+2\lambda n}\,gk+\overline{\theta+\lambda n+\mu n}\,fl+\overline{\theta+2\mu n}\,em}$$

$$4.\ \mathrm{B}=\frac{z^{\theta}\mathrm{R}^{\lambda}s^{\mu}-\theta ek\,\mathrm{A}\begin{array}{l}-\overline{\theta+2\lambda n}\,gk\\-\overline{\theta+\lambda n+\mu n}\,fl\\-\overline{\theta+2\mu n}\,em\end{array}\mathrm{C}\begin{array}{l}-\overline{\theta+2\lambda n+\mu n}\,gl\\-\overline{\theta+\lambda n+2\mu n}\,fm\end{array}\mathrm{D}\ -\overline{\theta+2\lambda n+2\mu n}\,gm\,\mathrm{E}}{\overline{\theta+\lambda n}\,fk+\overline{\theta+\mu n}\,el}$$

$$5.\ \mathrm{A}=\frac{z^{\theta}\mathrm{R}^{\lambda}s^{\mu}\begin{array}{l}-\overline{\theta+\lambda n}\,fk\\-\overline{\theta+\mu n}\,el\end{array}\mathrm{B}\begin{array}{l}-\overline{\theta+2\lambda n}\,gk\\-\overline{\theta+\lambda n+\mu n}\,fl\\-\overline{\theta+2\mu n}\,em\end{array}\mathrm{C}\begin{array}{l}-\overline{\theta+2\lambda n+\mu n}\,gl\\-\overline{\theta+\lambda n+2\mu n}\,fm\end{array}\mathrm{D}\ -\overline{\theta+2\lambda n+2\mu n}\,gm\,\mathrm{E}}{\theta ek}$$

IV. Si

ponendo $\frac{n}{\nu}=s$, & $z^s=x$, migrat in aliam ſibi æqualem, cujus Ordinata eſt $\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{e+fx^\nu+gx^{2\nu}+\&c.}|^\lambda$ (gg). Prop. IX. Cor. III & IV.

COROL. III.

Et Curva omnis, cujus Ordinata eſt $z^{\theta-1}\times\overline{a+bz^n+cz^{2n}+\&c.}$ $\times\overline{e+fz^n+gz^{2n}+\&c.}|^\lambda$, aſſumendo quantitatem quamvis pro ν, & ponendo $\frac{n}{\nu}=s$, & $z^s=x$, migrat in aliam ſibi æqualem, cujus Ordinata eſt $\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{a+bx^\nu+cx^{2\nu}+\&c.}\times\overline{e+fx^\nu+gx^{2\nu}+\&c.}|^\lambda$.

COROL. IV.

Et Curva omnis, cujus Ordinata eſt $z^{\theta-1}\times\overline{a+bz^n+cz^{2n}+\&c.}|$ $\times\overline{e+fz^n+gz^{2n}+\&c.}|^\lambda\times\overline{k+lz^n+mz^{2n}+\&c.}|^\mu$, aſſumendo quantitatem quamvis pro ν, & ponendo $\frac{\nu}{n}=s$, & $z^s=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{a+bx^\nu+cx^{2\nu}+\&c.}$ $\times\overline{e+fx^\nu+gx^{2\nu}+\&c.}|^\lambda\times\overline{k+lx^\nu+mx^{2\nu}+\&c}|^\mu$.

IV. Si R ex quatuor conſtet nominibus, S ex duobus.

$$1.\ E=\frac{z^\theta R^\lambda S^\mu - \overline{\theta+\lambda n}\,fk\,B - \overline{\theta+2\lambda n}\,gk\,C - \overline{\theta+3\lambda n}\,hk\,D \atop \qquad - \overline{\theta+\mu n}\,el\,B - \overline{\theta+\lambda n+\mu n}\,fl\,C - \overline{\theta+2\lambda n+\mu n}\,gl\,D}{\overline{\theta+3\lambda n+\mu n}\,hl}$$

(ee) Dummodo ſimul à nihilo generari inceperint. Aliter enim altera alterâ dato major erit (Geometr. Flux. Prop. II.) Arearum autem partes ſimul genitæ vel ſic etiam æquales erunt.

(ff) Vide Geometr. Analyt. Prop. VIII.

(gg) Ponatur enim $z^{\theta-1}\times\overline{e+fz^n+gz^{2n}+}|^\lambda,=y$, et ſit v Ordinata, x Abſciſſa curvæ cujuſdam, cujus Area areæ alterius, cujus Ordinata ſit y Abſciſſa z, ſemper ſit æqualis. Abſciſſarum igitur fluxiones inverſas Ordinatarum inter ſe proportiones gerunt (ex hâc Prop.) Hoc eſt $\dot{z}:\dot{x}=v:y$. Sed propter $z^s=x$, erit $\dot{x}=sz^{s-1}\dot{z}$. Quare $v:y=\dot{z}:sz^{s-1}\dot{z}=1:sz^{s-1}$. Quare $v=\frac{y}{sz^{s-1}}=\frac{1}{s}z^{\theta-s}\times\overline{e+fz^n+gz^{2n}+,}|^\lambda$. Sed propter $\frac{n}{\nu}=s$, erit $\frac{1}{s}=\frac{\nu}{n}$. Et propter $z^s=x$, erit $z=x^{\frac{1}{s}}$, et $z^\theta=x^{\frac{\theta}{s}}=x^{\frac{\nu\theta}{n}}$, et $z^{\theta-s}=x^{\frac{\nu\theta}{n}-1}=x^{\frac{\nu\theta-n}{n}}$. Et $z^n=x^{\frac{n}{s}}=x^\nu$. Ergo $z^{2n}=x^{2\nu}$; $z^{3n}=x^{3\nu}$; * * *; $z^{mn}=x^{m\nu}$. Quare $\frac{1}{s}z^{\theta-s}\times\overline{e+fz^n+gz^{2n}+,}|^\lambda=v=\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{e+fx^\nu+gx^{2\nu}+,}|^\lambda$. Q. E. D. Demonſtratio Cor. II.

Similem hujus Corollarii demonſtrationem in commentariis ſuis Stewartus attulit.

Prop. IX. Cor. V. & VI.

COROL. V.

Et Curva omnis cujus Ordinata eſt $z^{\theta-1} \times \overline{e+fz^n+gz^{2n}+\&c.}|^{\lambda}$, ponendo $\frac{1}{z}=x$, migrat in aliam ſibi æqualem, cujus Ordinata eſt $\frac{1}{x^{\theta+1}} \times \overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda}$: id eſt $\frac{1}{x^{\theta+1+n\lambda}} \times \overline{f+ex^n}|^{\lambda}$, ſi duo ſunt nomina in vinculo radicis: vel $\frac{1}{x^{\theta+1+2n\lambda}} \times \overline{g+fx^n+ex^{2n}}|^{\lambda}$, ſi tria ſunt nomina; & ſic deinceps.

COROL. VI.

Et Curva omnis cujus Ordinata eſt $z^{\theta-1} \times \overline{e+fz^n+gz^{2n}+\&c.}|^{\lambda} \times \overline{k+lz^n+mz^{2n}+\&c.}|^{\mu}$, ponendo $\frac{1}{z}=x$, migrat in aliam ſibi æqualem, cujus Ordinata eſt $\frac{1}{x^{\theta+1}} \times \overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda} \times \overline{k+lz^{-n}+mx^{-2n}+\&c}|^{\mu}$: id eſt $\frac{1}{x^{\theta+1+n\lambda+n\mu}} \times \overline{f+ex^n}|^{\lambda} \times \overline{l+kx^n}|^{\mu}$, ſi bina ſunt nomina in vinculis radicum: vel $\frac{1}{x^{\theta+1+2n\lambda+n\mu}} \times \overline{g+fx^n+ex^{2n}}|^{\lambda} \times \overline{l+kx}|^{\mu}$, ſi tria ſunt nomina in vinculo radicis prioris, ac duo in vinculo poſterioris: & ſic in aliis.

Et

(hh) Editionem principem cum Stewarto ſequor. Editio Joneſii habuit $\frac{\eta-\delta}{\alpha\delta-\beta\eta}$. Vitioſè.

Demonſtratio Cor. VII.

(ii) Hujus Corollarii hanc ferè demonſtrationem Stewartus attulit.

Hæ ſunt æquationes à Newtono poſitæ, præter illam ad curvam ſcilicet cujus abſciſſa z, ordinata y.

1. $s=\frac{\eta-\delta}{\eta}$. 2. $x=\frac{1}{s}z^s$. 3. $\lambda=\frac{\eta-\delta}{\alpha\delta+\beta\eta}$.

Ex quibus has alias deducere licet.

1. $s^{\frac{1}{s}}x^{\frac{1}{s}}=z$. 2. $s^{\frac{\beta}{s}}x^{\frac{\beta}{s}}=z^{\beta}$. 3. $s^{\frac{\delta}{s}}x^{\frac{\delta}{s}}=z^{\delta}$ (ex ſecundâ poſitarum). 4. $\eta-\eta s=\delta$ (ex primâ poſitarum). 5. $\frac{s}{\frac{1}{\eta}\alpha\delta+\beta}=\lambda$ (per primam et tertiam poſitarum). 6. $\frac{\alpha+\beta-s\alpha}{s}=\frac{1}{\lambda}$ (per 4^m & 5^m deductarum).

Jam cùm Area curvæ cujus Abſciſſa eſt z Ordinata y, alterius areæ ſit æqualis, cujus Abſciſſa x, Ordinata v, idcirco abſciſſarum Fluxiones inverſas ordinatarum inter ſe proportiones gerent. Hoc eſt, erit $\dot{z}:\dot{x}=v:y$. Sed, per æquationem poſitarum ſecundam, $\dot{z}:\dot{x}=1:z^{s-1}$. Ergo $v:y=1:z^{s-1}=z:z^s$. Sed $z:z^s=s^{\frac{1}{s}}x^{\frac{1}{s}}:sx$ (per æquationes deductarum primam et poſitarum ſecundam). Ergo $v:y=s^{\frac{1}{s}}x^{\frac{1}{s}}:sx$. Hinc æquationes aliæ venient. Nempe, 7. $y=s^{\frac{s-1}{s}}$

Et nota, quòd Arcæ duæ æquales, in noviſſimis hiſce duobus Corollariis, jacent ad contrarias partes Ordinatarum. Si area in alterutrâ Curvâ adjacet abſciſſæ, area huic æqualis, in alterâ Curvâ, adjacet abſciſſæ productæ. PROP. IX. COR. VII. & VIII.

COROL. VII.

Si relatio inter Curvæ alicujus Ordinatam, y, & Abſciſſam z, definiatur per æquationem quamvis affectam hujus formæ, $y^{\alpha}\times\overline{e+fy^{\eta}z^{\delta}+gy^{2\eta}z^{2\delta}+hy^{3\eta}z^{3\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{\eta}z^{\delta}+my^{2\eta}z^{2\delta}+\&c.}$ hæc figura, aſſumendo $s=\frac{\eta-\delta}{\eta}$, $x=\frac{1}{s}z^{s}$, & $\lambda=\frac{\eta-\delta}{\alpha\delta+\beta\eta}$ (hh), migrat in aliam ſibi æqualem, cujus Abſciſſa x, ex datâ Ordinatâ v, determinatur per æquationem non affectam; $\frac{1}{s}v^{\alpha\lambda}\times\overline{e+fv^{\eta}+gv^{2\eta}+\&c.}|^{\lambda}\times\overline{k+lv^{\eta}+mv^{2\eta}+\&c.}|^{-\lambda}=x$ (ii).

COROL. VIII.

Si relatio inter Curvæ alicujus Ordinatam, y, & Abſciſſam, z, definitur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha}\times\overline{e+fy^{\eta}z^{\delta}+gy^{2\eta}z^{2\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{\eta}z^{\delta}+my^{2\eta}z^{2\delta}+\&c.}$$
$$+z^{\gamma}\times\overline{p+qy^{\eta}z^{\delta}+ry^{2\eta}z^{2\delta}+\&c.}$$

hæc figura, aſſumendo $s=\frac{\eta-\delta}{\eta}$, $x=\frac{1}{s}z^{s}$, $\mu=\frac{\alpha\delta+\beta\eta}{\eta-\delta}$, & $\nu=\frac{\alpha\delta+\gamma\eta}{\eta-\delta}$,

$s^{\frac{s-1}{s}}x^{\frac{s-1}{s}}v$. 8. $y^{\alpha}=s^{\frac{s\alpha-\alpha}{s}}x^{\frac{s\alpha-\alpha}{s}}v^{\alpha}$. 9. $y^{\eta}=s^{\frac{s\eta-\eta}{s}}x^{\frac{s\eta-\eta}{s}}v$. Cæterùm $\frac{s\eta-\eta}{s}=-\frac{\delta}{s}$ (per æquationem deductarum quartam.) Unde noviſſima in hanc migrabit. 10. $y^{\eta}=s^{-\frac{\delta}{s}}x^{-\frac{\delta}{s}}v^{\eta}$. Ex hâc autem et tertiâ deductarum efficietur, 11. $y^{\eta}z^{\delta}=v^{\eta}$. Ac propterea $y^{2\eta}z^{2\delta}=v^{2\eta}$. $y^{3\eta}z^{3\delta}=v^{3\eta}$. $y^{4\eta}z^{4\delta}=v^{4\eta}$, eodemque uſque modo. Jam in æquationem ad curvam, cujus abſciſſa z, ordinata y, in hujus inquam æquationem à Newtono poſitam, pro z^{β}, y^{α}, $y^{\eta}z^{\delta}$, ſubſtituantur earum æſtimationes, æquationibus deductarum 2ª. 8ª. et 11ª expoſitæ; et æquatio illa in hanc tranſmutabitur. $s^{\frac{s\alpha-\alpha}{s}}x^{\frac{s\alpha-\alpha}{s}}v^{\alpha}\times\overline{e+fv^{\eta}+gv^{2\eta}+hv^{3\eta}+,}=s^{\frac{\beta}{s}}x^{\frac{\beta}{s}}\times\overline{k+lv^{\eta}+mv^{2\eta}+nv^{3\eta}+}|$. Utramque hujus æquationis partem dividat quantitas, $x^{\frac{s\alpha-\alpha}{s}}\times s^{\frac{\beta}{s}}\times\overline{k+lv^{\eta}+mv^{2\eta}+}|$. Fiet $s^{\frac{s\alpha-\alpha-\beta}{s}}v^{\alpha}\times\overline{e+fv^{\eta}+gv^{2\eta}+hv^{3\eta}+,}|\times\overline{k+lv^{\eta}+mv^{2\eta}+}|^{-1}=x^{\frac{\beta+\alpha-s\alpha}{s}}$. Hoc eſt, per æquationem deductarum 6^m. $s^{-\frac{1}{\lambda}}v^{\alpha}\times\overline{e+fv^{\eta}+gv^{2\eta}+hv^{3\eta}+,}|\times\overline{\ +lv^{\eta}+mv^{2\eta}+}|^{-1}=x^{\frac{1}{\lambda}}$. Hujus denique pars utraque evehatur ad gradum illum cujus index quantitas λ; fiet $\frac{1}{s}v^{\alpha\lambda}\times\overline{e+fv^{\eta}+gv^{2\eta}+hv^{3\eta}+}|^{\lambda}\times\overline{k+lv^{\eta}+mv^{2\eta}+,}|^{-\lambda}=x$. Q. E. D. DEMONSTRATIO COR. VII.

PROP. IX. COR. IX.

migrat in aliam ſibi æqualem, cujus Abſciſſa, x, ex datâ Ordinatâ, v, determinatur per æquationem minùs affectam;

$$v^{\alpha} \times \overline{e+fv^{n}+gv^{2n}+\&c.} = s^{\mu}x^{\mu} \times \overline{k+lv^{n}+mv^{2n}+\&c.}$$
$$+ s^{\nu}x^{\nu} \times \overline{p+qv^{n}+rv^{2n}+\&c}$$ (kk).

COROL. IX.

Curva omnis cujus Ordinata eſt $\pi z^{\theta-1} \times \overline{\nu e+\overline{\nu+\eta}fz^{n}+\overline{\nu+2\eta}gz^{2n}+\&c.} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$ in $\overline{a+b\times\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}|^{\tau}}|^{\omega}$, ſi ſit $\theta=\lambda\nu$, & aſſumantur $x=\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}|^{\pi}$, $\sigma=\frac{\tau}{\pi}$, & $\vartheta=\frac{\lambda-\pi}{\pi}$, migrat in aliam ſibi æqualem, cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$ (ll). Et nota, quòd Ordinata prior in hoc corollario evadit ſimplicior, ponendo $\lambda=1$, vel ponendo $\tau=1$; & efficiendo, ut radix dignitatis

DEMONSTRATIO COR. VIII.

(kk) Hujus Corollarii veritatem cum Stewarto ſic oſtendimus.

Hæ ſunt æquationes à Newtono poſitæ, præter illam ad Curvam ſcilicet cujus abſciſſa z, ordinata y. 1. $s=\frac{\eta-\delta}{\eta}$. 2. $x=\frac{1}{s}z^{s}$. 3. $\mu=\frac{\alpha\delta+\beta\eta}{\eta-\delta}$. 4. $\nu=\frac{\alpha\delta+\gamma\eta}{\eta-\delta}$. Quarum duæ primæ eædem ſunt ac duæ primæ poſitarum corollarii ſuperioris. Cæterùm quatuor illarum $\dot{z}$, $\dot{x}$, v, y eadem erit in hoc, atque in illo fuit, proportionis convenientia. Quare deductæ etiam Corollarii ſuperioris, quarum ope æquationis ad Curvam tranſmutationes efficiebantur, hæ inquam omnes, præter quintam et ſextam, in hoc etiam obtinebunt. Utpote quæ omnes, præter illas, duabus poſitarum primis, & analogiâ illâ nituntur, quam analogiam huic cum illo communem diximus. Loco autem 5ᵃ & 6ᵃ corollarii ſuperioris, venient $\frac{s}{\frac{1}{\eta}\alpha\delta+\beta}=\frac{1}{\mu}$ (ex primâ et tertiâ poſitarum) et

$\frac{\alpha+\beta-\alpha s}{s}=\mu$ (è noviſſimâ, et quartâ deductarum Corollarii ſuperioris).

Ex primâ autem et quartâ poſitarum elicietur $\frac{s}{\frac{1}{\eta}\alpha\delta+\gamma}=\frac{1}{\nu}$: et ex hâc et quartâ deductarum Corollarii ſuperioris, $\frac{\alpha+\gamma-\alpha s}{s}=\nu$. Præterea ex ſecundâ poſitarum elicietur, $z^{\gamma}=s^{\frac{\gamma}{s}}x^{\frac{\gamma}{s}}$. Jam in æquationem ad curvam cujus abſciſſa z, ordinata y, in hujus inquam æquationem à Newtono poſitam, pro y^{α}, z^{β}, $y^{\eta}z^{\delta}$ ſubſtituantur earum æſtimationes æquationibus deductarum ſuperioris 2ᵃ. 8ᵃ. et 11ᵃ. expoſitæ; et pro z^{γ} ſubſtituatur $s^{\frac{\gamma}{s}}x^{\frac{\gamma}{s}}$; et æquatio illa in hanc tranſmutabitur; $s^{\frac{s\alpha-\alpha}{s}}x^{\frac{s\alpha-\alpha}{s}}v^{\alpha}\times\overline{e+fv^{n}+gv^{2n}+hv^{3n}+,}=s^{\frac{\beta}{s}}x^{\frac{\beta}{s}}\times\overline{k+lv^{n}+mv^{2n}+,}+s^{\frac{\gamma}{s}}x^{\frac{\gamma}{s}}\times\overline{p+qv^{n}+rv^{2n}+}$. Hujus pars utraque à quantitate $s^{\frac{s\alpha-\alpha}{s}}x^{\frac{s\alpha-\alpha}{s}}$ dividatur; fiet $v^{\alpha}\times$

$$\overline{e+fv^{n}+gv^{2n}+hv^{3n}+,}=s^{\frac{\alpha+\beta-s\alpha}{s}}x^{\frac{\alpha+\beta-s\alpha}{s}}\times\overline{k+lv^{n}+mv^{2n}+,}$$
$$+s^{\frac{\alpha+\gamma-s\alpha}{s}}x^{\frac{\alpha+\gamma-s\alpha}{s}}\times\overline{p+qv^{n}+rv^{2n}+,}$$

Unde pro illis $\frac{\alpha+\beta-s\alpha}{s}$, $\frac{\alpha+\gamma-s\alpha}{s}$ ſcriptis μ, ν, veniet tandem æquatio quantitatum v, x, ab ipſo Newtono poſita. Q. E. D.

(ll) OSTEN-

tatis extrahi poſſit, cujus index eſt ω; vel etiam ponendo $\mu=-1$, & $\lambda=1=\tau=\sigma=\pi$, ut alios caſus prætercam. Prop. IX. Cor. X.

COROL. X.

Pro $ez^{\nu}+fz^{\nu+\eta}+gz^{\nu+2\eta}+$&c. $\nu ez^{\nu-1}+\overline{\nu+\eta}fz^{\nu+\eta-1}+\overline{\nu+2\eta}gz^{\nu+2\eta-1}+$&c. $k+lz^{\eta}+mz^{2\eta}+$ &c. et $\eta lz^{\eta-1}+2\eta mz^{2\eta-1}+$ &c. ſcribantur R, r, S, & s reſpectivè; & Curva omnis cujus Ordinata eſt $\overline{\pi Sr+\phi Rs}\times R^{\lambda-1}S^{\mu-1}\times\overline{aS^{-\nu}+bR^{\tau}}|^{\omega}$ (mm), ſi ſit $\frac{\mu-\nu\omega}{\lambda}=\frac{\upsilon}{\tau}=\frac{\phi}{\pi}$, $\frac{\tau}{\pi}=\sigma$, $\frac{\lambda-\pi}{\pi}=\vartheta$, & $R^{\pi}S^{\phi}=x$, migrat in aliam ſibi æqualem, cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. (nn) Et nota, quòd Ordinata prior evadit ſimplicior, ponendo unitates pro

(ll) Ostendimus quod Newtonus dicit hoc modo. Demonstratio. Cor. IX.

Hæ ſunt æquationes à Newtono poſitæ 1. $\theta=\lambda\nu$. 2. $x=\overline{ez^{\nu}+fz^{\nu+\eta}+gz^{\nu+2\eta}+,}|^{\tau}$. 3. $\sigma=\frac{\tau}{\pi}$. 4. $\vartheta=\frac{\lambda-\pi}{\pi}$. Ponantur præterea, 5. $\overline{\nu e+\overline{\nu+\eta}fz^{\eta}+\overline{\nu+2\eta}gz^{2\eta}+,}=R$. 6. $\overline{e+fz^{\eta}+gz^{2\eta}+,}=S$. 7. $\overline{ez^{\nu}+fz^{\nu+\eta}+gz^{\nu+2\eta}+}=\Sigma$. 8. $\overline{\nu ez^{\nu-1}+\overline{\nu+\eta}fz^{\nu+\eta-1}+\overline{\nu+2\eta}gz^{\nu+2\eta-1}+,}=\Gamma$.

Ex his æquationibus hæ aliæ deducentur.

1. $x=\Sigma^{\pi}$ (ex 2ª & 7ª poſitarum) Ergo 2. $\frac{\dot{x}}{\pi\Sigma^{\pi-1}}=\dot{\Sigma}$. 3. $\dot{\Sigma}=\Gamma\dot{z}$ (per 7m et 8m poſitarum) 4. $S=\frac{\Sigma}{z^{\nu}}$ (ex ſextâ et ſeptimâ poſitarum) Ex hâc autem, et eâ quæ primùm deducebatur, veniet 5. $S=x^{\frac{1}{\pi}}z^{-\nu}$.

Datæ autem curvæ Ordinata ſignificabitur hoc ſymbolo; $\pi z^{\theta-1}\times R\times S^{\lambda-1}\times\overline{a+b\Sigma^{\tau}}|^{\omega}$.

Curvæ tranſmutatæ, cujus abſciſſa x, Ordinata deſignetur literâ O.

Erit igitur $\dot{x}:\dot{z}=\pi z^{\theta-1}\times R\times S^{\lambda-1}\times\overline{a+b\Sigma^{\tau}}|^{\omega}:O$. Sed (per æquationes 2m & 3m deductarum) efficitur $\dot{x}:\dot{z}=\Gamma:\frac{1}{\pi\Sigma^{\pi-1}}$. Ergo $\pi z^{\theta-1}\times R\times S^{\lambda-1}\times\overline{a+b\Sigma^{\tau}}|^{\omega}:O=\Gamma:\frac{1}{\pi\Sigma^{\pi-1}}$. Ergo $O=\frac{z^{\theta-1}\times R}{\Gamma}\times\frac{S^{\lambda-1}}{\Sigma^{\pi-1}}\times\overline{a+b\Sigma^{\tau}}|^{\omega}$. Sed (per æquationem deductarum quintam) $S^{\lambda-1}=x^{\frac{\lambda-1}{\pi}}z^{\nu-\lambda\nu}$. Et (per æquationem deductarum primam) $\Sigma^{\pi-1}=x^{\frac{\pi-1}{\pi}}$. Quare $\frac{S^{\lambda-1}}{\Sigma^{\pi-1}}=x^{\frac{\lambda-\pi}{\pi}}z^{\nu-\lambda\nu}=x^{\vartheta}z^{\nu-\theta}$ (per æquationes poſitarum 4m & 1m). Hinc $O=\frac{z^{\nu-1}x^{\vartheta}\times R}{\Gamma}\times\overline{a+b\Sigma^{\tau}}|^{\omega}$. Sed $z^{\nu-1}\times R=\Gamma$ (per æquationes quintam et octavam poſitarum). Ergo $O=\frac{x^{\vartheta}\times\Gamma}{\Gamma}\times\overline{a+b\Sigma^{\tau}}|^{\omega}=x^{\vartheta}\times\overline{a+b\Sigma^{\tau}}|^{\omega}$. Sed $\Sigma^{\tau}=x^{\frac{\tau}{\pi}}$ (per æquationem deductarum primam) $=x^{\sigma}$ (per æquationem poſitarum tertiam). Ergo $O=x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. Q. E. D.

(mm) Stewarti emendationem ſequor. Editiones ipſius priores habuere $\overline{aS^{\upsilon}+bR^{\tau}}|^{\omega}$; vitioſè.

(nn) Hujus Corollarii veritatem cum Stewarto ſic oſtendimus.

Hæ ſunt æquationes à Newtono poſitæ.

1. $\overline{ez^{\nu}+fz^{\nu+\eta}+gz^{\nu+2\eta}}+,=R$.

2.

PROP. X. pro τ, υ, & λ vel μ, & faciendo, ut radix dignitatis extrahi poſſit, cujus index eſt ω; vel ponendo $\omega = -1$, vel $\mu = 0$.

PROP. X. PROB. III.

Invenire figuras ſimpliciſſimas, cum quibus Curva quævis geometricè comparari poteſt, cujus Ordinatim applicata, y, per æquationem non affectam ex datâ Abſciſſâ, z, determinatur.

CAS. I.

Sit Ordinata $az^{\theta-1}$, & Area erit $\frac{1}{\theta} az^{\theta}$, ut ex Prop. v. ponendo $b=0=c=d=f=g=h$, & $e=1$, facilè colligitur.

CAS. II.

Sit Ordinata $az^{\theta-1} \times \overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda-1}$; et ſi Curva cum figuris rectilineis geometricè comparari poteſt, quadrabitur per Prop. v. ponendo $b=0=c=d$. Sin minùs, convertetur in aliam curvam ſibi æqualem, cujus Ordinata eſt $\frac{a}{\eta} x^{\frac{\theta-\eta}{\eta}} \times \overline{e+fx+gx^2+\&c.}|^{\lambda-1}$ per Corol. 2. Prop. IX. Deinde ſi de dignitatum indicibus $\frac{\theta-\eta}{\eta}$ & $\lambda-1$ (per Prop. VII.) rejiciantur unitates, donec dignitates illæ fiant quàm minimæ, devenietur ad figuras ſimpliciſſimas, quæ hâc ratione colligi poſſunt. Dein harum unaquæque, per Corol. 5.

DEMONSTRATIO. 2. $\overline{\nu cz^{\nu-1} + \overline{\nu+\eta} fz^{\nu+\eta-1} + \overline{\nu+2\eta} gz^{\nu+2\eta-1} +}| = r$. 3. $\overline{k + lz^{\eta} + mz^{2\eta} +}| = s$. 4. $\overline{\eta lz^{\eta-1} + 2\eta mz^{2\eta-1} +,}| = s$. 5. $\frac{\mu-\iota\omega}{\lambda} = \frac{\upsilon}{\tau} = \frac{\phi}{\pi}$. 6. $\frac{\tau}{\pi} = \sigma$. 7. $\frac{\lambda-\pi}{\pi} = \vartheta$. 8. $R^{\pi} S^{\phi} = x$.

Ex his autem hæ aliæ deducentur. 1. $\dot{R} = r\dot{z}$. 2. $\dot{S} = s\dot{z}$. 3. $\pi R^{\pi-1} S^{\phi} \dot{R} + \phi R^{\pi} S^{\phi-1} \dot{S} = \dot{x}$.

Et ex his tribus veniet. 4. $\overline{r\pi R^{\pi-1} S^{\phi} + s\phi R^{\pi} S^{\phi-1}}| \times \dot{z} = \dot{x}$.

Jam Curvæ tranſmutatæ, cujus abſciſſa x, hujus Ordinata deſignetur literâ O. Et cùm Curvæ datæ cujus abſciſſa z, ordinata ſit $\overline{\pi s r + \phi R s} \times R^{\lambda-1} \times S^{\mu-1} \times \overline{aS^{-\upsilon} + bR^{\tau}}|^{\omega}$, ſive $\overline{r\pi S^{\mu} R^{\lambda-1} + s\phi R^{\lambda} S^{\mu-1}} \times \overline{aS^{-\upsilon} + bR^{\tau}}|^{\omega}$; erit $\dot{x} : \dot{z} = \overline{r\pi S^{\mu} R^{\lambda-1} + s\phi R^{\lambda} S^{\mu-1}}| \times \overline{aS^{-\upsilon} + bR^{\tau}}|^{\omega} : O$. Sed $\dot{x} : \dot{z} = r\pi R^{\pi-1} S^{\phi} + s\phi R^{\pi} S^{\phi-1} : 1$ (per æquationem deductarum quartam). Quare $r\pi R^{\pi-1} S^{\phi} + s\phi R^{\pi} S^{\phi-1} : 1 = \overline{r\pi S^{\mu} R^{\lambda-1} + s\phi R^{\lambda} S^{\mu-1}} \times \overline{aS^{-\upsilon} + bR^{\tau}}|^{\omega} : O$.

Unde $O = \frac{r\pi S^{\mu} R^{\lambda-1} + s\phi R^{\lambda} S^{\mu-1}}{r\pi R^{\pi-1} S^{\phi} + s\phi R^{\pi} S^{\phi-1}} \times \overline{aS^{-\upsilon} + bR^{\tau}}|^{\omega}$. Quantitatis autem fractæ denominator numeratorem algebraicè metitur, et quantitas diviſione facta erit $R^{\lambda-\pi} . S^{\mu-\phi}$. Quare $O = R^{\lambda-\pi} S^{\mu-\phi}$

×

5. Prop. IX. dat aliam, quæ nonnunquam ſimplicior eſt. Et ex his, per Prop. III. & Corol. 9 & 10. Prop. IX. inter ſe collatis, figuræ adhuc ſimpliciores quandoque prodeunt. Deniqne ex figuris ſimpliciſſimis aſſumptis, facto regreſſu, computabitur Area quæſita. PROP. X.

CAS. III.

Sit Ordinata $z^{\theta-1} \times \overline{a+bz^{n}+cz^{2n}+\&c.} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, & hæc figura, ſi quadrari poteſt, quadrabitur per Prop. V. Sin minùs, diſtinguenda eſt Ordinata in partes, $z^{\theta-1} \times a \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, $z^{\theta-1} \times bz^{n} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$; &c. et per Caſ. II. inveniendæ ſunt figuræ ſimpliciſſimæ, cum quibus figuræ, partibus illis reſpondentes, comparari poſſunt.

Nam Areæ figurarum, partibus illis reſpondentium, ſub ſignis ſuis + & − conjunctæ, component Aream totam quæſitam.

CAS. IV.

Sit Ordinata $z^{\theta-1} \times \overline{a+bz^{n}+cz^{2n}+\&c.} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$ in $\overline{k+lz^{n}+mz^{2n}+\&c}|^{\mu-1}$; et ſi curva quadrari poteſt, quadrabitur per Prop. VI. Sin minùs, convertetur in ſimpliciorem, per Corol. 4. Prop. IX. ac inde comparabitur cum figuris ſimpliciſſimis, per Prop. VIII. et Corol. 6, 9 & 10. Prop. IX. ut fit in Caſu 2. & 3.

$\times \overline{as^{-\nu}+bR^{\tau}}|^{\omega}$. Sed $R^{\tau} = x^{\frac{\tau}{\pi}} \times \Sigma^{-\frac{\phi\tau}{\pi}}$ (per æquationem poſitarum octavam). Et $\frac{\tau}{\pi} = \sigma$ (per ſextam poſitarum) et $\frac{\phi\tau}{\pi} = \nu$ (per quintam poſitarum). Quare $R^{\tau} = x^{\sigma} \times \Sigma^{-\nu}$. Et $O = R^{\lambda-\pi} S^{\mu-\tau} \times \overline{aS^{-\nu}+bx^{\sigma} \times S^{-\nu}}|^{\omega} = R^{\lambda-\pi} S^{\mu-\tau-\nu\omega} \times \overline{a+bx^{\sigma}}|^{\omega}$. Sed $\mu-\nu\omega = \frac{\lambda\phi}{\pi}$ (per æquationem poſitarum quintam). Quare $\mu-\phi-\nu\omega = \frac{\lambda\phi-\pi\phi}{\pi}$. Et $\frac{\lambda\phi-\pi\phi}{\pi} = \phi\vartheta$ (per æquationem poſitarum ſeptimam). Ergo $O = R^{\lambda-\pi}\Sigma^{\phi\vartheta} \times \overline{a+bx^{\sigma}}|^{\omega}$. Sed $R^{\lambda-\pi} = R^{\pi\vartheta}$ (per æquationem poſitarum ſeptimam). Et $R^{\pi\vartheta} = \frac{x^{\vartheta}}{\Sigma^{\phi\vartheta}}$ (per octavam poſitarum). Quare $R^{\lambda-\pi}\Sigma^{\phi\vartheta} = \frac{x^{\vartheta} \times \Sigma^{\phi\vartheta}}{\Sigma^{\phi\vartheta}} = x^{\vartheta}$. Et O $(= R^{\lambda-\pi}\Sigma^{\phi\vartheta} \times \overline{a+bx^{\sigma}}|^{\omega}) = x^{\vartheta} \times \overline{a+bx^{\sigma}}|^{\omega}$. Q. E. D. COR. X.

Unum moneo, quod tamen leviſſimum eſt; editiones hâc noſtrâ priores pro r, s, habuiſſe $\dot{r}$, $\dot{s}$, errore ut videtur ab editione principe in omnes poſteriores propagato.

CAS.

PROB. X. Cor. I. II. & III.

CAS. V.

Si Ordinata ex variis partibus conſtat, partes ſingulæ pro Ordinatis curvarum totidem habendæ ſunt, & Curvæ illæ, quotquot quadrari poſſunt, ſigillatim quadrandæ ſunt; earumque Ordinatæ de Ordinatâ totâ demendæ. Dein Curva, quam Ordinatæ pars reſidua deſignat, ſeorſim (ut in Caſu 2, 3 & 4.) cum figuris ſimpliciſſimis comparanda eſt, cum quibus comparari poteſt. Et ſumma Arearum omnium pro Areâ curvæ propoſitæ habenda eſt.

COROL. I.

Hinc etiam Curva omnis, cujus Ordinata eſt radix quadratica affecta æquationis ſuæ, cum figuris ſimpliciſſimis, ſeu rectilineis ſeu curvilineis, comparari poteſt. Nam radix illa ex duabus partibus ſemper conſtat, quæ ſeorſim ſpectatæ non ſunt æquationum radices affectæ.

Proponatur æquatio $a^2y^2+z^2y^2=2a^3y+2z^3y-z^4$, & extracta radix erit $y=\frac{a^3+z^3+a\sqrt{a^4+2az^3-z^4}}{aa+zz}$; cujus pars rationalis, $\frac{a^3+z^3}{aa+zz}$, & pars irrationalis, $\frac{a\sqrt{a^4+2az^3-z^4}}{aa+zz}$, ſunt Ordinatæ curvarum, quæ, per hanc propoſitionem, vel quadrari poſſunt, vel cum figuris ſimpliciſſimis comparari, cum quibus collationem geometricam admittunt.

COROL. II.

Et Curva omnis, cujus Ordinata per æquationem quamvis affectam definitur, quæ per Corol. 7. Prop. IX. in æquationem non affectam migrat, vel quadratur per hanc propoſitionem, ſi quadrari poteſt, vel comparatur cum figuris ſimpliciſſimis, cum quibus comparari poteſt. Et hâc ratione curva omnis quadratur, cujus æquatio eſt trium terminorum. Nam æquatio illa, ſi affecta ſit, tranſmutatur in non affectam per Corol. 7. Prop. IX ac deinde per Corol. 2 & 5. Prop. IX. in ſimpliciſſimam migrando, dat vel quadraturam figuræ, ſi quadrari poteſt, vel curvam ſimpliciſſimam quâcum comparatur.

COROL. III.

Et curva omnis, cujus Ordinata per æquationem quamvis affectam

feƈtam definitur, quæ per Corol. 8. Prop. IX. in æquationem quadraticam affeƈtam migrat; vel quadratur per hanc propoſitionem & hujus Corol. 1. ſi quadrari poteſt, vel comparatur cum figuris ſimpliciſſimis, cum quibus collationem geometricam admittit. Prop. X.

SCHOLIUM.

Ubi quadrandæ ſunt figuræ; ad regulas haſce generales ſemper recurrere nimis moleſtum eſſet: præſtat figuras, quæ ſimpliciores ſunt & magis uſui eſſe poſſunt, ſemel quadrare, & quadraturas in Tabulam referre; deinde Tabulam conſulere, quoties ejuſmodi Curvam aliquam quadrare oportet. Hujus autem generis ſunt Tabulæ duæ ſequentes; in quibus z denotat Abſciſſam, y Ordinatam reƈtangulam, & t Aream curvæ quadrandæ; & d, e, f, g, h, η ſunt quantitates datæ cum ſignis ſuis + & −.

TABULA

CURVARUM SIMPLICIORUM QUÆ QUADRARI POSSUNT.

CURVARUM FORMÆ			CURVARUM AREÆ
I		$dz^{\eta-1} = y$	$\frac{d}{\eta} z^{\eta} = t$
II		$\frac{dz^{\eta-1}}{e^2 + 2efz^{\eta} + f^2 z^{2\eta}} = y$	$\frac{dz^{\eta}}{\eta e^2 + \eta efz^{\eta}} = t$, vel $\frac{-d}{\eta ef + \eta f^2 z^{\eta}} = t$
III	1	$dz^{\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{2d}{3\eta f} R^3 = t$, exiſtente $R = \sqrt{e+fz^{\eta}}$
	2	$dz^{2\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{-4e+6fz^{\eta}}{15\eta f^2} dR^3 = t$
	3	$dz^{3\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{16e^2 - 24efz^{\eta} + 30f^2 z^{2\eta}}{105\eta f^3} dR^3 = t$
	4	$dz^{4\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{-96e^3 + 144e^2 fz^{\eta} - 180ef^2 z^{2\eta} + 210f^3 z^{3\eta}}{945\eta f^4} dR^3 = t$
IV	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{2d}{\eta f} R = t$
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{-4e+2fz^{\eta}}{3\eta f^2} dR = t$
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{16e^2 - 8efz^{\eta} + 6f^2 z^{2\eta}}{15\eta f^3} dR = t$
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{-96e^3 + 48e^2 fz^{\eta} - 36ef^2 z^{2\eta} + 30f^3 z^{3\eta}}{105\eta f^4} dR = t$

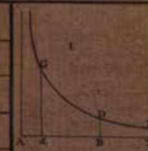
Between p. 378 and 379

TABULA

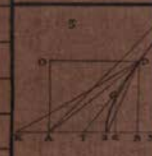
CURVARUM SIMPLICIORUM QUÆ CUM ELLIPSI ET HYPERBOLA COMPARARI POSSUNT.

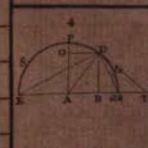
Sit jam αGD vel PGD vel GDS Sectio Conica cujus Area ad Quadraturam Curvæ propoſitæ requiritur, ſitq; ejus Centrum A, Axis Kα, Vertex α, Semiaxis conjugatus AP, datum Abſciſſæ principium A vel α vel α, Abſciſſa AB vel αB vel αB = x, Ordinata rectangula BD = v, et Area ABDP vel αBDG vel αBDG = s, exiſtente αG Ordinata ad punctum α. Jungantur KD, AD, αD; ducatur Tangens DT occurrens Abſciſſæ AB in T, & compleatur parallelogrammum ABDO. Et ſiquando ad quadraturam Curvæ propoſitæ requiruntur Areæ duarum Sectionum Conicarum, dicatur poſterioris Abſciſſa ξ, Ordinata Υ, et Area σ. Sit autem $\div$ differentia duarum quantitatum ubi incertum eſt utrum poſterior de priori an prior de poſteriori ſubduci debeat. Et in Forma ſexta ſcribatur p pro $\sqrt{ff-4eg}$.

Forma		Curvarum Formæ	Sectionis Conicæ: Abſciſſa	Sectionis Conicæ: Ordinata		Curvarum Areæ
I	1	$\frac{dz^{\eta-1}}{e+fz^\eta}=y$	$z^\eta=x$	$\frac{d}{e+fx}=v$		$\frac{1}{\eta}s=t=\frac{\alpha GDB}{\eta}$. Fig. 1.
	2	$\frac{dz^{2\eta-1}}{e+fz^\eta}=y$	$z^\eta=x$	$\frac{d}{e+fx}=v$		$\frac{d}{\eta f}z^\eta-\frac{e}{\eta f}s=t$
	3	$\frac{dz^{3\eta-1}}{e+fz^\eta}=y$	$z^\eta=x$	$\frac{d}{e+fx}=v$		$\frac{d}{2\eta f}z^{2\eta}-\frac{de}{\eta ff}z^\eta+\frac{ee}{\eta ff}s=t$
II	1	$\frac{dz^{\frac12\eta-1}}{e+fz^\eta}=y$	$\sqrt{\frac{d}{e+fz^\eta}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$		$\frac{2xv-4s}{\eta}=t=\frac{4}{\eta}ADG\alpha$. Fig. 2, 3, 4.
	2	$\frac{dz^{\frac32\eta-1}}{e+fz^\eta}=y$	$\sqrt{\frac{d}{e+fz^\eta}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$		$\frac{2d}{\eta f}z^{\frac12\eta}+\frac{4es-2exv}{\eta f}=t$
	3	$\frac{dz^{\frac52\eta-1}}{e+fz^\eta}=y$	$\sqrt{\frac{d}{e+fz^\eta}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$		$\frac{2d}{3\eta f}z^{\frac32\eta}-\frac{2de}{\eta ff}z^{\frac12\eta}+\frac{2eexv-4ees}{\eta ff}=t$
III	1	$\frac{d}{z}\sqrt{e+fz^\eta}=y$	$\frac{1}{z^\eta}=x^2$	$\sqrt{f+ex^2}=v$		$\frac{4de}{\eta f}\times\overline{\frac{v^3}{2ex}-s}=t=\frac{4de}{\eta f}$ in αGDT *vel in* APDB + TDB. Fig. 2, 3, 4.
		vel ſic	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{8de^2}{\eta f^2}\times\overline{s-\frac12xv-\frac{fv}{4e}+\frac{fvv}{4eex}}=t=\frac{8de^2}{\eta f^2}$ in αGDA $+\frac{fvv}{4eex}$. Fig. 2, 4.
	2	$\frac{d}{z^{\eta+1}}\sqrt{e+fz^\eta}=y$	$\frac{1}{z^\eta}=x^2$	$\sqrt{f+ex^2}=v$		$-\frac{2d}{\eta}s=t=\frac{2d}{\eta}$ APDB ſeu $\frac{2d}{\eta}$ αGDB. Fig. 2, 3, 4.
		vel ſic	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{4de}{\eta f}\times\overline{s-\frac12xv-\frac{fv}{2e}}=t=\frac{4de}{\eta f}\times\alpha GDK$. Fig. 2, 4.
	3	$\frac{d}{z^{2\eta+1}}\sqrt{e+fz^\eta}=y$	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$-\frac{2d}{\eta}s=t=\frac{2d}{\eta}\times$ αGDB *vel* BDPR. Fig. 4.
	4	$\frac{d}{z^{3\eta+1}}\sqrt{e+fz^\eta}=y$	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{3dfs-2dv^3}{3\eta ee}=t$
IV	1	$\frac{d}{z\sqrt{e+fz^\eta}}=y$	$\frac{1}{z^\eta}=x^2$	$\sqrt{f+ex^2}=v$		$\frac{4d}{\eta f}\times\overline{\frac12xv+s}=t=\frac{4d}{\eta f}$ in PAD *vel in* αGDA. Fig. 2, 3, 4.
		vel ſic	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{8de}{\eta f^2}\times\overline{s-\frac12xv-\frac{fv}{4e}}=t=\frac{8de}{\eta f^2}$ in αGDA. Fig. 2, 4.
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^\eta}}=y$	$\frac{1}{z^\eta}=x^2$	$\sqrt{f+ex^2}=v$		$\frac{2d}{\eta e}\times\overline{s-xv}=t=\frac{2d}{\eta e}$ in POD *vel in* AODGα. Fig. 2, 3, 4.
		vel ſic	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{4d}{\eta f}\times\overline{\frac12xv-s}=t=\frac{4d}{\eta f}$ in ADGα. Fig. 2, 4.
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^\eta}}=y$	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{d}{\eta e}\times\overline{3s-2xv}=t=\frac{d}{\eta e}$ in 3αDGA + ΔADB. Fig. 2, 4.
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^\eta}}=y$	$\frac{1}{z^\eta}=x$	$\sqrt{fx+ex^2}=v$		$\frac{10dfxv-15dfs-4dex^2v}{6\eta ee}=t$









Iohan. Senex sculp.

RESIDUUM TABULÆ CURVARUM SIMPLICIORUM QUÆ CUM ELLIPSI ET HYPERBOLA COMPARARI POSSUNT

		Curvarum Formæ	Sectionis Conicæ: Abſciſſa	Sectionis Conicæ: Ordinata	Curvarum Areæ
V	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x$	$\sqrt{\frac{d}{g}+\frac{ff-4eg}{4g^{2}}x^{2}}=v$	$\frac{xv-2s}{\eta}=t$.
		vel sic	$\sqrt{\frac{dz^{2\eta}}{e+fz^{\eta}+gz^{2\eta}}}=x$	$\sqrt{\frac{d}{e}+\frac{ff-4eg}{4e^{2}}x^{2}}=v$	$\frac{2s-xv}{\eta}=t$.
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x$; $fz^{\eta}+gz^{2\eta}=\xi$	$\sqrt{\frac{d}{g}+\frac{ff-4eg}{4g^{2}}x^{2}}=v$; $\frac{1}{e+\xi}=x$	$\frac{d\xi+2fs-fxv}{2\eta g}=t$.
VI	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^{\eta}}}=x$; $\sqrt{\frac{2dg}{f+p+2gz^{\eta}}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^{2}}=v$; $\sqrt{d+\frac{-f-p}{2g}\xi^{2}}=\tau$	$\frac{2xv-4s+2\xi\tau+4\sigma}{\eta p}=t$.
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{2dez^{\eta}}{fz^{\eta}-pz^{\eta}+2e}}=x$; $\sqrt{\frac{2dez^{\eta}}{fz^{\eta}+pz^{\eta}+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^{2}}=v$; $\sqrt{d+\frac{-f-p}{2e}\xi^{2}}=\tau$	$\frac{4s-2xv-4\sigma+2\xi\tau}{\eta p}=t$.
VII	1	$\frac{d}{z}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$; $\frac{1}{z^{\eta}}=\xi$	$\sqrt{e+fx+gx^{2}}=v$; $\sqrt{g+f\xi+e\xi^{2}}=\tau$	$\frac{8de^{2}\,[illegible]+2def\,[illegible]-4dfg\,xv\,[illegible]-8de^{2}\tau+4dfgs}{4\eta eg-\eta ff}=t$.
	2	$dz^{\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{d}{\eta}s=t=\frac{d}{\eta}\times aGDB$. Fig. 2, 3, 4.
	3	$dz^{2\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{d}{3\eta g}v^{3}-\frac{df}{2\eta g}s=t$.
	4	$dz^{3\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{6dgx-5df}{24\eta g^{2}}v^{3}+\frac{5df^{2}-4deg}{16\eta g^{2}}s=t$.
VIII	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{8dgs-4dgxv-2dfv}{4\eta eg-\eta ff}=t=\frac{8dg}{4\eta eg-\eta ff}\times aGDB\pm\Delta DBA$. Fig. 2, 4.
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{-4dfs+2dfxv+4dev}{4\eta eg-\eta ff}=t$.
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{\frac{3dff}{-4deg}s\;\frac{-3dff}{+4deg}xv-2defv}{4\eta eg^{2}-\eta ffg}=t$.
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{[illegible]\,s+[illegible]\,x^{2}v+[illegible]\,xv+[illegible]\,v}{[illegible]}=t$.
IX	1	$\frac{dz^{\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{[illegible]\,s-[illegible]\,xv+2df\frac{v}{x}}{\eta fh}=t$.
	2	$\frac{dz^{2\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{[illegible]\,s-[illegible]\,xv+\frac{2}{3}dh\frac{v^{3}}{x^{3}}-2dfg\frac{v}{x}}{\eta fh^{2}}=t$.
X	1	$\frac{dz^{\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{2xv-4s}{\eta f}=t=\frac{4}{\eta f}ADGa$. Fig. 2, 4.
	2	$\frac{dz^{2\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{4gs-2gxv+2d\frac{v}{x}}{\eta fh}=t$.
XI	1	$dz^{-\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$; $\sqrt{h+gz^{-\eta}}=\xi$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}}=v$; $\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^{2}}=\tau$	$\frac{2dxvz^{-\eta}-4dfs-4de\sigma}{\eta fg-\eta eh}=t$.
	2	$dz^{\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}}=v$	$\frac{2d}{\eta h}s=t$.
	3	$dz^{2\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}}=v$	$\frac{dhxv^{3}-[illegible]\,s}{[illegible]\,\eta fh^{2}}=t$.









Iohan. Senex sculp.

In Tabulis hiſce, ſeries curvarum cujuſque formæ utrinque in infinitum continuari poteſt. Scilicet in Tabulâ primâ, in numeratoribus Arearum formæ tertiæ & quartæ, numeri coefficientes initialium terminorum (2, −4, 16, −96, 768 (oo) &c.) generantur multiplicando numeros −2, −4, −6, −8, −10, &c. in ſe continuò (pp), & ſubſequentium terminorum coefficientes ex initialibus derivantur multiplicando ipſos gradatim; in Formâ quidem tertiâ, per $-\frac{3}{2}, -\frac{5}{4}, -\frac{7}{6}, -\frac{9}{8}, -\frac{11}{10}$, &c. in quartâ vero per $-\frac{1}{2}, -\frac{3}{4}, -\frac{5}{6}, -\frac{7}{8}, -\frac{9}{10}$, &c. Et denominatorum coefficientes 3, 15, 105, &c. prodeunt multiplicando numeros 1, 3, 5, 7, 9, &c. in ſe continuè. Prop. XI.

In ſecundâ vero Tabulâ, ſeries curvarum Formæ primæ, ſecundæ, quintæ, ſextæ, nonæ, & decimæ ope ſolius diviſionis, & Formæ reliquæ ope Propoſitionis tertiæ & quartæ, utrinque producuntur in infinitum.

Quinetiam hæ ſeries mutando ſignum numeri η variari ſolent. Sic enim *e.g.* Curva $\frac{d}{z}\sqrt{e+fz^{\eta}}=y$, evadit $\frac{d}{z^{\frac{1}{2}\eta+1}}\sqrt{f+ez^{\eta}}=y$ (qq).

PROP. XI. THEOR. VIII.

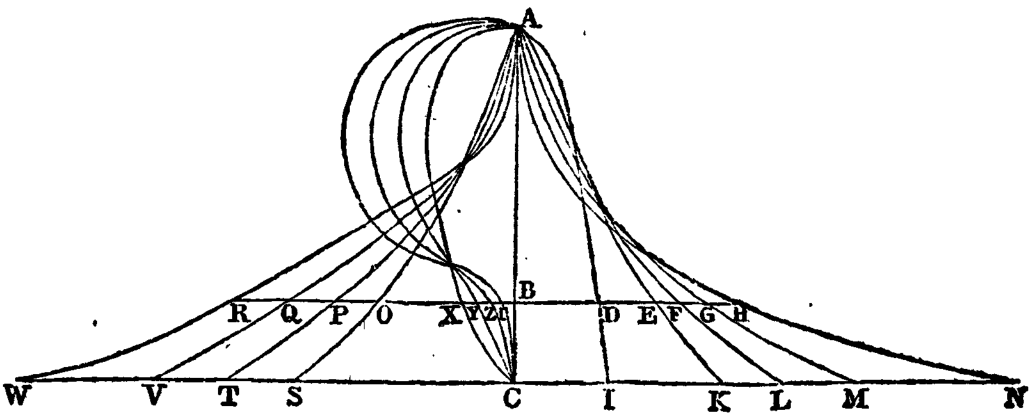
Sit ADIC Curva quævis Abſciſſam habens AB$=z$, & Ordinatam BD$=y$; & ſit AEKC Curva alia, cujus Ordinata, BE, æqualis eſt prioris areæ, ADB, ad unitatem applicatæ; & AFLC Curva tertia; cujus Ordinata, BF, æqualis eſt ſecundæ areæ, AEB, ad unitatem applicatæ; & AGMC Curva quarta, cujus Ordinata, BG, æqualis eſt tertiæ areæ, AFB, ad unitatem applicatæ; & AHNC Curva quinta, cujus Ordinata, BH, æqualis eſt quartæ areæ, AGB, ad unitatem applicatæ; & ſic deinceps in infinitum. Et ſunto *A*, *B*, *C*, *D*, *E*,

(oo) Editiones priores habuere 868. Vitioſè. Colſonus in Geometriâ Analyticâ hunc numerum emendatè dedit. Vid. Geometr. Analyt. C. x. Not. *a*.

(pp) Rectius ni fallor dixiſſet coefficientes illos generari multiplicando binarium in hanc ſeriem 1, −2, −4, −6, −8, −. Nimirum in Formâ tertiâ et quartâ Tabulæ primæ, coefficiens primi membri numeratoris areæ primæ binarius eſt, ſive factus è duobus, 2, 1, inter ſe multiplicatis. Areæ ſecundæ coefficiens ille numeratoris factus eſt ex tribus, 2, 1, −2; tertiæ, ex quatuor, 2, 1, −2, −4; quartæ, ex quinque 2, 1, −2, −4, −6 inter ſe multiplicatis, eodemque uſque modo progrediendum eſt.

(qq) Vide Geomet. Analyt. C. x.

PROP. XI.



&c. Areæ curvarum Ordinatas habentium y, zy, z^2y, z^3y, z^4y, &c. et Abſciſſam communem z.

Detur Abſciſſa quævis AC$=t$, ſitque BC$=t-z=x$, & ſunto P, Q, R,

(T) PROPOSITIONIS XI^æ. DEMONSTRATIO ROBINESII.

HUJUS Propoſitionis ſunt partes duæ; quarum Prima eſt hæc.

SIT Curva quæpiam ADI, cujus abſciſſa AB literâ z ſignificetur; ordinata BD, cuicuimodi ejus cum abſciſſâ ſit cognatio, literâ y. Tum, vocatâ hâc curvâ primâ, ſint aliæ curvæ AEK, AFL, AGM, AHN, quarum abſciſſa omnium quidem communis z, ſingularum ordinatæ BE, BF, BG, BH. Harum autem BE, ſecundæ curvæ ordinata, æqualis ſit areæ primæ ad unitatem applicatæ; Curvæ tertiæ ordinata BF æqualis ſit areæ ſecundæ ad unitatem applicatæ: Quartæ ordinata, BG, æqualis ſit areæ tertiæ ad unitatem applicatæ: et ſimili lege progreſſio uſque fiat.

Jam ſint aliæ etiam Curvæ, AOS, APT, AQV, ARW, eâdem abſciſſâ z, ordinatis BO, BP, BQ, BR; quarum BO, curvæ AOS ordinata, æqualis ſit quantitati zy: Ordinata BP curvæ APT æqualis ſit quantitati z^2y. Ordinata BQ curvæ AQV æqualis ſit quantitati z^3y. Ordinata BR curvæ ARW æqualis ſit quantitati z^4y. Tum literâ A deſignante aream ACI, literâ B aream ACS, literâ C aream ACT, literâ D aream ACV, literâ E Aream ACW, areæ ACI, ACK, ACL, ACM, ACN, ſic ordine erunt æſtimandæ.

$$\text{Area prima ACI} = \text{A}$$

$$\text{Secunda ACK} = z\text{A} - \text{B}.$$

$$\text{Tertia ACL} = \frac{z^2\text{A} - 2z\text{B} + \text{C}}{2}$$

$$\text{Quarta ACM} = \frac{z^3\text{A} - 3z^2\text{B} + 3z\text{C} + \text{D}}{6}$$

$$\text{Quinta ACN} = \frac{z^4\text{A} - 4z^3\text{B} + 6z^2\text{C} - 4z\text{D} + \text{E}}{24}$$

Eodemque deinceps uſque modo.

Harum formularum ea quidem eſt conditio, ut in omnibus poſt primam, index elatiſſimæ poteſtatis literæ z ſit numerus ille, qui curvæ cujuſque à primâ diſtantiam ſignificet; index autem elatiſſimus ille unitate gradatim minuat. Membrum primum multiplicatur cum A, ſecundum cum B, tertium cum C, eodemque deinceps uſque modo. Coefficientes iidem ſunt ac poteſtatis ejus à quâlibet duorum nominum radice, quæ poteſtas pari ſit gradu cum elatiſſimâ quantitatis z. Diviſor verò factus eſt ex tot numeris (1, 2, 3, 4, 5, 6) naturali ordine ſumendis, quot ſint unitates in numero elatiſſimam poteſtatem literæ z indicante.

Vel ſi curvæ, cujus æſtimationem inire oporteat, ſi hujus à primâ diſtantiam numerus n ſignificet,

R, *S*, *T* Areæ curvarum Ordinatas habentium x, xy, x^2y, x^3y, x^4y, &c. et Abſciſſam communem x. Prop. XI.

Terminentur autem hæ Areæ omnes ad Abſciſſam totam datam AC, nec non ad Ordinatam, poſitione datam & infinitè productam, CI.

Et erit Arearum ſub initio poſitarum

Prima ADIC $= A = P$.

Secunda AEKC $= tA - B = Q$.

Tertia AFLC $= \frac{t^2A - 2tB + C}{2} = \frac{1}{2}R$.

Quarta AGMC $= \frac{t^3A - 3t^2B + 3tC - D}{6} = \frac{1}{6}S$.

Quinta AHNC $= \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24} = \frac{1}{24}T$ ([11]).

COROL.

[11] licet, Area exquirenda ita quidem invenietur, ſi quantitas $\overline{z-1}|^n$ in ſeriem explicabitur, cujus membrum primum multiplicabitur cum A, ſecundum cum B, tertium cum C, quartum cum D; tum omnium ſumma dividetur numero à mutuâ horum multiplicatione facto; n, $n-1$, $n-2$, $n-3$, horum utique ſerie eò uſque continuatâ, ut in unitatem deſinat.

Altera Pars Propoſitionis hæc eſt.

Sint curvæ prima, ſecunda, tertia, et reliquæ, quæ ſuperiùs poſitæ ſunt. Jam literâ t deſignante abſciſſam totam AC, literâ verò x, partem ejus BC, ſcribi intelligantur curvæ CXA, CYA, CZA, CΓA, eâ quidem lege, ut ex ordinatis BX, BY, BZ, BΓ, illa BX æquetur quantitati xy; illa BY quantitati x^2y; illa BZ quantitati x^3y; BΓ quantitati x^4y. His poſitis, ſi Curvarum CIDA, CXA, CYA, CZA, CΓA, prima CIDA ſignificetur literâ P, ſecunda CXA literâ Q, tertia CYA literâ R, quarta CZA literâ S, quinta CΓA literâ T, Curvarum, quæ initio poſitæ ſunt, Areæ totæ ad hunc modum ordine æſtimandæ ſunt.

Prima AIC $=$ P

Secunda AKC $=$ Q

Tertia ALC $= \frac{1}{2}$ R

Quarta AMC $= \frac{1}{6}$ S

Quinta ANC $= \frac{1}{24}$ T

Nimirum areæ P, Q, R, S, T à numeris horum, 1, 2, 3, 4, 5 mutuâ multiplicatione factis dividendæ, tot ex iſtâ ſerie ac priùs ordine deſumptis.

DEMONSTRATIO PARTIS PRIMÆ.

Cum Curvæ AEK, AFL, AGM, AHN, alia ex aliâ, eâ lege procreatæ ſint, ut abſciſſam z communem omnes habeant, ordinatam unaquæque prioris aream; illud nos oſtendere oportet, formularum illarum, A, $zA - B$, $\frac{zzA - 2zB + C}{2}$, unamquamque Curvæ cujuſdam Aream deſignare, cujus Ordinatam formula præcedens deſignaverit, cùm abſciſſa ſit z.

Ponatur igitur $m = n - 1$.

Prop. XI. Cor.

COROL.

Unde ſi Curvæ quarum ordinatæ ſunt y, zy, z^2y, z^3y, &c. vel

Propoſitionis XI^æ.

Frunt igitur $\frac{z^m A - mz^{m-1} B + m \times \frac{m-1}{2} z^{m-2} C -,}{m \times m-1 \times m-2 \times,}$ et $\frac{z^n A - nz^{n-1} B + n \times \frac{n-1}{2} z^{n-2} C -}{n \times n-1 \times n-2}$, erunt hæ, ex iis quas ſuprà poſuimus formulis duæ aliquæ inter ſe proximæ. Oſtendere igitur nos oportet, harum formularum poſteriorem curvæ cujuſdam Aream ſignificare, quæ ordinatam quidem à priore deſignatam habeat, abſciſſam verò z. Hujus autem areæ fluxio facta erit multiplicando illam $\dot{z}$ cum formulâ priore. Oſtendere igitur nos oportet hujus fluxionis fluentem per formulam poſteriorem deſignari. Vel, quod eodem redit, poſterioris formulæ fluxionem æqualem eſſe illi $\dot{z}$ cum formulâ priore multiplicatæ.

Formulæ poſterioris fluxiò conſtat ex quantitatibus, $nz^{n-1}\dot{z}A - n \times \overline{n-1}\, z^{n-2}\dot{z}B + n \times \frac{n-1}{2} \times \overline{n-2}\, z^{n-3}\dot{z}C -,$ $+ z^n\dot{A} - nz^{n-1}\dot{B} + n \times \frac{n-1}{2} z^{n-2}\dot{C}$, ex his, inquam, cum facto, $n \times n-1 \times n-2 \times$, diviſis.

Tota quantitas dividenda è duabus conſtat partibus. Quarum prima conflata eſt è fluxionibus poteſtatum literæ z cum quantitatibus A, B, C, multiplicatis; altera ex illarum A, B, C fluxionibus in poteſtates literæ z multiplicatis. Horum autem, $z^n\dot{A}$, $z^{n-1}\dot{B}$, $z^{n-2}\dot{C}$, ex quibus partem alteram dividendæ componi diximus, horum unumquodque æquatur quantitati $z^n\dot{z}y$; cùm ex ipſâ Curvarum procreandarum lege, quarum areæ deſignantur literis A, B, C, efficietur $\dot{A} = \dot{z}y$. $\dot{B} = \dot{z}zy$. $\dot{C} = \dot{z}z^2y$. Dividendæ igitur pars altera illa huic æqualis erit, $z^n\dot{z}y \times 1 - n + n \times \frac{n-1}{2} - n \times \frac{n-1}{2} \times \frac{n-2}{3} +$: vel huic, $z^n\dot{z}y \times \overline{1-1}|^n$. Nihilo igitur. Et tota fluxio huic fit æqualis $\frac{nz^{n-1}\dot{z}A - n \times \overline{n-1}\, z^{n-2}\dot{z}B + n \times \frac{n-1}{2} \times \overline{n-2}\, z^{n-3}\dot{z}C -}{n \times n-1 \times n-2 \times n-3 \times,}$. Deleto igitur n, qui quantitatem utramque, dividentem et dividendam, metitur, et pro illis $n-1$, $n-2$, $n-3$, his m, $m-1$, $m-2$ ſubſtitutis, fluxio de quâ agitur hâc formâ comparebit; $\dot{z} \times \frac{z^m A - mz^{m-1} B + m \times \frac{m-1}{2} z^{m-2} C -}{m \times m-1 \times m-2 -}$. Formulæ igitur poſterioris fluxio æqualis erit illi $\dot{z}$ in formulam priorem multiplicatæ. Unde conſequetur, quamlibet è formulis illis curvæ cujuſdam Aream deſignare, quæ formulam priorem Ordinatam habeat, cùm Abſciſſa ejus ſit z. Q. E. D.

ALTERIUS PARTIS DEMONSTRATIO.

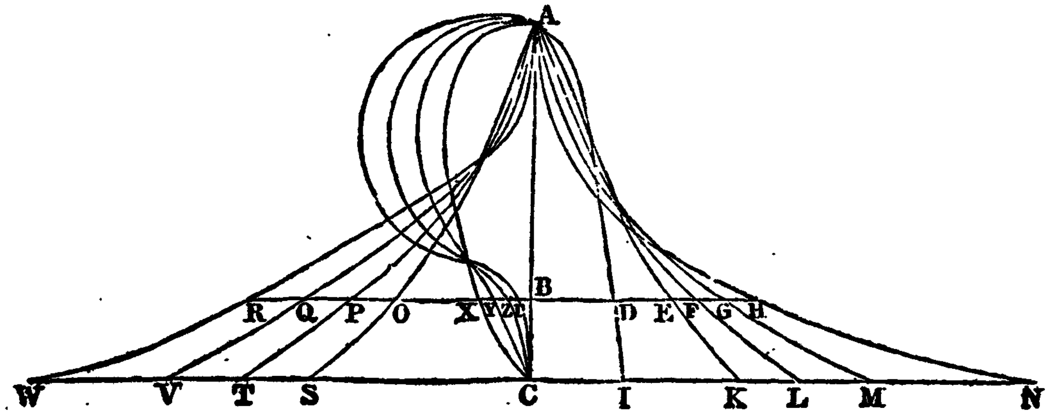
Ex Curvis initio poſitis, ADI, AEK, AFL, AGM, AHN deſumatur illa, cujus à primâ diſtantiam numerus n ſignificat. Illud jam nos oſtendere oportet; illius Aream quæ Abſciſſam habeat BC, ſeu x, Ordinatam $x^n y$, ſi à facto illo $n \times n-1 \times n-2 \times n-3 \times * * * * 1$ diviſa ſit, æquari deſumptæ illius Areæ, modò CB toti CA ſit æqualis, ſive x fiat t.

Areis ABD, ABO, ABP, ABQ, ABR decreſcentibus, alias illas, BCID, BCSO, BCTP, BCVQ, BCWR, augeri manifeſtum eſt. Unde conſequetur illorum decrementa, horum eſſe incrementa; illorum fluxiones negatas horum fluxiones eſſe. Hoc eſt ſi area ABD deſignetur literâ a, area ABO literâ b; area ABP, literâ c; area ABQ, litera d; area ABR, literâ e; tum area BCID, literâ α; area BCSO, literâ β; area BCTP, literâ γ; area BCVQ, literâ δ; area BCWR, literâ ε; erit $\dot{\alpha} = -\dot{a} : \dot{\beta} = -\dot{b} : \dot{\gamma} = -\dot{c} : \dot{\delta} = -\dot{d} : \dot{\varepsilon} = -\dot{e}$.

Jam

vel y, xy, x^2y, x^3y, &c. quadrari poſſunt, quadrabuntur etiam Curvæ ADIC, AEKC, AFLC, AGMC, &c. et habebuntur Ordinatæ BE, BF, BG, BH, Areis curvarum proportionales. Prop. XI. Cor.

Jam Curvæ cujus Abſciſſa x, Ordinata $x^n y$, hujus fluxio erit $\dot{x}x^n y = \dot{x}y \times \overline{t-z}|^n$, cùm ſit $x = t - z$. Vel cùm incrementa illius x decrementis ipſius z ſint æqualia, ac propterea cùm ſit $\dot{x} = -\dot{z}$, fluxio Robinesii Demonstratio



Curvæ, quam modò diximus, huic æqualis erit, $-\dot{z}y \times \overline{t-z}|^n = -\dot{z}y \times \overline{t^n - nt^{n-1}z + n \times \frac{n-1}{2} t^{n-2}z^2 -}$,

$= -t^n\dot{z}y + nt^{n-1}\dot{z}zy - n \times \frac{n-1}{2} t^{n-2}\dot{z}z^2y = t^n \times \overline{-\dot{a}} - nt^{n-1} \times \overline{-\dot{b}} + n \times \frac{n-1}{2} t^{n-2} \times \overline{-\dot{c}+}$,

$= \overline{t^n\dot{\alpha} - nt^{n-1}\dot{\beta} + n \times \frac{n-1}{2} t^{n-2}\dot{\gamma} -}$. Sumptiſque fluentibus, Area curvæ cujus Abſciſſa x, vel BC, Ordinata $x^n y$, æqualis erit huic quantitati $t^n\alpha - nt^{n-1}\beta + n \times \frac{n-1}{2} t^{n-2}\gamma -$. Sed cùm x ſit t, hoc eſt cùm BC æqualis evadit toti AC, areæ α, β, γ, totis illis A, B, C æquales ſunt: quod ſatis apertum eſt. Si igitur illa x eſt t, hoc eſt ſi CB eſt æqualis toti CA, Curvæ cujus Abſciſſa x, Ordinata $x^n y$, hujus Area huic ſit æqualis, t^nA $- nt^{n-1}$B $+ n \times \frac{n-1}{2} \times t^{n-2}$C: quæ ſit ex illa $\overline{t-1}|^n$ in ſeriem explicatâ, membro ſeriei primo cum A multiplicato, ſecundo cum B, tertio cum C.—Sed quantitas quæ ſit ex illâ $\overline{t-1}|^n$ quo diximus modo, hæc ſi cum facto $n \times n-1 \times n-2 \times n-3 \times * \times * \times 1$ dividetur, æquabitur areæ curvæ cujus diſtantiam à primâ numerus n ſignificaverit (per Cas. 1.) Hujus igitur areæ æquabitur illius area, cum facto $n \times n-1 \times n-2 \times n-3 \times * * \times 1$ diviſa, cujus abſciſſa x, ordinata $x^n y$: modò abſciſſa x toti AC ſit æqualis. Hoc eſt Q, ſive Area curvæ cujus Abſciſſa x Ordinata xy, ſi x æqualis ſit rectæ AC, æquabitur areæ ſecundæ AKC: $\frac{1}{2}$ R, ſive ſemiſſis areæ cujus Abſciſſa x, Ordinata x^2y, æqualis erit areæ tertiæ ALC: $\frac{1}{6}$ S, ſive pars ſexta ejus areæ, cujus Abſciſſa x, Ordinata x^3y, æqualis erit areæ quartæ AMC: $\frac{1}{24}$ T, ſive pars viceſima quarta ejus Areæ, cujus Abſciſſa x Ordinata x^4y areæ quintæ ANC æqualis; eodemque deinceps uſque modo; dummodo id in omnibus ſervetur, ut x ſit æqualis ipſi AC. Q. E. C.

Atque hæc eſt reconditiſſimæ Propoſitionis Robineſii Demonſtratio; cujus prima adumbratio anno 1727 inter Acta Philoſophicæ Societatis noſtræ Regiæ Londinenſis edita eſt; ſpecies perfectior anno 1733 in Epitome Actorum illâ, quam Viri docti concinnârunt Reid & Gray. Anglico ſermone ab auctore conſcriptam nos in Latinum convertimus. Dignam, credo, omnes judicabunt, cui id laboris noſtri nos impenderemus. Certè Stewarto Abredonenſi tantoperè eam placuiſſe video, ut in ſuos in hunc librum Commentarios transferendam judicârit, mutilam tamen ac haud in meliùs immutatam, & nullâ Robineſii mentione factâ.

SCHOL.

SCHOLIUM.

Quantitatum fluentium Fluxiones eſſe primas, ſecundas, tertias, quartas, aliaſque diximus ſuprà. Hæ fluxiones ſunt ut termini ſerierum infinitarum convergentium.

Ut ſi z^n ſit quantitas fluens, & fluendo evadat $\overline{z+o}|^n$, deinde reſolvatur in ſeriem convergentem $z^n + nozn^{n-1} + \frac{nn-n}{2} ooz^{n-2} + \frac{n^3-3nn+2n}{6} o^3z^{n-3} +$ &c. terminus primus hujus ſeriei z^n erit quantitas illa fluens, ſecundus noz^{n-1} erit (ff) ejus incrementum primum, ſeu differentia prima, cui naſcenti proportionalis eſt ejus Fluxio prima; tertius $\frac{nn-n}{2} ooz^{n-2}$ erit ejus incrementum ſecundum, ſeu differentia ſecunda, cui naſcenti proportionalis eſt ejus Fluxio ſecunda; quartus $\frac{n^3-3nn+2n}{6} o^3z^{n-3}$ erit ejus incrementum tertium, ſeu differentia tertia, cui naſcenti Fluxio tertia proportionalis eſt; & ſic deinceps in infinitum.

Exponi autem poſſunt hæ Fluxiones per Curvarum Ordinatas BD, BE, BF, BG, BH, &c. *(Vide fig. pag.* 380.*)*

Ut ſi Ordinata BE $(=\frac{ADB}{1})$ ſit quantitas fluens, erit ejus Fluxio prima ut Ordinata BD.

Si BF $(=\frac{AEB}{1})$ ſi quantitas fluens, erit ejus Fluxio prima ut Ordinata BE, & Fluxio ſecunda ut Ordinata BD.

Si BH $(=\frac{AGB}{1})$ ſit quantitas fluens, erunt ejus Fluxiones prima, ſecunda, tertia & quarta, ut Ordinatæ BG, BF, BE, BD reſpectivè.

Et hinc in æquationibus, quæ quantitates tantùm duas incognitas involvunt, quarum una eſt quantitas uniformiter fluens, & altera eſt fluxio quælibet quantitatis alterius fluentis, inveniri poteſt fluens illa altera per quadraturam curvarum. Exponatur enim fluxio ejus per Ordinatam BD; & ſi hæc ſit fluxio prima, quæratur area ADB = BE × 1; ſi fluxio ſecunda, quæratur area AEB

(ff) Lege, *erit* UT *ejus incrementum primum, tertius* * *erit* UT *ejus incrementum ſecundum, quartus* * *erit* UT *ejus incrementum tertium.* Hæc Emendatio ipſius quidem Newtoni eſt, in Recenſione Controverſiæ inter Leibnitſium & Keillium, &c. minimè verò Keillii, id quod Stewarto perſuaſum fuiſſe video. Ignorabat utique Stewartus Recenſionem illam ab ipſo Newtono profectam. Quòd ſi quis dubitare velit, locus hic, vel ſic emendatus, ſitne ſatis perſpicuus, id quod Stewartus ſe dubitare dicit; eum vel meridie dubitare, ſitne lux, jubebo.

(tt) Nempe

AEB = BF × 1; ſi fluxio tertia, quæratur area AFB = BG × 1; &c. et Generale.
Area inventa erit exponens fluentis quæſitæ.

Sed & in æquationibus, quæ fluentem & ejus fluxionem primam ſine alterâ fluente, vel duas ejuſdem fluentis fluxiones, primam & ſecundam, vel ſecundam & tertiam, vel tertiam & quartam, &c. ſine alterutrâ fluente involvunt: inveniri poſſunt fluentes per quadraturam curvarum. Sit æquatio $aa\dot v = av + vv$, exiſtente $v = $ BE, $\dot v = $ BD, $z = $ AB, & $\dot z = 1$; & æquatio illa, complendo dimenſiones fluxionum, evadet $aa\dot v = av\dot z + vv\dot z$, ſeu $\frac{aa\dot v}{av+vv} = \dot z$.

Jam fluat v uniformiter, & ſit ejus fluxio $\dot v = 1$, & erit $\frac{aa}{av+vv} = \dot z$, & quadrando curvam, cujus Ordinata eſt $\frac{aa}{av+vv}$, & Abſciſſa v, habebitur fluens z. Adhæc ſit æquatio $aa\ddot v = a\dot v + \dot v v$, exiſtente $v = $ BF, $\dot v = $ BE, $\ddot v = $ BD, & $z = $ AB; & per relationem inter $\ddot v$ & $\dot v$, ſeu BD & BE, invenietur relatio inter AB & BE, ut in exemplo ſuperiore. Deinde, per hanc relationem, invenietur relatio inter AB & BF, quadrando curvam AEB.

Æquationes, quæ tres incognitas quantitates involvunt, aliquando reduci poſſunt ad æquationes, quæ duas tantùm involvunt; &, in his caſibus, fluentes invenientur ex fluxionibus ut ſuprà. Sit æquatio $a - bx^m = cxy^n\dot y + dy^{2n}\dot y\dot y$: ponatur $y^n\dot y = \dot v$, & erit $a - bx^m = cx\dot v + d\dot v\dot v$. Hæc æquatio, quadrando curvam cujus abſciſſa eſt x & ordinata $\dot v$, dat aream v [tt]; & æquatio altera $y^n\dot y = \dot v$, regrediendo ad fluentes, dat $\frac{1}{n+1}y^{n+1} = v$: unde habetur fluens y.

Quinetiam in æquationibus, quæ tres incognitas involvunt, & ad æquationes quæ duas tantùm involvunt, reduci non poſſunt, fluentes quandoque prodeunt per quadraturam curvarum. Sit æquatio $\overline{ax^m + bx^n}|^p = rex^{r-1}y^s + sex^r\dot y y^{s-1} - f\dot y y^t$, exiſtente $\dot x = 1$; et pars poſterior, $rex^{r-1}y^s + sex^r\dot y y^{s-1} - f\dot y y^t$, regrediendo ad fluentes, fit $ex^r y^s - \frac{f}{t+1}y^{t+1}$, quæ proinde eſt ut Area curvæ, cujus abſciſſa eſt x & ordinata $\overline{ax^m + bx^n}|^p$; & inde datur fluens y.

[tt] Nempe per reſolutionem æquationis quadraticæ $a - bx^m = cx\dot v + d\dot v\dot v$ veniet $\dot v = \frac{\pm\sqrt{4d^2a - 4d^2bx^m + c^2x^2} - cx}{2d}$. Si igitur detur Area curvæ cujus Abſciſſa x, Ordinata $\frac{\sqrt{4d^2a - 4d^2bx^m + c^2x^2} - cx}{2d}$, dabitur v.

Scholium Generale.

Sit æquatio $\dot{x} \times \overline{ax^m + bx^n}|^p = \frac{d\dot{y}y^{\eta-1}}{\sqrt{e+fy^\eta}}$. Et fluens, cujus fluxio eſt $\dot{x} \times \overline{ax^m + bx^n}|^p$, erit ut Area curvæ, cujus abſciſſa eſt x & ordinata eſt $\overline{ax^m + bx^n}|^p$. Item fluens, cujus fluxio eſt $\frac{d\dot{y}y^{\eta-1}}{\sqrt{e+fy^\eta}}$, erit ut Area curvæ, cujus abſciſſa eſt y & ordinata $\frac{dy^{\eta-1}}{\sqrt{e+fy^\eta}}$, id eſt (per Caſum 1. Formæ quartæ Tab. I.) ut Area $\frac{2d}{\eta f}\sqrt{e+fy^\eta}$. Pono ergo $\frac{2d}{\eta f}\sqrt{e+fy^\eta}$ æqualem Areæ curvæ, cujus abſciſſa eſt x & ordinata $\overline{ax^m + bx^n}|^p$, & habebitur fluens y.

Et nota quòd fluens omnis, quæ ex fluxione primâ colligitur, augeri poteſt vel minui quantitate quâvis non fluente. Quæ ex fluxione ſecundâ colligitur, augeri poteſt vel minui quantitate quâvis, cujus fluxio ſecunda nulla eſt. Quæ ex fluxione tertiâ colligitur, augeri poteſt vel minui quantitate quâvis, cujus fluxio tertia nulla eſt. Et ſic deinceps in infinitum.

Poſtquam fluentes ex fluxionibus collectæ ſunt, ſi de veritate concluſionis dubitatur, fluxiones fluentium inventarum viciſſim colligendæ ſunt, & cum fluxionibus, ſub initio propoſitis, comparandæ. Nam ſi prodeunt æquales, concluſio rectè ſe habet: ſin minùs, corrigendæ ſunt fluentes ſic, ut earum fluxiones fluxionibus ſub initio propoſitis æquentur. Nam & fluens pro lubitu aſſumi poteſt, & aſſumptio corrigi, ponendo fluxionem fluentis aſſumptæ æqualem fluxioni propoſitæ, & terminos homologos inter ſe comparando.

Et his principiis via ad majora ſternitur.

ARTIS ANALYTICÆ

SPECIMINA

VEL

GEOMETRIA ANALYTICA.

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In hoc libro edendo tribus uſi ſumus MSS; in quibus et Auctoris Autographus erat. Alium, ignoti cujuſdam ſcribæ manu exaratum, nobiſcum communicavit honoratiſſimus Dominus Carolus Cavendiſh; cùm ipſe eum olim à Joneſio acceperat. Tertium de Cavendiſiano, ut videtur, propriâ manu exſcripſerat vir Doctiſſimus Jacobus Wilſon, editor ille operum Robineſii. Hunc nobis tradidit Joannes Nourſe, Bibliopola Regius, vir elegantioris Matheſeos et amantiſſimus et ipſe minimè imperitus. Diviſio Libri in Capita à nobis eſt.

GEOMETRIA ANALYTICA.

CAPUT PRIMUM.

DE SERIEBUS INFINITIS.

ANIMADVERTENTI pleroſque Geometras, poſthabitâ ferè Veterum Syntheticâ Methodo, Analyticæ excolendæ plurimùm incumbere, et ejus ope tot tantaſque difficultates ſuperâſſe; ut penè omnia, extra Curvarum Quadraturas, et ſimilia quædam nondum penitùs enodata, videantur exhauſiſſe; placuit ſequentia, quibus campi analytici terminos expandere, juxtà & Curvarum doctrinam promovere poſſem, in gratiam diſcentium breviter compingere.

2. Cùm in Numeris et Speciebus operationes computandi perſimiles ſint, neque differre videantur niſi in characteribus, quibus quantitates, in iſtis definitè, in his indefinitè deſignantur: demiror, quòd doctrinam de numeris decimalibus nuper inventam (ſi Quadraturam Hyperbolæ per N. Mercatorem demas) nemini in mentem venerit, Speciebus itidem accommodare; præſertim cùm ad præclariora viam aperiat. Hujus autem de Speciebus doctrinæ, cùm eodem modo, ad Algebram relata ſit, ac doctrina decimalium Numerorum ad vulgarem Arithmeticam; Operationes, Additio, Subtractio, Multiplicatio, Diviſio et Extractio radicum, exinde addiſci poſſunt: modo Lector utriuſque, et Arithmeticæ et Algebræ vulgaris, peritus fuerit, et noverit correſpondentiam inter decimales numeros, ac terminos Algebraicos in infinitum continuatos; ſcilicet quòd ſingulis numerorum locis, proportione decimali dextrorſùm perpetuò decreſcentibus, correſpondent ſinguli ſpecierum termini, ſecundum ſeriem dimenſionum numeratorum vel denominatorum uniformi progreſſione in infinitum continuatam, prout factum in ſequentibus, ordinati. Et quemadmodum commoditas decimalium in eo conſiſtit, ut fractiones omnes et ra-

dicales, in eos reductæ, quodammodo naturam integrorum induant: ſic etiam infinitarum ſpecierum commoditas eſt, quòd per eas abſtruſiorum terminorum genera, quales ſunt fractiones à compoſitis quantitatibus denominatæ, compoſitarum radices, et radices affectarum æquationum, poſſunt ad ſimplicium genus reduci; ad infinitas nempe fractionum ſeries numeratores ac denominatores ſimplices habentium, in quibus nullæ ſunt aliorum difficultates propemodum inſuperabiles. Imprimis itaque reductiones aliarum quantitatum ad hujuſmodi terminos, et methodos computandi minùs obvias oſtendam; dein hanc Analyſin ad ſolutiones Problematum applicabo.

3. Reductiones per diviſionem et extractionem radicum è ſequentibus exemplis, cum ſimilibus operandi modis in Arithmeticâ decimali et ſpecioſâ collatis, eluceſcet.

Exempla reductionum per Diviſionem.

4. Propoſitâ $\frac{aa}{b+x}$. Divide aa per $b+x$ ad hunc modum.

$$b+x)\,aa+00\left(\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}+\frac{a^2x^4}{b^5}-\frac{a^2x^5}{b^6}\ \&c.\right.$$

$$aa+\frac{a^2x}{b}$$

$$-\frac{a^2x}{b}+0$$

$$-\frac{a^2x}{b}-\frac{a^2x^2}{b^2}$$

$$+\frac{a^2x^2}{b^2}+0$$

$$+\frac{a^2x^2}{b^2}+\frac{a^2x^3}{b^3}$$

$$-\frac{a^2x^3}{b^3}+00$$

$$-\frac{a^2x^3}{b^3}-\frac{a^2x^4}{b^4}$$

$$+\frac{a^2x^4}{b^4}\ \&c.$$

Et prodit $\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}$ &c. quæ ſeries in infinitum continuata tantùm valet ac $\frac{aa}{b+x}$. Vel poſito x primo diviſoris termino, hoc modo $x+b)\,aa($ prodibit $\frac{aa}{x}-\frac{a^2b}{x^2}+\frac{a^2b^2}{x^3}-\frac{a^2b^3}{x^4}$ &c.

5. Ad eundem modum fractio $\frac{1}{1+x^2}$ reducitur ad $1-x^2+x^4-x^6+x^8$ &c. Vel ad $x^{-2}-x^{-4}+x^{-6}-x^{-8}$ &c.

6. Et

6. Et fractio $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ ad $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^{2}+34x^{\frac{5}{2}}$ &c. DIVISIO ET EXTRACT. RAD.

7. Ubi obiter notandum eſt, quòd uſurpo x^{-1}, x^{-2}, x^{-3}, x^{-4}, &c. pro $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$, et $x^{\frac{1}{2}}$, $x^{\frac{3}{2}}$, $x^{\frac{5}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{2}{3}}$, &c. pro $\sqrt{x}$, $\sqrt{x^3}$, $\sqrt{x^5}$, $\sqrt[3]{x}$, $\sqrt[3]{x^2}$; et $x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$, $x^{-\frac{1}{4}}$, &c. pro $\frac{1}{\sqrt{x}}$, $\frac{1}{\sqrt[3]{x^2}}$, $\frac{1}{\sqrt[4]{x}}$, &c. Idque ob analogiam rei, quæ deprehendi poteſt ex hujuſmodi geometricis progreſſionibus x^3, $x^{\frac{5}{2}}$, x^2, $x^{\frac{3}{2}}$, x, $x^{\frac{1}{2}}$, x^0, ſive 1, $x^{-\frac{1}{2}}$, x^{-1}, $x^{-\frac{3}{2}}$, x^{-2}, &c.

Ad hunc modum pro $\frac{aa}{x}-\frac{aab}{x^2}+\frac{aab^2}{x^3}$ &c. ſcribi poteſt $aax^{-1}-aabx^{-2}+aab^2x^{-3}$ &c.

Et ſic vice $\sqrt{aa-xx}$ ſcribi poteſt $\overline{aa-xx}|^{\frac{1}{2}}$; et $\overline{aa-xx}|^{2}$ vice quadrati ex $aa-xx$; et $\overline{\frac{aab-y^3}{by+yy}}|^{\frac{1}{3}}$ vice $\sqrt[3]{\frac{aab-y^3}{by+yy}}$; & ſic in aliis.

Unde meritò poteſtates diſtingui poſſunt in affirmativas et negativas, integras et fractas ([a]).

Exempla reductionum per Extractionem radicum.

8. Propoſitâ $aa+xx$, radicem ejus ut ſequitur extrahas

$$aa+xx\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}-\frac{21x^{12}}{1024a^{11}}\ \&c.\right.$$

$$\begin{array}{l}
aa \\
\hline
0\ +xx \\
\quad xx+\frac{x^4}{4a^2} \\
\hline
\quad -\frac{x^4}{4a^2} \\
\quad -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
\qquad +\frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
\qquad \frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\
\hline
\qquad\quad -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}}\ \&c. \\
\qquad\quad -\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8}+\frac{5x^{12}}{512a^{10}} \\
\hline
\qquad\qquad \frac{7x^{10}}{128a^8}-\frac{7x^{12}}{512a^{10}} \\
\qquad\qquad \frac{7x^{10}}{128a^8}+\frac{7x^{12}}{256a^{10}} \\
\hline
\qquad\qquad\quad -\frac{21x^{12}}{512a^{10}}\ \&c.
\end{array}$$

([a]) Vide Arithmetic. Univerſ. Cap. 1. Not. *b*.

Et prodit $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$ &c. Ubi notandum, quòd circa finem operis eos omnes terminos negligo, quorum dimenſiones tranſcenderent dimenſiones ultimi termini, ad quem cupio quotientem ſolummodo produci, puta $\frac{x^{12}}{a^{11}}$.

9. Poteſt etiam ordo terminorum inverti ad hunc modum; $x^2 + a^2$. Et radix erit $x + \frac{a^2}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5}$ &c.

Sic ex $a^2 - x^2$ radix eſt $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

Et ex $x - x^2$ eſt $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$ &c.

Et ex $a^2 + bx - x^2$ eſt $a + \frac{bx}{2a} - \frac{x^2}{2a} - \frac{b^2x^2}{8a^3} -$ &c.

Et ex $\frac{1 + ax^2}{1 - bx^2}$ eſt $\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6}$ &c. Factâque inſuper diviſione, fit

$$\begin{array}{llll} 1 + \frac{1}{2}b\,x^2 & + \frac{3}{8}b^2\,x^4 & + \frac{5}{16}b^3\,x^6 & \&c. \\ \;\;\; + \frac{1}{2}a & + \frac{1}{4}ab & + \frac{3}{16}ab^2 & \\ & - \frac{1}{8}a^2 & - \frac{1}{16}a^2b & \\ & & + \frac{1}{16}a^3 & \end{array}$$

10. Operationes verò per debitam præparationem non raro abbreviari poſſunt; ut in allato exemplo, ad extrahendam $\sqrt{\frac{1 + ax^2}{1 - bx^2}}$, ſi non eadem fuiſſet numeratioris ac denominatoris forma, utrumque multiplicaſſem per $\sqrt{1 - bx^2}$, & ſic prodiiſſet $\frac{\sqrt{1 \begin{smallmatrix} +a \\ -b \end{smallmatrix} x^2 + abx^4}}{1 - bx^2}$; et reliquum opus perficeretur extrahendo radicem numeratoris tantùm, ac dividendo per denominatorem.

11. Ex hiſce, credo, manifeſtum eſt, quo pacto radices aliæ poſſunt extrahi, et quælibet compoſitæ quantitates, quibuſcunque radicibus vel denominatoribus perplexæ; ut hîc videre eſt, $x^3 + \frac{\sqrt{x - \sqrt{1 - x^2}}}{\sqrt[3]{ax^2 + x^3}} - \frac{\sqrt[5]{x^3 + 2x^5 - x^{\frac{3}{2}}}}{\sqrt[3]{x + xx - \sqrt{2x - x^{\frac{2}{3}}}}}$, in ſeries infinitas ſimplicium terminorum reduci.

CAPUT SECUNDUM.

De affectarum Æquationum reductione.

PROPOSITIS verò affectis Æquationibus, modus, quo radices earum ad hujuſmodi ſeries poſſint reduci, obnixiùs explicari debet;

bet; idque cùm earum doctrina, quam hactenus in numeris exposuerunt Mathematici, per ambages (ſuperfluis etiam operationibus adhibitis) tradatur, ut in ſpecimen operis in ſpeciebus non debeat adhiberi. Imprimis itaque Numeroſam affectarum æquationum Reſolutionem compendioſè tradam; dein Specioſam ſimiliter explicabo.

RESOLUTIO NUMEROSA

2. Proponatur æquatio $y^3-2y-5=0$, reſolvenda; et ſit 2 numerus, utcunque inventus, qui minùs quàm decimâ ſui parte differt à radice quæſitâ. Tum pono $2+p=y$, et pro y ſubſtituo $2+p$ in æquationem; et inde nova prodit $p^3+6p^2+10p-1=0$, cujus radix p exquirenda eſt, ut quotienti addatur. Nempe, neglectis p^3+6p^2 ob parvitatem, $10p-1=0$, ſive $p=0,1$ ad veritatem proximè accedet. Scribo itaque 0,1 in quotiente, & ſuppono $0,1+q=p$, et hunc ejus fictitium valorem, ut antè, ſubſtituo, et prodit $q^3+6,3q^2+11,23q+0,061=0$. Et cùm $11,23q+0,061=0$, veritatem appropinquet, ſive ferè ſit $q=-0,0054$ (dividendo nempe 0,061 per 11,23 donec tot eliciantur figuræ quot loca primis figuris hujus et principalis quotientis excluſivè intercedunt, quemadmodum hîc duo ſunt inter 2 & 0,005) ſcribo $-0,0054$, in inferiori parte quotientis, ſiquidem negativa ſit; et ſupponens $-0,0054+r=q$, hunc ut priùs ſubſtituo. Et ſic operationem ad placitum produco, pro more ſubjecti diagrammatis.

		$+2,10000000$
		$-0,00544852$
		$2,09455148$ &c.
$2+p=y$	y^3	$+8+12p+6p^2+p^3$
	$-2y$	$-4-2p$
	-5	-5
Summa		$-1+10p+6p^2+p^3$
$0,1+q=p$	$+p^3$	$+0,001+0,03q+0,3q^2+q^3$
	$+6p^2$	$+0,06 \quad +1,2 \quad +6,$
	$+10p$	$+1, \quad +10,$
	$-p$	$-1,$
Summa		$+0,061+11,23q+6,3q^2+q^3$
$-0,0054+r=q$	q^3	$-0,000000157464+0,000098748r-0,0162r^2+r^3$
	$+6,3q^2$	$+0,000183708 \quad -0,06804 \quad +6,3$
	$+11,23q$	$-0,060642 \quad +11,23$
	$+0,061$	$+0,061$
Summa		$+0,0005416 \quad +11,162r$
$-0,00004852+s=r$		

Opus verò ſub fine, præſertim in æquationibus plurium dimenſionum, hâc methodo multùm abbreviabitur. Determinato quouſque velis radicem extrahi, tot loca poſt primam figuram coefficientis penultimi

Caput Secundum.

mi termini æquationum, in dextrâ parte diagrammatis reſultantium, adnumera, quot ſuperſunt loca in quotiente complenda, et ſubſequentes decimales neglige. In ultimo verò termino decimales poſt tot plura loca neglige, quot in quotiente complentur loca decimalia. Inque antepenultimo termino neglige omnes poſt tot pauciora loca. Et ſic deinceps arithmeticè progrediendo per intervallum iſtud locorum; ſive quod perinde eſt, tot figuras paſſim elidendo, quot in penultimo termino; modò depreſſiſſima earum loca ſint in arithmeticâ progreſſione juxta ſeriem terminorum; aut circulis compleri ſubintelligantur, ubi res aliter eveniat. Sic in exemplo jam poſito, ſi cupiam ut quotiens ad octavum tantùm decimalem locum compleatur; inter ſubſtituendum, $00054+r$ pro q, ubi quatuor loca decimalia in quotiente complentur, ac totidem ſuperſunt complenda, potui figuras in inferioribus quinque locis omiſſiſſe, quas eapropter lineolâ tranſverſim notavi; imo primum terminum r^3, etſi coefficientem, 99999 habuiſſet, potui tamen penitus omiſiſſe. Expunctis itaque figuris iſtis, pro ſubſequente operatione prodit ſumma $0,0005417+11,162r$; quæ, per diviſionem aduſque præſcriptum terminum peractam, dat $-0,00004852$ pro r, quod quotientem ad optatam periodum complet.

Denique negativam partem quotientis ab affirmativâ ſubduco, et oritur $2,09455148$ quotiens abſoluta.

3. Præterea notandum eſt, quòd, ſub initio operis, ſi dubitarem an $0,1=p$ ad veritatem ſatis accederet, vice $10p-1=0$ finxiſſem $6p^2+10p-1=0$, et ejus radicis nihilo propioris primam figuram in quotiente ſcripſiſſem. Et hoc modo ſecundam vel etiam tertiam quotientis figuram explorare convenit, ubi in æquatione ſecundariâ, circa quam verſaris, quadratum coefficientis penultimi termini non ſit decies majus, quàm factus ex ultimo termino ducto in coefficientem termini antepenultimi. Quinimo laborem plerumque minues, præſertim in æquationibus plurimarum dimenſionum, ſi figuras omnes quotienti addendas hoc modo (id eſt, extrahendo minorem radicum ex tribus ultimis terminis æquationis ejus ſecundariæ) quæras. Sic enim figuras duplo plures in quotiente quâlibet vice lucraberis.

CAPUT

CAPUT TERTIUM.

De Specioſâ Æquationum Reſolutione.

HIS in numeris ſic oſtenſis, conſimiles operationes in ſpeciebus explicandæ reſtant, de quibus convenit ſequentia prænoſcere.

I. Quòd è ſpeciebus coefficientibus aliqua præ reliquis (ſi ſint plures) inſignienda ſit, ea nempe, quæ eſt, aut fingi poteſt eſſe, omnium longè minima, vel maxima, vel datæ quantitati viciniſſima: cujus rei cauſa eſt, ut ob ejus dimenſiones, in numeratoribus vel denominatoribus terminorum quotientis, perpetim auctas, illi termini continuò minores, et inde quotiens radici propinquior evadat; ſicut antè de ſpecie x, in exemplis reductionum per diviſionem et extractionem radicum, manifeſtum eſſe poteſt. Pro iſthâc verò ſpecie in ſequentibus ut plurimùm uſurpabo etiam x vel z, quemadmodum et y, p, q, r, s, &c. pro ſpecie radicali extrahendâ.

II. Siquando fractiones complexæ, vel ſurdæ quantitates, in æquatione propoſitâ, vel pòſt in operatione occurrant, tolli debent per methodos analyſtis ſatis notas. Quemadmodum ſi habeatur $y^3 + \frac{b^2}{b-x} y^2 - x^3 = 0$, multiplico per $b-x$, et ex facto $by^3 - xy^3 + b^2y^2 - bx^3 + x^4 = 0$, valorem y elicio. Vel poſſum fingere $y \times \overline{b-x} = v$, et ſic ſcribendo $\frac{v}{b-x}$ pro y, orietur $v^3 + b^2v^2 - b^3x^3 + 3b^2x^4 - 3bx^5 + x^6 = 0$: dein, extractâ radice v, divido quotientem per $b-x$ ut obtineatur valor y. Item ſi proponatur $y^3 - xy^{\frac{1}{2}} + x^{\frac{4}{3}} = 0$, fingo $y^{\frac{1}{2}} = v$, et $x^{\frac{1}{3}} = z$; et ſic, ſcribendo v^2 pro y, et z^3 pro x, oritur $v^6 - z^3v + z^4 = 0$; quâ æquatione reſolutâ, reſtituo y et x. Scilicet radix invenietur, $v = z + z^3 + 6z^5$ &c. et, reſtitutis y et x, orietur $y^{\frac{1}{2}} = x^{\frac{1}{3}} + x + 6x^{\frac{5}{3}}$ &c. et quadrando $y = x^{\frac{2}{3}} + 2x^{\frac{4}{3}} + 13x^2$ &c.

Ad eundem modum ſiquæ ſint negativæ dimenſiones ipſorum x & y, tollo, multiplicando per eaſdem x & y. Sic habito $x^3 + 3x^2y^{-1} - 2x^{-1} - 16y^{-3} = 0$, multiplico per x & y^3; oriturque $x^4y^3 + 3x^3y^2 - 2y^3 - 16x = 0$. Et habito $x = \frac{aa}{y} - \frac{2a^3}{y^2} + \frac{3a^4}{y^3}$, duco in y^3, et oritur $xy^3 = a^2y^2 - 2a^3y + 3a^4$. Et ſic de cæteris.

III. Æquatione ſic præparatâ, opus ab inventione primi termini

mini quotientis initium ſumit; de quâ, ut et conſimili ſubſequentium terminorum inventione, hæc eſto Regula generalis, cùm ſpecies indefinita (x vel z) parva eſſe fingitur, ad quem caſum cæteri duo caſus ſunt reducibiles.

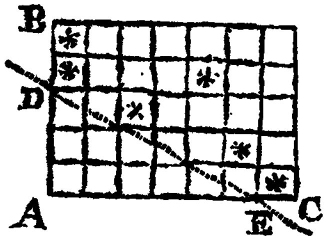
E terminis, in quibus ſpecies radicalis (y, p, q vel r, &c.) non reperitur, ſelige depreſſiſſimum reſpectu dimenſionum indefinitæ ſpeciei (x vel z, &c.) dein alium terminum, in quo ſit alia ſpecies radicalis, ſelige; talem nempe, ut progreſſio dimenſionum utriuſque præfatæ ſpeciei, à termino priùs aſſumpto ad hunc terminum continuata, quàm maximè poteſt deſcendat, vel minimè aſcendat. Et ſiqui ſint alii termini, quorum dimenſiones cum hâc progreſſione, ad arbitrium continuatâ, conveniant, eos etiam ſelige. Deinque ex his ſelectis terminis tanquam nihilo æqualibus quære valorem dictæ ſpeciei radicalis, et quotienti appone.

2. Cæterùm ut hæc Regula magis elucefcat, placuit inſuper ope ſequentis diagrammatis exponere. Deſcripto angulo recto BAC, latera ejus BA, AC, divido in partes æquales, et inde normales erigo diſtribuentes angulare ſpatium in æqualia quadrata vel parallelogramma, quæ concipio denominata eſſe à dimenſionibus ſpecierum x et y, prout vides in fig. 1. inſcriptas. Deinde cùm æquatio aliqua proponitur, parallelogramma, ſingulis ejus terminis correſpondentia, inſignio notâ aliquâ. Et Regulâ ad duo, vel fortè plura, ex inſignitis parallelogrammis applicatâ, quorum unum ſit humillimum in columnâ ſiniſtrâ juxta AB, et alia ad Regulam dextrorſum ſita, cæteraque omnia, non contingentia Regulam, ſupra eam jaceant: ſeligo terminos æquationis per parallelogramma contingentia regulam deſignatos, et inde quæro quantitatem quotienti addendam.

Fig. 1.

B	x^4	x^4y	x^4y^2	x^4y^3	x^4y^4	x^4y^5	
	x^3	x^3y	x^3y^2	x^3y^3	x^3y^4	x^3y^5	
	x^2	x^2y	x^2y^2	x^2y^3	x^2y^4	x^2y^5	
	x	xy	xy^2	xy^3	xy^4	xy^5	
A	o	y	y^2	y^3	y^4	y^5	C

Fig. 2.



3. Sic ad extrahendam radicem y ex $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$; parallelogramma, hujus terminis reſpondentia, ſigno

no notâ aliquâ *, ut vides in Schem. 2. Dein applico regulam DE ad inferiorem è locis ſignatis in ſiniſtrâ columnâ, eamque ab inferioribus ad ſuperiora dextrorſum gyrare facio, donec alium ſimiliter, vel fortè plura, è reliquis ſignatis locis cæperit attingere; videoque loca ſic attacta eſſe x^3, x^2y^2 et y^6. E terminis itaque $y^6-7a^2x^2y^2+6a^3x^3$ tanquam nihilo æqualibus (et inſuper ſi placet reductis ad $v^6-7v^2+6=0$, ponendo $y=v\sqrt{ax}$) quæro valorem y; et invenio quadruplicem $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$ et $-\sqrt{2ax}$: quorum quemlibet pro initio quotientis accipere liceat, prout è radicibus quampiam extrahere decretum eſt. Resolutio Speciosa.

Sic ex $y^5-by^2+9bx^2-x^3=0$, ſeligo $-by^2+9bx^2$, et inde obtineo $+3x$ pro initiali termino quotientis.

Et ex $y^3+axy+aay-x^3-2a^3=0$. Seligo $y^3+a^2y-2a^3$, et radicem ejus, $+a$, ſcribo in quotiente.

Et ex $x^2y^5-3c^4xy^2-c^5x^2+c^7=0$. Seligo $x^2y^5+c^7$, quod exhibet $-\sqrt[5]{\frac{c^7}{x^2}}$ pro initio quotientis. Et ſic de cæteris.

4. Cæterùm invento hoc termino, ſi is contingat eſſe negativæ poteſtatis, æquationem per eandem indefinitæ ſpeciei poteſtatem deprimo, eò, ut non opus ſit inter ſolvendum deprimere; et inſuper ut Regula de ſuperfluis terminis elidendis, mox tradenda, aptè poſſit adhiberi. Sic propoſito $8z^6y^3+az^6y^2-27a^9=0$, cujus quotiens exordiri debet à $\frac{3a^3}{zz}$, deprimo per zz; ut fiat, $8z^4y^3+az^4y^2-27a^9z^{-2}=0$, antequam ſolutionem ineo.

5. Subſequentes quotientum termini eâdem methodo ex æquationibus ſecundariis, inter operandum prodeuntibus, eruuntur; ſed ut plurimùm leviori curâ: res enim peragi ſolet, dividendo depreſſiſſimum è terminis, cum indefinitè parvâ ſpecie (x, xx, x^3, &c.) abſque ſpecie radicali (p, q, r, &c.) affectis, per quantitatem, quâcum ſpecies illa radicalis, unius tantùm dimenſionis, abſque alterâ indefinitâ ſpecie, afficitur; et exitum ſcribendo in quotiente. Sic in exemplo ſequente, termini $\frac{x}{4}$, $\frac{x^2}{64a}$, $\frac{131x^3}{512a^2}$, &c. eliciuntur dividendo a^2x, $\frac{1}{16}ax^2$, $\frac{131}{128}x^3$ &c. per $4aa$.

6. His præmiſſis, reſtat ut praxin Reſolutionis exhibeam.

Sit itaque $y^3+a^2y+axy-2a^3-x^3=0$, æquatio reſolvenda; et ex e-terminis $y^3+aay-2a^3=0$, æquatione fictitiâ juxta tertium è præ-

 miſſis,

CAPUT TERTIUM.

miſſis, elicio $y-a=0$, et ſcribo $+a$ in quotiente. Deinde cùm $+a$ non accuratè valeat y, pono $a+p=y$, et pro y, in terminis æquationis in margine ſcriptis, ſubſtituo $a+p$, terminoſque reſultantes (p^3+3ap^2+axp &c.) rurſum ſcribo in margine; ex quibus iterum, juxta tertium è præmiſſis, excerpo terminos $+4a^2p+a^2x=0$ pro æquatione fictitiâ; quæ cùm exhibeat $p=-\frac{1}{4}x$, pono $-\frac{1}{4}x$ in quotiente. Præterea cùm $-\frac{1}{4}x$ non accuratè valet p, ſcribo $-\frac{1}{4}x+q=p$, et pro p, in terminis marginalibus, ſubſtituo $-\frac{1}{4}x+q$, terminoſque reſultantes ($q^3-\frac{3}{4}xq^2+3aq^2$ &c.) iterum ſcribo in margine; ex quibus denuò, juxta Regulam præfatam, ſeligo terminos $4a^2q-\frac{1}{16}ax^2=0$, pro æquatione fictitiâ; quæ cùm exhibeat $q=\frac{x^2}{64a}$, ſcribo $+\frac{x^2}{64a}$ in quotiente. Porro cùm $\frac{x^2}{64a}$ non accurate valeat q, pono $\frac{x^2}{64a}+r=q$, et pro q, in terminis marginalibus, ſubſtituo $\frac{x^2}{64a}+r$, et ſic opus ad placitum produco, prout indicat ſubjectum diagramma.

$y^3+a^2y-2a^3+axy-x^3=0$		
$y=a-\frac{x}{4}+\frac{xx}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$,		
$+a+p=y$	$+y^3$	$+a^3+3a^2p+3ap^2+p^3$
	$+a^2y$	$+a^3+a^2p$
	$+axy$	$+a^2x+axp$
	$-2a^3$	$-2a^3$
	$-x^3$	$-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$
	$+3ap^2$	$+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$
	$+4a^2p$	$-a^2x+4a^2q$
	$+axp$	$-\frac{1}{4}ax^2+axq$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a}+r=q$	q^3	*
	$-\frac{3}{4}xq^2$	*
	$+3aq^2$	$+\frac{3x^4}{4096a}+\frac{3}{32}x^2r+3ar^2$
	$+4a^2q$	$+\frac{1}{16}ax^2+4a^2r$
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3-\frac{1}{2}axr$
	$+\frac{3}{16}x^2q$	$+\frac{3x^4}{1024a}+\frac{3}{16}x^2r$
	$-\frac{1}{16}ax^2$	$-\frac{1}{16}ax^2$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$
$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$		

7. Quòd ſi quotientem ad certam uſque periodum produci cupiam, ut x nempe, in ultimo ejus termino, ultra datum dimenſionum numerum non aſcendat, terminos inter ſubſtituendum ſemper omitto, quos nulli deinceps uſui fore prævideam. Cujus rei Regula eſto, quòd poſt primum terminum, ex quâlibet quantitate in margine collaterali reſultantem, non addantur plures dextrorſum, quàm iſtius primo

mò reſultantis termini dimenſio à periodicâ, ſive maximâ dimenſione quotientis deficit gradibus. Ut in hoc exemplo, ſi cupiam ut quotiens (ſive x in quotiente) ad quatuor tantùm dimenſiones aſcendat, omitto omnes terminos poſt x^4, & poſt x^3 pono unicum tantùm. Terminos itaque poſt notam * delendos eſſe concipe: et opere ſic continuato, donec ultimò ad terminos $\frac{15x^4}{4096a} - \frac{131}{128}x^3 + 4a^2r - \frac{1}{2}axr + \frac{9}{32}x^2r$ deveniatur, in quibus p, q, r vel s, &c. reſiduum radicis extrahendæ, ſit unicæ tantùm dimenſionis; tot terminos $\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ per diviſionem elicies, quot, ad complendum quotientem, deeſſe videbis. Atque ita tandem obtinebitur $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$. RESOLUTIO SPECIOSA.

8. Plenioris illuſtrationis gratiâ, dedi aliud exemplum, reſolvendo $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0$; ubi proponitur inventio quotientis ad quintam tantùm dimenſionem, terminique ſuperflui poſt notam (&c.) negliguntur.

$y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.		
$z + p = y$	$+\frac{1}{5}y^5$	$+\frac{1}{5}z^5$, &c.
	$-\frac{1}{4}y^4$	$-\frac{1}{4}z^4 - z^3p$ &c.
	$+\frac{1}{3}y^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$ &c.
	$-\frac{1}{2}y^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$ &c.
	$+ y$	$+ z + p$
	$- z$	$- z$
$\frac{1}{2}z^2 + q = p$	$+ zp^2$	$+\frac{1}{4}z^5$ &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$ &c.
	$+ z^3p$	$-\frac{1}{2}z^5$ &c.
	$+ z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$- zp$	$-\frac{1}{2}z^3 - zq$
	$+ p$	$+\frac{1}{2}z^2 + q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

9. Atque ita ſi cupiam æquationem $\frac{63y^{11}}{2816} + \frac{35}{1152}y^9 + \frac{5}{112}y^7 + \frac{3}{40}y^5 + \frac{1}{6}y^3 + y - z = 0$, aduſque nonam tantùm dimenſionem quotientis reſolvi, ante opus initum negligo terminum $\frac{63}{2816}y^{11}$; deinde, inter operandum, negligo etiam omnes terminos poſt z^9, poſt z^7 pono unicùm, ac duos tantùm poſt z^5; eò quòd percipio quotientem ubique per gradus binarum unitatum (hoc modo z, z^3, z^5, &c.) debere aſcendere. Tandemque prodit $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$.

10. Et hinc patet artificium, quo æquationes in infinitum affectæ, vel utcunque multis, numerovè infinitis terminis conſtantes, poſſunt ſolvi. Scilicet omnes termini ante opus initum debent negligi, in quibus dimenſio ſpeciei indefinitè parvæ, non

 affectæ

affectæ cum radicali ſpecie, tranſcendit maximam dimenſionem in quotiente deſideratam; vel ex quibus, ſubſtituendo pro radicali ſpecie primum terminum quotientis, ope teſſellatæ tabulæ inventum, non niſi ejuſmodi tranſcendentes termini poſſunt emergere. Sic in exemplo noviſſimo terminos omnes ſupra y^9, quamvis infinitè progrederentur, omiſiſſem. Et ſic in hâc æquatione

$$0 = \begin{cases} -8 + z^2 - 4z^4 + 9z^6 - 16z^8 \text{ \&c.} \\ +y \text{ in } z^2 - 2z^4 + 3z^6 - 4z^8 \text{ \&c.} \\ -y^2 \text{ in } z^2 - z^4 + z^6 - z^8 \text{ \&c.} \\ +y^3 \text{ in } z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \frac{1}{4}z^8 \text{ \&c.} \end{cases}$$

ut radix cubica ad quatuor tantùm dimenſiones ipſius z extrahatur, omitto omnes in infinitum terminos poſt $+y^3$ in $z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6$, et poſt $-y^2$ in $z^2 - z^4 + z^6$, et poſt $+y$ in $z^2 - 2z^4$, et poſt $-8 + z^2 - 4z^4$. Et hanc tantùm æquationem, $\frac{1}{3}z^6y^3 - \frac{1}{2}z^4y^3 + z^2y^3 - z^6y^2 + z^4y^2 - z^2y^2 - 2z^4y + z^2y - 4z^4 + z^2 - 8 = 0$, reſolvendam ſumo, ſiquidem $2z^{-\frac{2}{3}}$ (primus nempe quotientis terminus) pro y in reliquâ æquatione, per $z^{\frac{4}{3}}$ depreſſâ, ſubſtitutus, dat plures ubique quàm quatuor dimenſiones.

11. Quæ de altioribus æquationibus dixi, ad quadraticas etiam applicari poſſunt. Quemadmodum ſi hujus

$$0 = \begin{cases} yy \\ -y \text{ in } a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} \text{ \&c.} \\ + \frac{x^4}{4a^2} \end{cases}$$

radicem ad uſque periodum x^6 deſiderem, mitto terminos in infinitum poſt $-y$ in $a + x + \frac{x^2}{a}$; et iſthanc tantùm, $y^2 - ay - xy - \frac{x^2}{a}y + \frac{x^4}{4a^2} = 0$ (ſive id fiat hâc lege; $y = \frac{1}{2}a + \frac{1}{2}x + \frac{x^2}{2a} - \sqrt{\frac{1}{4}a^2 + \frac{1}{2}ax + \frac{3}{4}x^2 + \frac{x^3}{2a}}$, ut ſolet, ſive expeditiùs per methodum de affectis æquationibus jam traditam) reſolvo: et exit $y = \frac{x^4}{4a^3} - \frac{x^5}{4a^4} *$, ultimo deſiderato termino exiſtente nullo.

12. Poſtquam verò radices ad convenientem periodum extractæ ſunt, poſſunt aliquando ex analogiâ ſeriei obſervatâ, ad placitum produci. Sic hanc $z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c. (radicem æquationis infinitæ $z = y + \frac{1}{2}y^2 + \frac{1}{3}y^3$ &c.) perpetuò produces, dividendo ultimum

ultimum terminum per hos ordine numeros, 2, 3, 4, 5, 6, 7, &c. Et hanc, $z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c. dividendo per hos 2×3, 4×5, 6×7, 8×9, &c. Et hanc $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. multiplicando per hos $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{6}$, $-\frac{5}{8}$, $-\frac{7}{10}$, &c. Et sic in aliis ([b]). RESOLUTIO SPECIOSA.

13. Cæterùm in inventione primi termini quotientis, et nonnunquam ſecundi tertiive, difficultas etiamnum enodanda ſupereſt. Poteſt enim valor ejus, ſecundum præcedentia quæſitus, eſſe ſurda, ſive inextricabilis, radix æquationis multipliciter affectæ. Quod cùm accidit, modò non ſit inſuper impoſſibilis, illum literâ aliquâ deſignabis, dein operabere tanquam ſi cognitùm haberes. Quemadmodum in exemplo $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, ſi radix hujus $y^3 + a^2y - 2a^3 = 0$, fuiſſet ſurda, vel ignota; finxiſſem quamlibet b pro eâ ponendam eſſe, et reſolutionem (puta ad tertiam dimenſionem quotientis) ut ſequitur perfeciſſem.

$$y^3 + aay + axy - 2a^3 - x^3 = 0. \quad \text{Sit } cc = 3b^2 + a^2.$$

$$y = b - \frac{abx}{c^2} + \frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} - \frac{a^5bx^3}{c^6} + \frac{a^5b^3x^3}{c^{10}} \text{ \&c.}$$

$b + p = y$	$+y^3$	$+b^3 + 3b^2p + 3bp^2 + p^3$
	$+axy$	$+abx + axp$
	$+aay$	$+aab + aap$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$\frac{-abx}{cc} + q = p$	p^3	$-\frac{a^3b^3x^3}{c^6}$ &c.
	$+3bp^2$	$+\frac{3a^2b^3x^2}{c^4} \quad \frac{6ab^2x}{c^2}q$, &c.
	$+axp$	$-\frac{a^2bx^2}{c^2} + axq$
	$+ccp$	$-abx + ccq$
	$-x^3$	$-x^3$
	$+abx$	$+abx$

$$c^2 + ax - \frac{6ab^2x}{cc}\Big)\, \frac{a^4bx^2}{c^4} + x^3 + \frac{a^3b^3x^3}{c^6}\, \Big(\frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} \text{ \&c.}$$

Scribens b in quotiente, ſuppono $b+p=y$, & pro y ſubſtituo ut vides: unde prodit $p^3 + 3p^2b +$ &c. rejectis terminis $b^3 + a^2b - 2a^3$, qui nihilo ſunt æquales, propterea quòd b ſupponitur radix hujus $y^3 + a^2y - 2a^3 = 0$. Deinde termini $3b^2p + a^2p + abx$ dant $\frac{-abx}{3b^2 + a^2}$ quotienti apponendum, & $\frac{-abx}{3b^2 + a^2} + q$ ſubſtituendum pro p.

([b]) Vide Analyſ. per Æquat. Infinit. Cap. VIII.

 Brevitatis

CAPUT TERTIUM.

Brevitatis autem gratiâ ſcribo cc pro $3b^2+a^2$, cavendo tamen ut $3bb+aa$ reſtituatur, ubi terminos ſic abbreviari poſſe percipiam. Completo opere, aſſumo numerum aliquem pro a, et hanc $y^3+a^2y-2a^3=0$, ſicut de numerali æquatione oſtenſum ſuprà, reſolvo, et quemlibet ejus radicem, modò tres haberet, pro b ſubſtituo. Vel potius hujuſmodi æquationes à ſpeciebus, ut poſſum, libero, præſertim ab indefinitâ; idque pro more quam volui innuere, pag. 399, lin, 6 & 7; et pro cæteris tantùm, ſiquæ ſuperſint indelebiles, pono numeros. Sic $y^3+a^2y-2a^3=0$, liberabitur ab a dividendo radicem per a, fietque $y^3+y-2=0$, cujus inventa radix ducta in a ſubſtitui debet pro b.

14. Hactenus indefinitam ſpeciem ſuppoſui parvam eſſe. Quòd ſi datæ quantitati vicina ſupponatur, pro indefinitè parvâ differentiâ, pono ſpeciem aliquam, et hâc ſubſtitutâ, ſolvo ut antè. Quemadmodum in $\frac{1}{5}y^5-\frac{1}{4}y^4+\frac{1}{3}y^3-\frac{1}{2}y^2+y-a-x=0$, cognito vel ficto x eſſe ejuſdem propè quantitatis ac a, pono z differentiam inter ea; et ſcribendo $a+z$ vel $a-z$ pro x, orietur $\frac{1}{5}y^5-\frac{1}{4}y^4+\frac{1}{3}y^3-\frac{1}{2}y^2+y-$ vel $+z=0$, ſolvendum ut in præcedentibus.

15. Sin autem ſpecies illa ſupponatur indefinitè magna, pro reciproco ejus indefinitè parvo, pono ſpeciem aliquam; quâ ſubſtitutâ ſolvo ut antè. Sic habito $y^3+y^2+y-x^3=0$: ubi x cognoſcitur, vel fingitur eſſe valdè magnum, pro reciprocè parvo $\frac{1}{x}$ pono z; et ſubſtituto $\frac{1}{z}$ pro x, orietur $y^3+y^2+y-\frac{1}{z^3}=0$, cujus radix eſt $\frac{1}{z}-\frac{1}{3}-\frac{2}{9}z+\frac{7}{81}z^2+\frac{14}{729}z^4$ &c. et x ſi placet reſtituto, fit $y=x-\frac{1}{3}-\frac{2}{9x}+\frac{7}{81x^2}+\frac{14}{729x^4}$ &c.

16. Siquando ex aliquâ harum trium ſuppoſitionum res non omnino, aut non commodè, ſuccedat, ad aliam recurri poteſt. Sic in $y^4-x^2y^2+xy^2+2y^2-2y+1=0$, cùm primus terminus obtineri deberet, fingendo $y^4+2y^2-2y+1=0$, quæ tamen nullam admittit poſſibilem radicem, tento quid fiet aliter; quemadmodum ſi fingam x parùm differre à $+2$, ſive eſſe $2+z=x$, ſubſtituendo $2+z$ vice x prodibit $y^4-z^2y^2-2zy^2-2y+1=0$, et quotiens exordietur ab $+1$. Vel ſi fingam x indefinitè magnam eſſe, ſive $\frac{1}{x}=z$, obtinebitur $y^4-\frac{y^2}{z^2}+\frac{y^2}{z}+2y^2-2y+1=0$, et $+z$ pro initio quotientis.

 Et

Et hâc ratione ſecundum varias hypotheſes procedendo, licebit variis modis extrahere, ac deſignare radices. RESOLUTIO SPECIOSA.

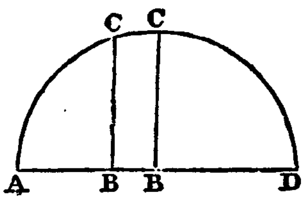
17. Quòd ſi cupias explorare, quot modis id poteſt fieri, tentabis quænam quantitates, pro indefinitâ ſpecie in æquationem propoſitam ſubſtitutæ, efficient diviſibilem per y + vel − aliquâ quantitate, vel per y ſolùm. Id quod, verbi gratiâ, in æquatione $y^3+axy+a^2y-x^3-2a^3=0$, eveniet ſubſtituendo $+a$, vel $-a$, vel $-2a$, vel $-\overline{2a^3}|^{\frac{1}{3}}$, &c. pro x. Atque ita poſſis commodè ſupponere quantitatem x parùm ab $+a$, vel $-a$, vel $-2a$, vel $-\overline{2a^3}|^{\frac{1}{3}}$, differre, et inde radicem propoſitæ æquationis tot modis extrahere. Imo et fortaſſe tot aliis modis fingendo differentias iſtas eſſe indefinitè magnas. Quinetiam ſi aliam atque aliam è ſpeciebus radicem definientibus pro indefinitâ adhibeas, poſſis aliis adhunc fortaſſe modis propoſitum conſequi; et etiamnum aliis, ſubſtituendo valores quâcunque ratione fictos (quales ſunt $az+bz^2$, $\frac{a}{b+x}$, $\frac{a+cz}{b+z}$, &c.) pro indefinitâ ſpecie, et in æquatione reſultante operando ſicut in præcedentibus.

18. Cæterùm ut concluſionum veritas conſtet, quotientes nempe ſic extractas, dum producuntur, ita propriùs ad radicem accedere, ut minùs tandem quâvis datâ quantitate differant, adeoque in infinitum productas non omnino differre: perpende, quòd quantitates in ſiniſtrâ columnâ dextræ partis diagrammatum, ſunt ultimi termini æquationum quarum p, q, r, s, &c. exiſtunt radices; et inde quòd, ipſis evaneſcentibus, illæ p, q, r, s id eſt differentiæ inter quotientem, et quæſitam radicem ſimul evaneſcunt: adeoque quotiens tunc non differt à radice. Quamobrem ſub initio operis, ſi terminos in dictâ columnâ ſeſe omnes deſtruere videas, conclude quotientem, eatenus extractam, eſſe juſtam radicem. Sin aliter, videbis tamen terminos, in quibus indefinitè parva ſpecies eſt pauciorum dimenſionum, id eſt, longè maximos, è columnâ iſtâ perpetuò tolli; ut tandem non reſtant niſi datâ quâvis quantitate minores, et proinde non majores nihilo, cùm opus infinitè producitur. Quare Quotiens infinitè extracta fiet etiam juſta radix.

19. Etſi denique ſpecies, quas hactenus, perſpicuitatis gratiâ, ſuppoſui indefinitè parvas eſſe, quantumvis magnæ ſupponantur, tamen

Caput Tertium.

tamen veræ erunt quotientes, ut minùs citò ad juſtam radicem convergant, quemadmodum ex analogiâ rei conſtet. Sed hîc radicum termini, maximæque et minimæ quantitates ſpectandæ veniunt: nam infinitarum cum finitis æquationibus communia ſunt hujuſmodi ſymptomata. Radix autem in his maxima fit vel minima, quando maxima vel minima eſt differentia ſummæ affirmativorum terminorum à ſummâ negativorum, ac terminatur, cùm indefinita quantitas, quam ideo parvam eſſe non immeritò finxi, non poteſt major ſumi, quin magnitudo radicis in infinitum proſiliet; hoc eſt, fiet impoſſibilis: verbi gratiâ, poſito ACD ſemicirculo ſuper diametro AD deſcripto, et BC ordinatim applicata. Dic $AB=x$, $BC=y$, $AD=a$. Et erit $y=(\sqrt{ax-x^2})=\sqrt{ax}-\frac{x}{2a}\sqrt{ax}-\frac{x^2}{8a^2}\sqrt{ax}-\frac{x^3}{16a^3}\sqrt{ax}$ &c. ut ſuprà. Fit ergo BC, ſive y, maxima cùm $\sqrt{ax}$ maximè ſuperat omnes $\frac{x}{2a}\sqrt{ax}+\frac{x^2}{8a^2}\sqrt{ax}+\frac{x^3}{16a^3}\sqrt{ax}$; id eſt, cùm ſit $x=\frac{1}{2}a$; terminabitur autem cùm ſit $x=a$: quia ſi ſumas x majorem quàm a ſumma omnium terminorum $-\frac{x}{2a}\sqrt{ax}-\frac{x^2}{8a^2}\sqrt{ax}-\frac{x^3}{16a^3}\sqrt{ax}$ &c. erit infinita. Eſt et alius terminus, cùm ponitur $x=0$, propter impoſſibilitatem radicalis $\sqrt{-ax}$; quibus terminis correſpondent ſemicirculi limites A et D.



CAPUT QUARTUM.

Doctrina Fluxionum.

HACTENUS de modis computandi, quorum poſthac frequens erit uſus. Jam reſtat, ut, in illuſtrationem Artis Analyticæ, tradam aliquot Problematum ſpecimina, qualia præſertim natura Curvarum miniſtrabit. Sed imprimis obſervandum venit, quòd hujuſmodi difficultates poſſunt omnes ad hæc duo tantùm Problemata reduci, quæ circa ſpatium motu locali, utcunque accelerato vel retardato, deſcriptum ponere licebit.

I. Spatii longitudine continuò (ſive ad omne tempus) datâ, Celeritatem Motûs ad tempus propoſitum invenire.

II. Celeritate

II. Celeritate Motûs continuò datâ, longitudinem deſcripti ſpatii ad tempus propoſitum invenire. DOCTRINA FLUXIONUM.

Sic in æquatione $xx=y$. Si y deſignat ſpatii longitudinem, ad quodlibet tempus quod aliud ſpatium x uniformi celeritate $\dot{x}$ creſcendo menſurat et exhibet, deſcriptam: tunc $2\dot{x}x$ deſignabit celeritatem, quâ ſpatium y, ad idem temporis momentum, deſcribi pergit; et contrà: et hinc eſt, quòd in ſequentibus conſiderem quantitates, quaſi generatæ eſſent per incrementum continuum, ad modum ſpatii quod mobile percurrendo deſcribit.

2. Cùm autem Temporis nullam habemus æſtimationem, niſi quatenùs id per æquabilem motum localem exponitur et menſuratur, et præterea cùm quantitates ejuſdem tantùm generis inter ſe conferri poſſunt, et earum incrementi et decrementi celeritates inter ſe; eapropter ad tempus formaliter ſpectatum in ſequentibus haud reſpiciam: ſed è propoſitis quantitatibus, quæ ſunt ejuſdem generis, aliquam æquabili fluxione augeri fingam, cui cæteræ tanquam tempori referantur, adeoque cui nomen temporis analogicè tribui mereatur. Si quando itaque vocabulum Temporis in ſequentibus occurrat (quemadmodum perſpicuitatis et diſtinctionis gratiâ nonnunquam intertexui) eo nòmine non Tempus formaliter ſpectatum ſubintelligi debet; ſed illa alia quantitas, cujus æquabili incremento, ſive fluxione, tempus exponitur et menſuratur.

3. Quantitates autem quas ut ſenſim creſcentes indefinitè conſidero, quo diſtinguam ab aliis quantitatibus, quæ in æquationibus quibuſcunque pro determinatis et cognitis habendæ ſunt ac initialibus literis a, b, c, &c. deſignantur, poſthac denominabo Fluentes, ac deſignabo finalibus literis v, x, y et z. Et celeritates, quibus ſingulæ à motu generante fluunt et augentur, quas poſſim fluxiones vel ſimpliciter celeritates vocitare, deſignabo literis $\dot{v}$, $\dot{x}$, $\dot{y}$ et $\dot{z}$, nempe pro celeritate quantitatis v ponam $\dot{v}$, et ſic pro celeritatibus aliarum quantitatum x, y et z ponam $\dot{x}$, $\dot{y}$ et $\dot{z}$ reſpectivè.

4. His præmiſſis, è veſtigio rem aggredior; imprimis duorum jam modo propoſitorum Problematum ſolutionem exhibiturus.

PROB.

PROBLEMA PRIMUM.

PROBLEMA PRIMUM.

Relatione quantitatum Fluentium inter ſe datâ, Fluxionum relationem determinare.

SOLUTIO.

ÆQUATIONEM, quâ data relatio exprimitur, diſpone ſecundum dimenſiones alicujus fluentis quantitatis, puta x, ac terminos ejus multiplica per quamlibet Arithmeticam progreſſionem; ac deinde per $\frac{\dot{x}}{x}$. Et hoc opus in quâlibet fluenti quantitate ſeorſim inſtitue. Dein omnium factorum ſummam pone nihilo æqualem, et habes æquationem deſideratam ([c]).

5. EXEMP. 1. Si quantitatum x et y relatio ſit $x^3-ax^2+axy-y^3=0$, terminos primò ſecundum x, ac deinde ſecundum y diſpoſitos multiplico ad hunc modum.

Mult.	x^3	$-\ ax^2$	$+axy$	$-y^3$	Mult.	$-\ y^3$	$+axy$	$-axx$ $+\ x^3$	
per	$\frac{3\dot{x}}{x}$	$\frac{2\dot{x}}{x}$	$\frac{\dot{x}}{x}$	0	per	$\frac{3\dot{y}}{y}$	$\frac{\dot{y}}{y}$	0	
fit	$3\dot{x}x^2$	$-2\dot{x}ax$	$+\dot{x}ay$	$-\ *$	fit	$-3\dot{y}y^2$	$+ax\dot{y}$	$*$	

Et factorum ſumma eſt $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$: æquatio quæ dat relationem inter fluxiones $\dot{x}$ et $\dot{y}$. Nempe ſi aſſumas x ad arbitrium, æquatio $x^3-ax^2+axy-y^3=0$ dabit y. Quibus determinatis erit $\dot{x}:\dot{y}::3y^2-ax:3x^2-2ax+ay$.

6. EXEMP. 2. Si quantitatum x, y et z, relatio ſit $2y^3+x^2y-2cyz+3yz^2-z^3=0$

Mult.	$2y^3+x^2y$ $-2czy$ $+3z^2y$	$-z^3$		Mult.	x^2y+	$2y^3$ $-2cyz$ $+3yz^2$ $-\ z^3$	Mult.	$-z^3$	$+3yz^2$	$-2cyz$	$+x^2y$ $+2y^3$
per	$\frac{2\dot{y}}{y}$	0	$-\frac{\dot{y}}{y}$	per	$\frac{2\dot{x}}{x}$	0	per	$\frac{3\dot{z}}{z}$	$\frac{2\dot{z}}{z}$	$\frac{\dot{z}}{z}$	0
fit	$4\dot{y}y^2$	$*$	$+\frac{\dot{y}z^3}{y}$	fit	$2\dot{x}xy+$	$*$	fit	$-3\dot{z}z^2$	$+6\dot{z}yz$	$-2cz\dot{y}$	$*$

Quare fluendi celeritatum $\dot{x}$, $\dot{y}$ et $\dot{z}$ relatio eſt $4\dot{y}y^2+\frac{\dot{y}z^3}{y}+2\dot{x}xy-3\dot{z}z^2+6\dot{z}yz-2cz\dot{y}=0$.

([c]) Vide Librum de Quad. Curv. Prop. I.

7. Cæterùm

6. Cæterùm cùm tres ſint hîc fluentes quantitates x, y & z, deberet alia inſuper æquatio dari, quâ relatio inter ipſas, ut et inter earum fluxiones, penitus determinetur. Quemadmodum ſi ponitur $x+y-z=0$; exinde fluxionum alia relatio juxta regulam erit $\dot{x}+\dot{y}-\dot{z}=0$. Confer jam haſce cum præcedentibus æquationibus, eliminando quamlibet è tribus quantitatibus, et quamlibet etiam è tribus earum fluxionibus, et reliquorum relationes penitius determinatas obtinebis.

INVENTIO FLUXIONUM.

8. Siquando in æquatione propoſitâ inſint fractiones complexæ, aut ſurdæ quantitates, pro ſingulis pono totidem literas, eaſque fingens deſignare quantitates fluentes, operor ut antè. Dein ſupprimo et extermino literas aſcriptitias, ut hîc videre eſt.

9. EXEMP. 3. Si quantitatum x et y relatio ſit $yy-aa-x\sqrt{aa-xx}$: pro $x\sqrt{aa-xx}$ ſcribo z; et inde habeo duas æquationes $y^2-a^2-z=0$, et $aaxx-x^4-z^2=0$: quarum prior ut antè dabit $2\dot{y}y-\dot{z}=0$, pro relatione celeritatum $\dot{y}$ et $\dot{z}$; et poſterior dabit $2a^2\dot{x}x-4\dot{x}x^3-2\dot{z}z=0$, ſive $\frac{aa\dot{x}x-2\dot{x}x^3}{z}=\dot{z}$, pro relatione celeritatum $\dot{x}$ et $\dot{z}$. Jam $\dot{z}$ ſuppreſſo, fiet $2\dot{y}y-\frac{aa\dot{x}x+2\dot{x}x^3}{z}=0$: dein reſtituto $x\sqrt{aa-xx}$ pro z, habebitur $2\dot{y}y-\frac{aa\dot{x}+2\dot{x}x^2}{\sqrt{aa-xx}}=0$, relatio inter $\dot{x}$ et $\dot{y}$, quæ quærebatur.

10. EXEMP. 4. Si $x^3-ay^2+\frac{by^3}{a+y}-x^2\sqrt{ay+x^2}=0$ deſignat relationem inter x et y: pono $z=\frac{by^3}{b+y}$, et $v=x^2\sqrt{ay+x^2}$, et inde nactus ſum tres æquationes, $x^3-ay^2+z-v=0$, $az+yz-by^3=0$, et $ax^4y+x^6-vv=0$. Prima dat $3\dot{x}x^2-2a\dot{y}y+\dot{z}-\dot{v}=0$. Secunda dat $a\dot{z}+\dot{z}y+\dot{y}z-3b\dot{y}y^2=0$, et tertia dat $4a\dot{x}x^3y+6\dot{x}x^5+a\dot{y}x^4-2\dot{v}v=0$, pro relationibus celeritatum $\dot{v}$, $\dot{x}$, $\dot{y}$ et $\dot{z}$. Ipſorum vero $\dot{z}$ et $\dot{v}$ valores per ſecundam ac tertiam inventos (nempe $\frac{3b\dot{y}y^2-\dot{y}z}{a+y}$ pro $\dot{z}$, et $\frac{4a\dot{x}x^3y+6\dot{x}x^5+a\dot{y}x^4}{2v}$ pro $\dot{v}$) ſubſtituo in primam, et oritur $3\dot{x}x^2-2a\dot{y}y+\frac{3b\dot{y}y^2-\dot{y}z}{a+y}-\frac{4a\dot{x}x^3y+6\dot{x}x^5+a\dot{y}x^4}{2v}=0$. Et vice z et v reſtitutis valoribus $\frac{by^3}{a+y}$ et $x^2\sqrt{ay+xx}$, prodit æquatio quæſita, $3\dot{x}x^2-2a\dot{y}y+\frac{3ab\dot{y}y^2+2b\dot{y}y^3}{a^2+2ay+y^2}-\frac{4a\dot{x}xy+6\dot{x}x^3+a\dot{y}xx}{2\sqrt{ay+xx}}=0$, quâ relatio celeritatum $\dot{x}$ et $\dot{y}$ deſignatur.

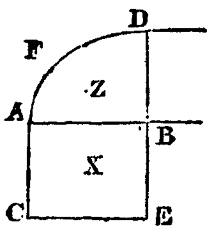
11. Quo pacto in aliis caſibus operandum eſt, quemadmodum

PROBLEMA PRIMUM. cùm in æquatione propoſitâ reperiuntur ſurdi denominatores, radicales cubicæ, radicales intra radicales, ut $\sqrt{ax+\sqrt{aa-xx}}$ aut alii ejuſmodi perplexi termini, ex his credo manifeſtum eſſe.

12. Quinimo ſi in æquatione quantitates involventur, quæ nullâ ratione geometricâ determinari et exprimi poſſunt, quales ſunt Areæ vel Longitudines curvarum; tamen relationes fluxionum haud ſecus inveſtigantur, prout in exemplo ſequente conſtabit.

Præparatio in Exempl. 5.

13. Pono BD ordinatam eſſe in angulo recto ad AB, et quòd ADH ſit Curva, quæ per relationem inter AB et BD, æquatione quâlibet exhibitam, definitur. AB vero dicatur x, et curvæ area ADB ad unitatem applicata dicatur z. Dein erige perpendiculum AC æquale unitati, et per C due CE parallelam AB et occurrentem BD in E, et concipiendo has duas ſuperficies, ADB & ACEB, genitas eſſe per motum rectæ BED, manifeſtum erit,



quòd earum fluxiones (hoc eſt fluxiones quantitatum $1\times z$ & $1\times x$, ſive quantitatum z & x) ſunt inter ſe ut BD et BE, lineæ generantes. Eſt ergo $\dot{z}:\dot{x}::$ BD : BE, ſive 1; adeoque $\dot{z}=\dot{x}\times$ BD.

14. Et hinc fit, quòd z in æquatione quâlibet deſignante relationem inter x et aliam quamvis fluentem quantitatem y involvi poteſt, et tamen fluxionum $\dot{x}$ & $\dot{y}$ relatio nihilo minùs invenire.

15. EXEMP. 5. Quemadmodum ſi proponatur $z^2+axz-y^4=0$, pro deſignandâ relatione inter x et y, ut et $\sqrt{ax-xx}=$ BD, pro curvâ determinandâ, quæ proin erit circulus: æquatio $z^2+axz-y^4=0$, ſicut in præcedentibus, dabit $2\dot{z}z+a\dot{z}x+a\dot{x}z-4\dot{y}y^3=0$, pro relatione celeritatum $\dot{x}$, $\dot{y}$ & $\dot{z}$. Et præterea, cùm ſit $\dot{z}=\dot{x}\times$ BD ſive $=\dot{x}\sqrt{ax-xx}$, pro eo ſubſtitue hunc valorem, et orietur $2\dot{x}z+a\dot{x}x\sqrt{ax-xx}+a\dot{x}z-4\dot{y}y^3=0$ æquatio definiens relationem celeritatum $\dot{x}$ et $\dot{y}$.

16. DEMONSTR. Fluentium quantitatum momenta (i. e. earum partes indefinitè parvæ, quarum additamento per ſingula temporis

temporis indefinitè parva ſpatia augentur) ſunt ut fluendi celeritates. Quare ſi cujuſvis, ut x, momentum per factum ex ejus celeritate $\dot{x}$, et infinitè parvâ quantitate o (i. e. per $\dot{x}o$) deſignetur, cæterorum v, y, z momenta per $\dot{v}o$, $\dot{y}o$, $\dot{z}o$, deſignabuntur, ſiquidem $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, & $\dot{z}o$. Sunt inter ſe ut $\dot{v}$, $\dot{x}$, $\dot{y}$, & $\dot{z}$. Inventio Fluxionum.

Jam cùm quantitatum fluentium (ut x & y) momenta (ut $\dot{x}o$ & $\dot{y}o$) ſint additamenta infinitè parva, quibus illæ quantitates, per ſingula temporis infinitè parva intervalla, augentur; ſequitur, quòd quantitates illæ x et y, poſt quodlibet infinitè parvum temporis intervallum, futuræ ſunt $x+\dot{x}o$, et $y+\dot{y}o$. Et inde æquatio, quæ relationem quantitatum fluentium ad omne tempus indifferenter deſignat, æquè deſignabit relationem inter $x+\dot{x}o$, et $y+\dot{y}o$, ac inter x et y: adeo ut $x+\dot{x}o$ et $y+\dot{y}o$ pro quantitatibus iſtis, vice x et y, in dictam æquationem ſubſtitui poſſunt.

Detur itaque quælibet æquatio $x^3-ax^2+axy-y^3=o$, et ſubſtitue $x+\dot{x}o$ pro x, et $y+\dot{y}o$ pro y, et emerget.

$$\left.\begin{array}{llll} x^3+3\dot{x}ox^2 & +3\dot{x}^2oox & +\dot{x}^3o^3 & \\ -ax^2-2a\dot{x}ox & -a\dot{x}^2oo & & \\ +axy+a\dot{x}oy+ & a\dot{y}ox & +a\dot{x}\dot{y}oo & \\ -y^3-3\dot{y}oy^2 & -3\dot{y}^2ooy & -\dot{y}^3o^3 & \end{array}\right\} = o$$

Jam ex hypotheſi ſunt $x^3-ax^2+axy-y^3=o$; quibus deletis, et reliquis terminis per o diviſis, reſtabunt $3\dot{x}x^2+3\dot{x}^2ox+\dot{x}^3oo-2a\dot{x}x-a\dot{x}\dot{x}o+a\dot{x}y+a\dot{y}x+a\dot{x}\dot{y}o-3\dot{y}y^2-3\dot{y}^2oy-\dot{y}^3oo=o$. Et inſuper, cùm o ſupponitur eſſe infinitè parvum, eò ut momenta quantitatum deſignare poſſit; termini per illud multiplicati, reſpectu cæterorum, nihil valebunt. Rejicio itaque, et reſtat $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y+a\dot{y}x-3\dot{y}y^2=o$, ut ſuprà in Exempl. 1.

Hinc obſervare eſt, quòd termini non multiplicati per o ſemper evaneſcent, ut et illi multiplicati per o pluſquam unius dimenſionis: et quòd reliquorum terminorum, per o diviſorum, ea ſemper erit forma, quam juxta regulam habere debent. Id quod volui oſtendere.

17. Et hoc monſtrato, cætera, quæ regula involvit, facilè conſequentur; quemadmodum quòd in æquatione propoſitâ plures fluentes quantitates involvi poſſunt, et quòd termini non modò

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per numerum dimenſionum quantitatum fluentium, ſed per quaſlibet alias Arithmeticas Progreſſiones multiplicari poſſunt; dummodo in operatione, juxta quamlibet fluentem quantitatem, ſit eadem terminorum differentia, et progreſſio ſecundum eundem cujuſque dimenſionum ordinem diſponatur ([d]). Et his conceſſis, quæ præterea in exemplis 3, 4 & 5, docentur per ſe manifeſta ſunt.

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18. *Expoſitâ æquatione Fluxiones quantitatum involvente relationem quantitatum inter ſe invenire.*

Solutio Particularis.

Cum hoc Problema ſit præcedentis converſum, contrario modo ſolvi debet: utpote terminos per $\dot{x}$ multiplicatos diſponendo ſecundum dimenſiones ipſius x, dividendoque per $\frac{\dot{x}}{x}$, ac dein per progreſſionem; atque idem opus in terminis per $\dot{v}$, $\dot{y}$ vel $\dot{z}$ multiplicatis inſtituendo, et reſultantium ſummam, rejectis terminis redundantibus, ponendo æqualem nihilo.

19. Exempl. Sic expoſitâ æquatione $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$; operor ad hunc modum.

Divido	$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$	divido	$-3\dot{y}y^2 + a\dot{y}x$
per $\frac{\dot{x}}{x}$ & fit	$3x^3 - 2ax^2 + axy$	per $\frac{\dot{y}}{y}$ fit	$-3y^3 \quad + ayx$
dein div. per	$3 \quad . \quad 2 \quad . \quad 1$	div. per	$3 \quad . \quad 1$
et fit	$x^3 \quad - ax^2 \quad + axy$	fit	$-y^3 \quad + axy$

Et ſumma $x^3 - ax^2 + axy - y^3 = 0$, erit relatio deſiderata quantitatum

([d]) Nimirum ſi Æquationem quamlibet propoſitam, puta $x^3 - ax^2 + axy - y^3 = 0$, in quantitatem $x^m y^n$ multiplicaveris, tranſmutaveris in hanc aliam; $x^{m+3}y^n - ax^{m+2}y^n + ax^{m+1}y^{n+1} - x^m y^{n+3} = 0$ quæ eaſdem quas prior illa quantitatum x, y inter ipſas relationes, alio licet modo, exprimet.

Jam ex tranſmutatâ, multiplicationem membrorum cum ſeriebus illis arithmeticis, quas tranſmutatæ indices conſtituunt, inſtituendo, hanc elicies fluxionum æquationem: $\overline{m+3}\, y^n x^{m+2}\dot{x} - \overline{m+2}.\, ay^n x^{m+1}\dot{x} + \overline{m+1}.\, ay^{n+1}x^m\dot{x} - my^{n+3}x^{m-1}\dot{x} - \overline{n+3}.\, x^m y^{n+2}\dot{y} + \overline{n+1}.ax^{m+1}y^n\dot{y} - n.ax^{m+2}y^{n-1}\dot{y} + nx^{m+3}y^{n-1}\dot{y} = 0$.

Quæ membris ejus omnibus à quantitate $x^m y^n$ diviſis, in hanc mutabitur; $\overline{m+3}.\, x^2\dot{x} - \overline{m+2}.ax\dot{x} + \overline{m+1}.ay\dot{x} - my^3x^{-1}\dot{x} - \overline{n+3}\,y^2\dot{y} + \overline{n+1}\,ax\dot{y} - nax^2y^{-1}\dot{y} + nx^3y^{-1}\dot{y} = 0$: quam minori quidem negotio è datâ deduxiſſes, ſi datam, ſecundum indices literæ x primùm, tum literæ y, ordinnatm, in ſeries, primùm $m+3.\ m+2.\ m+1.\ m$, tum $n+3.\ n+2.\ n+1.\ n$, multiplicâſſes, et reliqua peregiſſes ex præceptis Newtoni. Hæc autem æquatio, fluxionum $\dot{x}$, $\dot{y}$ relationes mutuas

tum x et y. Ubi obſervandum venit, quòd etſi terminus axy bis reſultavit, tamen non pono bis in hâc ſummâ, $x^3 - ax^2 + axy - y^3$ ſed redundantem terminum rejicio. Et ſic ubicunque terminus aliquis bis reſultat, aut ſæpiùs, ſi de pluribus fluentibus quantitabus agitur, ſemel tantùm in ſummâ terminorum ſcribo.

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20. Sunt et aliæ circumſtantiæ, quas artificis ingenio pro re natâ obſervandas eſſe remitto; nam ſupervacaneum eſſet his multa verba impendere; ſiquidem Problema non ſemper poteſt hoc artificio ſolvi. Addo tamen, quòd poſtquam artifex relationem fluentium quantitatum hâc methodo adeptus eſt, ſi juxta Prob. 1. poteſt regredi ad expoſitam æquationem fluxiones involventem, rectè operatus eſt; ſin ſecùs vitiosè. Sic in exemplo propoſito, ubi æquationem $x^3-ax^2+axy-y^3=0$ adeptus ſum, ſi relatio inter $\dot{x}$ et $\dot{y}$ ope primi Problematis viciſſim inde requiratur, obtinebitur æquatio expoſita $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$. Unde conſtat æquationem $x^3-ax^2+axy-y^3=0$, rectè inventam fuiſſe. At ſi æquatio $x\dot{x}-\dot{x}y+\dot{y}a=0$, exponeretur, et inde præſcriptâ methodo elicerem $\frac{1}{2}x^2-xy+ay=0$, pro relatione inter x et y, vitioſa foret operatio; ſiquidem exinde per Prob. 1. viciſſim produceretur $\dot{x}x-\dot{x}y-\dot{y}x+\dot{y}a=0$, quæ differt ab æquatione primò expoſitâ.

21. Hæc itaque perfunctoriè notata prætermittens, ſolutionem generalem aggredior.

Preparatio in Generalem Solutionem.

Et imprimis obſervandum eſt, quòd in expoſitâ æquatione ſymbola fluxionum, cùm ſint quantitates diverſi generis à quantitatibus quarum ſunt fluxiones, in ſingulis terminis debent ad

mutuas æquè exprimit, licet alio quidem modo, ac ſimplicior illa, quam è datâ, multiplicando in ſeries quas datæ indices conſtituunt, deducere licet; quæ hæc erit, $3x\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0$.

Similiter in omni caſu, ſive in ſeries quas poteſtatum indices conſtituunt, ſive in illas quas conſtituunt iidem indices datis quibuſvis quantitatibus, m, n aucti, æquationem propoſitam multiplices, & reliqua, quæ Newtonus præcepit, exſequaris, idem planè effeceris. Æquationes varias quidem efformaveris, ſed quæ variis modis idem omnes ſignificabunt; luculentè tamen aliæ, aliæ magis obſcurè. *Colſon ad Locum.*

Cæterùm rationem hanc inutiliter implicitam, quâ initio uſus eſt, Newtonus poſtea meritò repudiavit; multiplicationibus in ſeries indicum contentus. Cùm reverâ multiplicationibus illis in indices datis quantitatibus auctos vix aliud efficiatur, niſi ut æquatio fluxionalis, ex occultâ quaſi æquationis propoſitæ multiplicatione, formâ implicatiore prodeat. *Vide Wilſon. Appendix to Robins's Works.*

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æque-multas dimenſiones aſcendere (e). Et ubi res aliter ſe habet; alia alicujus fluentis quantitatis Fluxio ſubintelligi debet eſſe unitas, per quam termini depreſſiores toties multiplicantur, ut in omnibus ſymbola fluxionum ad eundem dimenſionum gradum aſcendant. Quemadmodum ſi exponitur æquatio $\dot{x}+\dot{x}\dot{y}x-xx=0$, tertiæ alicujus fluentis quantitatis, ut z, fluxio $\dot{z}$ ſubintelligi debet eſſe unitas; per quam primus terminus $\dot{x}$ ſemel, et ultimus xx bis multiplicetur, ut fluxiones inibi ad æque-multas dimenſiones, ac in ſecundo termino, $\dot{x}\dot{y}x$, aſcendant: quaſi expoſita æquatio ex hâc $\dot{z}\dot{x}+\dot{x}\dot{y}x-\dot{z}\dot{z}x^2=0$ derivata fuiſſet, ponendo $\dot{z}=1$. Et ſic in æquatione $\dot{y}x=yy$ debes imaginari $\dot{x}$ eſſe unitatem, per quam terminus yy multiplicatur.

22. Æquationes autem, in quibus duæ tantùm ſunt fluentes quantitates, quæ ad æque-multas dimenſiones paſſim aſcendunt, ſemper poſſunt ad talem formam reduci, ut ex unâ parte habeatur ratio fluxionum (velut $\frac{\dot{y}}{\dot{x}}$ vel $\frac{\dot{x}}{\dot{y}}$ vel $\frac{\dot{z}}{\dot{x}}$ &c.) et ex alterâ parte valor ejus rationis ſimplicibus terminis algebraicis deſignatus, ſicut hîc videre eſt: $\frac{\dot{y}}{\dot{x}}=2+2x-y$. Et ubi æquationibus præcedens particularis ſolutio non ſatisfacit, requiritur ut ad hanc formam reducas.

23. Quamobrem cùm in illius rationis valore terminus aliquis à compoſitâ quantitate denominetur, vel ſit radicalis, vel ſi ratio illa ſit æquationis radix affectæ: reductio vel per diviſionem, vel extractionem radicis, vel æquationis affectæ reſolutionem, inſtitui debet; prout in ſuperioribus oſtenſum eſt.

24. Quem-

(e) NIMIRUM huic *Fluxionis* vocabulo Newtonus duas ferè res ſubjecit: magnitudinum mutabilium mutandi Celeritates, et Magnitudines cujuſcunque generis, quibus celeritates illæ geometricè exhibeantur. Priore quidem ſenſu, qui primarius eſt, Fluxiones diverſi generis ſunt res, ac quantitates illæ, quarum Fluxiones ſunt. Quæ verò poſteriore ſenſu magnitudines Geometricæ Fluxiones dici ſolent, eas ſanè in eodem genere ſumi, in quo ipſarum ſint Fluentes, nihil vetat. Quanquam verò notis fluxionum algebraicis hæ ſæpiùs ſignificari exiſtimandæ ſunt, quàm celeritates illæ, quas repræſentant; in omni tamen Æquatione Fluxionali, ex quâ Fluentium relationes eliciendæ ſint, oportet membra ſingula notis fluxionum pariter implicari: proinde ac ſi magnitudinum fluentium, atque illarum, quæ fluendi celeritates exponant, genus neceſſariò diverſum eſſet. Aliter enim æquatio relationes Fluxionum mutuas non purè exprimet; ſed permiſtè quodammodo magnitudinum quibus geometricè fluxiones exponantur, harum, inquam, permiſtè inter ipſas relationes et cum fluentibus. Ut æquatio $b^2\dot{x}+\dot{x}\dot{y}x-ax^2=0$, relationes quidem indicat, quæ rectis fortè $\dot{x}$, $\dot{y}$ inter ipſas, et cum aliâ rectâ x, intercedant; minimè verò illas, quæ, nullâ tertiæ x ratione habitâ, duarum $\dot{x}$, $\dot{y}$ ſint peculiares. Itaque hujuſmodi æquatio non eſt verè Fluxionalis.

24. Quemadmodum ſi exponitur $\dot{y}a-\dot{y}x-\dot{x}a+\dot{x}x-\dot{x}y=0$: hæc imprimis reductione vel fit $\frac{\dot{y}}{\dot{x}}=1+\frac{y}{a-x}$, vel $\frac{\dot{x}}{\dot{y}}=\frac{a-x}{a-x+y}$. Et in priori caſu, ſi terminum $\frac{y}{a-x}$, à compoſitâ quantitate $a-x$ denominatum, reduco ad infinitam ſeriem ſimplicium terminorum, $\frac{y}{a}+\frac{xy}{aa}+\frac{x^2y}{a^3}+\frac{x^3y}{a^4}$ &c. dividendo numeratorem y per denominatorem $a-x$, obtinebo $\frac{\dot{y}}{\dot{x}}=1+\frac{y}{a}+\frac{xy}{a^2}+\frac{x^2y}{a^3}+\frac{x^3y}{a^4}$ &c. cujus ope relatio inter x et y determinanda eſt. Inventio Fluentium.

25. Sic expoſitâ $\dot{y}\dot{y}=\dot{x}\dot{y}+\dot{x}\dot{x}xx$, ſive $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x}}=\frac{\dot{y}}{\dot{x}}+xx$; et ulteriori reductione $\frac{\dot{y}}{\dot{x}}=\frac{1}{2}\pm\sqrt{\frac{1}{4}+xx}$; radicem quadraticam è terminis $\frac{1}{4}+xx$ extraho; et obtineo infinitam ſeriem $\frac{1}{2}+xx-x^4+2x^6-5x^8+14x^{10}$ &c. quam pro $\sqrt{\frac{1}{4}+xx}$ ſubſtituendo, prodit $\frac{\dot{y}}{\dot{x}}=1+x^2-x^4+2x^6-5x^8$ &c. vel $\frac{\dot{y}}{\dot{x}}=-x^2+x^4-2x^6+5x^8$ &c. prout $\sqrt{\frac{1}{4}+xx}$ additur vel ſubducitur à $\frac{1}{2}$.

26. Atque ita ſi exponitur $\dot{y}^3+ax\dot{x}^2\dot{y}+a^2\dot{x}^2\dot{y}-\dot{x}^3x^3-2\dot{x}^3a^3=0$, ſive $\frac{\dot{y}^3}{\dot{x}^3}+ax\frac{\dot{y}}{\dot{x}}+aa\frac{\dot{y}}{\dot{x}}-x^3-2a^3=0$, extraho radicem cubicam affectam, et prodit $\frac{\dot{y}}{\dot{x}}=a-\frac{x}{4}+\frac{xx}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$ &c. prout videre eſt Cap. III. § 6.

27. Cæterùm hîc obſervandum venit, quòd terminos ſolummodo pro compoſitis habeo, qui ex parte fluentium quantitatum componuntur. Terminos ubi nulla eſt, niſi ex parte datarum quantitatum, compoſitio, pro ſimplicibus habeo: ſiquidem ad ſimplices reduci poſſunt, fingendo æquales eſſe aliis datis. Sic

nalis. Verè quidem Fluxionalis eſſet, ſi rectarum $\dot{y}$, $\dot{x}$, eas relationes purè exprimeret, quæ ex eo illis intervenerint, quòd earum celeritatum proportiones, quibus celeritatibus rectæ mutabiles y, x mutantur, inter ſe ſemper ſervent. Has verò purè ſi exprimerit, nullam planè æquatio illa indicaret rectæ $\dot{y}$, vel $\dot{x}$, ad illam x relationem; quæ inde unde diximus nulla quidem naſci poteſt; ſiquidem Motûs celeritas et res mota genere ſint alieniſſima. At verò ex æquatione non verè fluxionali mutuas fluentium relationes eruere fruſtra aggreſſus eris, niſi artificio aliquo ad formam verè fluxionalem priùs ea revocata ſit. Sanè ſi pro æquatione, $b^2\dot{x}+\dot{x}\dot{y}x-a\dot{x}^2=0$, hanc aliam ſumas, $b^2\dot{x}\dot{z}+b\dot{x}\dot{y}x-\frac{ax^2\dot{z}\dot{z}}{b}=0$, quæ, poſito utramque b, z unitati æqualem eſſe, in illam priorem migraverit, ex hâc quidem trium x, y, z relationes mutuas, viâ mox tradendâ, explorare liceat, et per æquationem quandam exponere. Tum ſi illius z ad illam x, vel ad illam y, vel ad duas x, y ſimul, relationes aliæ per æquationem aliam, vel datam vel pro arbitrio fingendam, exprimantur, ex duabus æquationibus tertia utique exſiſtet, quâ duæ x, y quomodo inter ſe affectæ ſint, nullâ tertiæ z ratione habitâ, indicatum erit.

quantitates

PROBLEMA SECUNDUM.

quantitates $\frac{ax+bx}{c}$, $\frac{x}{a+b}$, $\frac{bcc}{ax+bx}$, $\frac{b^4}{ax^2+bx^2}$, $\sqrt{ax+bx}$ &c. pro ſimplicibus; habeo ſiquidem ad ſimplices $\frac{\varepsilon x}{c}$, $\frac{x}{\varepsilon}$, $\frac{bcc}{\varepsilon x}$, $\frac{b^4}{\varepsilon x^2}$, $\sqrt{\varepsilon x}$, ſive $\varepsilon^{\frac{1}{2}}x^{\frac{1}{2}}$ &c. reduci poſſunt, fingendo eſſe $a+b=\varepsilon$.

28. Præterea, quo fluentes quantitates à ſe invicem clariùs diſtinguantur, Fluxionem quæ, in Numeratore rationis diſponitur, ſive antecedentem rationis, haud impropriè Relatam quantitatem nominare poſſum, et alteram ad quam refertur, Correlatam; ut et fluentes quantitates iiſdem reſpectivè nominibus inſignire. Et quo ſequentia promptiùs intelligantur, poſſis imaginari Correlatam quantitatem eſſe Tempus, vel potiùs quamvis æquabiliter fluentem quantitatem, quâ tempus exponitur et menſuratur; et alteram, ſive Relatam quantitatem, eſſe Spatiùm, quod mobile, utcunque acceleratum vel retardatum, in illo tempore tranſigit: et quòd Problematis intentio eſt, ut è celeritate motûs ad omne tempus datâ, ſpatium, in toto tempore tranſactum, determinetur.

29. Cæterùm Æquationes, reſpectu hujus Problematis, in tres ordines diſtingui convenit.

1. In quibus duæ quantitatum Fluxiones et alterutra tantùm Fluens quantitas involvuntur.

2. In quibus duæ involvuntur fluentes quantitates unà cum Fluxionibus earum.

3. Quæ plures duabus quantitatum Fluxionibus complectuntur.

Et his præmiſſis Problematis confectionem ſecundum hoſce tres caſus aggrediar.

SOLUTIONIS CASUS PRIMUS.

30. Fluentem quantitatem, quam unicè æquatio complectitur, ſuppone Correlatam eſſe, et æquatione perinde diſpoſitâ (hoc eſt faciendo ut ex unâ parte habeatur fluxionis alterius ad hujus fluxionem ratio, et valor ejus in ſimplicibus terminis ex alterâ) multiplica valorem rationis fluxionum per Correlatam quantitatem; dein ſingulos ejus terminos divide per numerum dimenſionum, quibus illa quantitas inibi afficitur; et quod oritur valebit alteram fluentem quantitatem.

4 31. Sic

31. Sic expoſitâ $\dot{y}\dot{y}=\dot{x}\dot{y}+\dot{x}\dot{x}xz$, ſuppono x eſſe correlatam quantitatem; et, æquatione perinde reductâ, habebitur $\frac{\dot{y}}{\dot{x}}=1+x^2-x^4+2x^6$, &c. Jam duco valorem $\frac{\dot{y}}{\dot{x}}$ in x, et oritur $x+x^3-x^5+2x^7$, &c. quos terminos ſigillatim per numerum dimenſionum divido, et exitum $x+\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{2}{7}x^7$, &c. pono $=y$. Et iſthâc æquatione deſiderata relatio inter x et y determinatur. INVENTIO FLUENTIUM.

32. Sic habitâ $\frac{\dot{y}}{\dot{x}}=a-\frac{x}{4}+\frac{x^2}{64a}+\frac{131x^3}{512a^2}$, &c. prodibit $y=ax-\frac{x^2}{8}+\frac{x^3}{192a}+\frac{131x^4}{2048a^2}$, &c. pro determinandâ relatione inter x & y.

33. Et ſic æquatio $\frac{\dot{y}}{\dot{x}}=\frac{1}{x^3}-\frac{1}{x^2}+\frac{a}{x^{\frac{1}{2}}}-x^{\frac{1}{2}}+x^{\frac{3}{2}}$, dat $y=-\frac{1}{2x^2}+\frac{1}{x}+2ax^{\frac{1}{2}}-\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}$. Nam valorem $\frac{\dot{y}}{\dot{x}}$ duc in x, et fit $\frac{1}{xx}-\frac{1}{x}+ax^{\frac{1}{2}}-x^{\frac{3}{2}}+x^{\frac{5}{2}}$, ſive $x^{-2}-x^{-1}+ax^{\frac{1}{2}}-x^{\frac{3}{2}}+x^{\frac{5}{2}}$. Quibus terminis per numerum dimenſionum diviſis, emergit valor aſſignatus y.

3. Ad eundem modum æquatio $\frac{\dot{x}}{\dot{y}}=\frac{2b^2c}{\sqrt{ay^3}}+\frac{3y^2}{a+b}+\sqrt{by+cy}$ dat $x=-\frac{4b^2c}{\sqrt{ay}}+\frac{y^3}{a+b}+\frac{2}{3}\sqrt{by^3+cy^3}$. Nam valore $\frac{\dot{x}}{\dot{y}}$ ducto in y, oritur $\frac{2b^2c}{\sqrt{ay}}+\frac{3y^3}{a+b}+\sqrt{by^3+cy^3}$, ſive $2b^2ca^{-\frac{1}{2}}y^{-\frac{1}{2}}+\frac{3}{a+b}y^3+\sqrt{b+c}\times y^{\frac{3}{2}}$. Et inde prodit valor x, dividendo per numerum dimenſionum cujuſque termini.

35. Atque ita $\frac{\dot{y}}{\dot{z}}=z^{\frac{2}{3}}$ dat $y=\frac{3}{5}z^{\frac{5}{3}}$. Et $\frac{\dot{y}}{\dot{x}}=\frac{ab}{cx^{\frac{1}{3}}}$ dat $y=\frac{3abx^{\frac{2}{3}}}{2c}$. At æquatio $\frac{\dot{y}}{\dot{x}}=\frac{a}{x}$ dat $y=\frac{a}{0}$. Nam $\frac{a}{x}$ ductum in x fit a; quo per numerum dimenſionum (qui nullus eſt) diviſo, prodit $\frac{a}{0}$, quantitas infinita pro valore y.

36. Quamobrem ſiquando conſimilis terminus, cujus denominator involvit correlatam quantitatem unius tantùm dimenſionis, in valore $\frac{\dot{y}}{\dot{x}}$ reperiatur, pro correlatâ quantitate ſubſtitue ſummam vel differentiam inter eandem et aliam quamvis datam quantitatem, pro arbitrio aſſumptam. Nam quantitatum fluentium juxta prodeuntem æquationem eadem erit inter ſe fluendi relatio, ac juxta æquationem primò expoſitam; et infinita quantitas Relata hoc pacto parte infinitâ diminuetur, et evadet finita, ſed terminis tamen numero infinitis conſtans.

37. Æquatione itaque $\frac{\dot{y}}{\dot{x}}=\frac{a}{x}$ expoſitâ, ſi pro x ſcribam $b+x$,

 quantitatem

PROB. II. quantitatem b pro lubitu aſſumens, prodibit $\frac{\dot{y}}{\dot{x}} = \frac{a}{b+x}$; factâque diviſione $\frac{\dot{y}}{\dot{x}} = \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c. Et inde Regula, ut in ſuperioribus, dabit $y = \frac{ax}{b} - \frac{ax^2}{2b^2} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4}$, &c. relationem inter x et y.

38. Sic etiam habitâ æquatione $\frac{\dot{y}}{\dot{x}} = \frac{2}{x} + 3 - xx$; ſi, propter terminum $\frac{2}{x}$, ſcribam $1+x$ pro x, emerget $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - xx$; terminoque $\frac{2}{1+x}$ in infinitam ſeriem, $2 - 2x + 2x^2 - 2x^3 + 2x^4$, &c; redacto, erit $\frac{\dot{y}}{\dot{x}} = 4 - 4x + x^2 - 2x^3 + 2x^4$, &c. Adeoque juxta regulam obtinebitur $y = 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c. relatio inter x et y.

39. Atque ita ſi proponitur $\frac{\dot{y}}{\dot{x}} = x^{-\frac{3}{2}} + x^{-1} - x^{\frac{1}{2}}$, quia terminum x^{-1} (ſive $\frac{1}{x}$) ineſſe video, tranſmuto x; quemadmodum pro eo ſubſtituendo $1-x$; et oritur $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. Terminus autem $\frac{1}{1-x}$ valet $1 + x + x^2 + x^3$, &c. Et $\sqrt{1-x}$ valet $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$ &c. adeoque $\frac{1}{\sqrt{1-x}}$ ſive $\frac{1}{1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3, \&c.}$ valet $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3$, (g) &c. Quamobrem (valoribus hiſce ſubſtitutis) erit $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{11}{8}x^3$, (g) &c. Et inde per regulum fit $y = x + x^2 + \frac{1}{2}x^3 + \frac{11}{32}x^4$, (g) &c. Et ſic in aliis.

40. Hujuſmodi etiam tranſmutatione Fluentis quantitatis æquatio in aliis caſibus nonnunquam commodè reduci poterit. Quemadmodum ſi exponitur $\frac{\dot{y}}{\dot{x}} = \frac{c^2x}{c^3 - 3c^2x + 3cx^2 - x^3}$; pro x ſcribo $c-x$; et obtineo $\frac{\dot{y}}{\dot{x}} = \frac{c^3 - c^2x}{x^3}$, ſive $\frac{c^3}{x^3} - \frac{c^2}{x^2}$, et inde per regulam $y = -\frac{c^3}{2x^2} + \frac{c^2}{x}$. At harum tranſmutationum uſus in ſequentibus magis eluceſcet.

SOLU-

(g) Hæc ad Calculorum fidem emendata promo: Codices MSS. habent $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3$ &c. $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{27}{16}x^3$ &c. $y = x + x^2 + \frac{1}{2}x^3 + \frac{27}{64}x^4$ &c. Vitioſè quidem. Nam

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 +,$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 +,$$

Ergo

SOLUTIONIS CASUS SECUNDUS.

INVENTIO FLUENTIUM.

Præparatio.

41. Hæc itaque de æquationibus involventibus unicam tantùm fluentem quantitatem. Cùm verò utraque involvitur, Æquatio imprimis ad præſcriptam formam redigenda eſt; efficiendo ſcilicet, ut ex unâ parte habeatur fluxionum ratio æqualis aggregato ſimplicium terminorum ex alterâ.

42. Et præterea ſiquæ ſunt in æquationibus, ſic reductis, fractiones, quæ denominantur à fluenti quantitate, à denominatoribus iſtis liberari debent, per tranſmutationem ejus fluentis quantitatis paulo antè commemoratam.

43. Sic expoſitâ æquatione $\dot{y}ax-\dot{x}aa-\dot{x}xy=0$, ſive $\frac{\dot{y}}{\dot{x}}=\frac{y}{a}+\frac{a}{x}$ (propter terminum $\frac{a}{x}$) aſſumo b ad arbitrium, et pro x vel ſcribo $b+x$, vel $b-x$, vel $x-b$. Quemadmodum ſi ſcribam $b+x$, fiet $\frac{\dot{y}}{\dot{x}}=\frac{y}{a}+\frac{a}{b+x}$. Adeoque termino $\frac{a}{b+x}$ in infinitam ſeriem, per diviſionem, reducto, erit $\frac{\dot{y}}{\dot{x}}=\frac{y}{a}+\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}$ &c.

44. Et ad eundem modum expoſitâ æquatione $\frac{\dot{y}}{\dot{x}}=3y-2x+\frac{x}{y}-\frac{2y}{x^2}$, ſi (propter terminos $\frac{x}{y}$ & $\frac{2y}{x^2}$) ſcribam $1-y$ pro y, et $1-x$ pro x, orietur $\frac{\dot{y}}{\dot{x}}=1-3y+2x+\frac{1-x}{1-y}+\frac{2y-2}{1-2x+x^2}$. Terminus autem $\frac{1-x}{1-y}$, per infinitam diviſionem, dat $1-x+y-xy+y^2-xy^2+y^3-xy^3$, &c. ac terminus $\frac{2y-2}{1-2x+xx}$, per ſimilem diviſionem, dat $2y-2+4xy-4x+6x^2y-6x^2+8x^3y-8x^3+10x^4y-10x^4$, &c. Quare eſt $\frac{\dot{y}}{\dot{x}}=-3x+3xy+y^2-xy^2+y^3-xy^3$, &c. $+6x^2y-6x^2+8x^3y-8x^3+10x^4y-10x^4$, &c.

$$\text{Ergo } \frac{1}{\sqrt{1-x}}+1-x=2+\tfrac{3}{2}x+\tfrac{11}{8}x^2+\tfrac{21}{16}x^3+,$$

$$\text{Sed } \sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-,$$

$$\text{Ergo } \frac{1}{\sqrt{1-x}}+\frac{1}{1-x}-\sqrt{1-x}=\frac{\dot{y}}{\dot{x}}=1+2x+\tfrac{3}{2}x^2+\tfrac{11}{8}x^3+,$$

$$\text{Et } y=x+x^2+\tfrac{1}{2}x^3+\tfrac{11}{32}x^4+,$$

Cæterum hæ ſeries, ſicut alia nonnulla, quæ apud Colſonum et Editores exteros mendoſa ſunt, rectè ſe habent in editione alterâ Londinenſi, formâ minore, anonymo interprete; quæ tamen typothetarum erroribus tota ſcatet.

PROB. II.

REGULA.

45. Æquatione, cùm opus eſt, ſic præparatâ, terminos ordina juxta dimenſiones fluentium quantitatum, ponendo imprimis non affectos Relatâ quantitate; dein affectos minimâ ejus dimenſione, & ſic deinceps. Terminos etiam, in his ſingulis claſſibus, juxta dimenſiones alterius Correlatæ quantitatis pariter diſpone; eoſque in prima claſſe (i. e. quos relata quantitas non afficit) ſcribe in ſerie collaterali dextrorſum pergente, et cæteros in ſeriebus deſcendentibus in ſiniſtrâ columnâ, prout indicant ſubſequentia diagrammata. Opere ſic inſtituto, primum ſive depreſſiſſimum è terminis in primâ claſſe duc in Correlatam quantitatem, divideque per numerum dimenſionum, et in quotiente pro initiali termino valoris Relatæ quantitatis repone. Hunc deinde in æquationis terminos, in ſiniſtrâ columnâ diſpoſitos, pro relatâ quantitate ſubſtitue, et è terminis proximè depreſſiſſimis ſecundum quotientis terminum, eâdem ratione, quâ primum, elicies. Et eâdem operatione, ſæpiùs repetitâ, quotientem ad arbitrium producere poſſis. Sed res exemplo clariùs patebit.

46. EXEMPL. I. Exponatur æquatio $\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + x^2 + xy$, cujus terminos, $1 - 3x + x^2$, non affectos Relatâ quantitate, y, vides in ſupremâ ſerie collateraliter diſpoſitos; cæteroſque y et xy, in ſiniſtrâ columnâ.

	$+1 - 3x + xx$
$+y$	$* + x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$
$+xy$	$* \; * + x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$
Summa	$1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$
$y =$	$x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$

Et imprimis terminum initialem, 1, duco in Correlatam quantitatem x, fitque x; quam per numerum dimenſionum 1 diviſum repono in ſubſcriptâ quotiente. Dein hoc termino pro y in marginalibus ſubſtituto, vice $+y$ et $+xy$, obtineo x & x^2; quos è regione dextrorſum ſcribens, ex omnibus excerpo depreſſiſſimos terminos, $-3x$ & $+x$; quorum aggregatum, $-2x$, ductum in x fit $-2x^2$, et per numerum dimenſionum, 2, diviſum dat $-x^2$ pro ſecundo termino valoris y in quotiente. Hoc proinde termino ad complendum valorem y in marginalibus $+y$ & xy adſcito, oriuntur præterea $-xx$ & $-x^3$, terminis priùs oriundis, $+x$ & $+xx$, adnectendi. Quo facto, iterum terminos proximè depreſſiſſimos, $+x^2 - x^2$ & $+x^2$, in unam ſummam, x^2, colligo; et inde, ut priùs, tertium terminum, $\frac{1}{3}x^3$, in valore y reponendum elicio.

cio. Iterumque $\frac{1}{3}x^3$ in marginalium terminorum valores adſcito, è proximè depreſſiſſimis $\frac{1}{3}x^3$ & $-x^3$, in unum aggregatis, elicio $-\frac{1}{6}x^4$ quartum terminum valoris y. Et ſic in infinitum. INVENTIO FLUENTIUM.

47. EXEMPL. 2. Ad eundem modum, ſi relationem inter x & y, habitâ æquatione $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. cujus terminorum ſeries infinitè progredi ſubintelligitur, determinare oportet; pono 1 in capite, reliquoſque terminos in ſiniſtrâ columnâ, et opus deinde proſequor pro more adjuncti diagrammatis.

	$+1$						
$+\frac{y}{a}$	*	$+\frac{x}{a}$	$+\frac{x}{2a^2}$	$+\frac{x^3}{2a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$	&c.
$+\frac{xy}{a^2}$	*	*	$+\frac{x^2}{a^2}$	$+\frac{x^3}{2a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$	&c.
$+\frac{x^2y}{a^3}$	*	*	*	$+\frac{x^3}{a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$	&c.
$+\frac{x^3y}{a^4}$	*	*	*	*	$\frac{x^4}{a^4}$	$+\frac{x^5}{2a^5}$	&c.
$+\frac{x^4y}{a^5}$	*	*	*	*	*	$+\frac{x^5}{a^5}$	&c.
Summa	1	$+\frac{x}{a}$	$+\frac{3x^2}{2a^2}$	$+\frac{2x^3}{a^3}$	$+\frac{5x^4}{2a^4}$	$+\frac{3x^5}{a^5}$	&c.
	$y=x+\frac{x^2}{2a}+\frac{x^3}{2a^2}+\frac{x^4}{2a^3}+\frac{x^5}{2a^4}+\frac{x^6}{2a^5}$ &c.						

Ubi propoſitum eſt mihi elicere valorem y aduſque ſex dimenſiones x; et eâ de causâ terminos omnes, quos propoſito nihil conducere prævideo, inter operandum miſſos facio; ſicut innuit nota &c. quam ſeriebus interciſis adnexui.

48. EXEMPL. 3. Pari methodo ſi proponitur æquatio $\frac{\dot{y}}{\dot{x}} = +3x + 3xy + yy - xyy + y^3 - xy^3 + y^4 - xy^4$ &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4 + 12x^5y - 12x^5$ &c. et valorem y ad uſque ſeptem dimenſiones x eruere inſtitutum eſt, terminos, ut in adjuncto diagrammate, in ordinem redigo, et operor ſicut in præcedentibus; hoc tantùm excepto, quòd cùm hîc, in ſiniſtrâ columnâ, y non tantùm unius, ſed etiam duarum ac trium dimenſionum exiſtit, vel etiam plurium, prout valorem y ultra gradum x^7 extrahere ſtatuam; ſubjicio quadratum et cubum valoris y eatenus gradatim productum, ut cùm in valoribus marginalium terminorum dextrorſum gradibus inſcribuntur, termini tot dimenſionum emergant, quot ad ſequentem operationem

	$-3x$	$-6x^2$	$-8x^3$	$-10x^4$	$-12x^5$	$-14x^6$	&c.
$+3xy$	*	*	$-\frac{9}{2}x^3$	$-6x^4$	$-\frac{75}{8}x^5$	$-\frac{273}{16}x^6$	&c.
$+6x^2y$	*	*	*	$-9x^4$	$-12x^5$	$-\frac{75}{4}x^6$	&c.
$+8x^3y$	*	*	*	*	$-12x^5$	$-16x^6$	&c.
$+10x^4y$ &c.	*	*	*	*	*	$-15x^6$	&c.
$+y^2$	*	*	*	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$	&c.
$-xy^2$	*	*	*	*	$-\frac{9}{4}x^5$	$-6x^6$	&c.
$+y^3$ &c.	*	*	*	*	*	$-\frac{27}{8}x^6$	&c.
Summa	$-3x$	$-6x^2$	$-\frac{25}{2}x^3$	$-\frac{91}{4}x^4$	$-\frac{333}{8}x^5$	$-\frac{367}{5}x^6$	
$y = -\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4 - \frac{91}{20}x^5 - \frac{111}{16}x^6 - \frac{367}{35}x^7$ &c.							
$y^2 = +\frac{9}{4}x^4 + 6x^5 + \frac{107}{8}x^6$ &c.							
$y^3 = -\frac{27}{8}x^6$ &c.							

PROB. II. nem requiri percipio. Et hâc methodo prodit tandem $y = -\frac{3}{2}x^2 - 6x^3 - \frac{25}{8}x^4$ &c. æquatio deſiderata. Qui valor cùm ſit negativus, patet alterum è quantitatibus x et y decreſcere dum altera increſcit. Atque idem pariter concludi debet, cùm fluxionum altera affirmativa ſit, et altera negativa.

49. EXEMPL. 4. Haud ſecus cùm Relata quantitas fractis dimenſionibus afficitur, poſſis valorem ejus extrahere.

Veluti ſi proponitur $\frac{\dot{x}}{\dot{y}} = \frac{1}{2}y - 4y^2 + 2yx^{\frac{1}{2}} - \frac{4}{5}x^2 + 7y^{\frac{5}{2}} + 2y^3$, ubi x in termino $2yx^{\frac{1}{2}}$ (ſive $2y\sqrt{x}$) fractâ dimenſione, $\frac{1}{2}$, afficitur: ejus $x^{\frac{1}{2}}$ valorem è valore x paulatim elicio, extrahendo nempe radicem quadraticam, ſicut in inferiori parte diagrammatis videre eſt; eò ut in marginalis termini $2yx^{\frac{1}{2}}$ valorem gradatim transferri et inferi poſſit. Et ſic tandem adipiſcor æquationem $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} + \frac{8}{9}y^{\frac{9}{2}} - \frac{41}{100}y^5$ &c. quâ x reſpectu y indefinitè determinatur. Et ſic in aliis quibuſcunque caſibus operari licet.

	$+\frac{1}{2}y. - 4y^2 + 7y^{\frac{5}{2}} + 2y^3$
$2yx^{\frac{1}{2}}$	$*. + y^2. * - 2y^3 + 4y^{\frac{7}{2}} - 2y^4$ &c.
$-\frac{4}{5}x^2$	$*. \quad *. * \quad * \quad * - \frac{1}{16}y^4$ &c.
Summa	$+\frac{1}{2}y. - 3y^2 + 7y^{\frac{5}{2}} * + 4y^{\frac{7}{2}} - 4\frac{1}{16}y^4$ &c.
	$x = +\frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} * + \frac{8}{9}y^{\frac{9}{2}} - \frac{41}{100}y^5$ &c.
	$x^{\frac{1}{2}} = +\frac{1}{2}y - y^2 + 2y^{\frac{5}{2}} - y^3$ &c.
	$x^2 = \frac{1}{16}y^4$ &c.

50. Cæterùm dixi haſce ſolutiones infinitis modis præſtari poſſe. Et hoc fiet, ſi non tantùm initialem quantitatem ſupremæ ſeriei, ſed et aliam quamvis datam quantitatem, pro primo termino quotientis ad arbitrium aſſumas; ac deinde opereris ut in præcedentibus. Sic in primo præcedentium exemplorum; ſi pro primo termino valoris y aſſumas 1, et pro y, in terminis marginalibus ($+y$ et $+xy$) ſubſtituas, reliquamque operationem, cujus ſpecimen adjunxi, ſicut in præcedentibus proſequaris, ipſius y alius exurget valor $1 + 2x + x^3 + \frac{1}{4}x^4$ &c.

	$+1 - 3x + xx$
$+y$	$+1 + 2x \quad * \quad + x^3 + \frac{1}{4}x^4$ &c.
$+xy$	$* + x + 2x^2 \quad * \quad + x^4$ &c.
Summa	$+2 \quad * + 3x^2 + x^3 + \frac{5}{4}x^4$
	$y = 1 + 2x \quad * + x^3 + \frac{1}{4}x^4 + \frac{1}{}x^5$ &c.

51. Et ſic etiam alius atque alius exurget, aſſumendo 2, 3, vel alium quemvis numerum pro primo ejus termino. Vel ſi ſymbolum aliquod, ut a, pro illo termino indefinitè deſignando uſurpes, eâdem operandi methodo, quam hîc etiam deſignatam habes,

	$+1 - 3x + x^2$
$+y$	$+a + x - x^2 + \frac{1}{3}x^3$ &c.
	$+ ax + ax^2 + \frac{2}{3}ax^3$ &c.
$+xy$	$* + ax + x^2 - x^3$ &c.
	$+ ax^2 + ax^3$
Summa	$+1 - 2x + x^2 - \frac{2}{3}x^3$
	$+a + 2ax + 2ax^2 + \frac{5}{3}ax^3$
$y = a$	$+x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4$ &c.
	$+ax + ax^2 + \frac{2}{3}ax^3 + \frac{5}{12}ax^4$

habes, elicies tandem $y = a + x + ax - x^2 + ax^2 + \frac{1}{3}x^3 + \frac{2}{3}\hat{a}x^3$ &c. Quâ inventâ, poſſis pro a ſubſtituere, 1, 2, 0, $\frac{1}{2}$, aut quemvis numerum, et ſic relationem inter x et y modis infinitis obtinere. INVENTIO FLUENTIUM.

52. Et nota, quòd ubi quantitas elicienda afficitur fractâ dimenſione, ut in præcedentium exemplorum quarto vides, convenit plerumque unitatem, vel alium quemvis aptum numerum, pro primo ejus termino adhibere. Imo hoc neceſſe eſt, ubi radix, ad fractæ illius dimenſionis valorem obtinendum, propter negativum ſignum, nequit aliâs extrahi; ut et ubi nulli ſunt termini in primâ ſive capitali claſſe reponendi, ex quibus initialis ille terminus eliciatur.

53. Sic tandem hoc moleſtiſſimum et omnium difficillimum Problema, ubi duæ tantùm fluentes quantitates, unà cum earum fluxionibus, in æquatione comprehenduntur, abſolvi. Sed præter generalem methodum, quâ omnes difficultates complexus ſum, ſunt aliæ plerumque contractiores, quibus opus aliquando ſublevari poſſit, et quarum aliqua ſpecimina ex abundanti perſtringere, fortè non erit ingratum.

54. (I) Siquando itaque quantitas elicienda ſit alicubi negatæ dimenſionis, non eſt abſolutè neceſſarium, ut æquatio propterea ad aliam formam reducatur. Sic enim expoſitâ æquatione $\frac{\dot{y}}{\dot{x}} = \frac{1}{y} - x^2$, ubi y eſt unius negatæ dimenſionis, poſſim equidem ad aliam formam reducere, veluti ſcribendo $1 + y$ pro y; ſed expeditior erit ſolutio, quam in annexo diagrammate deſignatam habes; ubi aſſumpto 1 pro initio valoris y, cæteros ejus terminos ut in præcedentibus extraho; et interea valorem $\frac{1}{y}$ exinde per diviſionem, paulatim inſtitutam, elicio, et inſero in valorem marginalis termini.

	$* \; * - xx$
$\frac{1}{y}$	$1 - x + \frac{3}{2}xx$ &c.
Summa	$1 - x + \frac{1}{2}xx$
	$y = 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x$ &c.
	$\frac{1}{y} = 1 - x + \frac{3}{2}x^2$ &c.

55. (II) Neque ſemper opus eſt ut alterius fluentis quantitatis dimenſiones ſint paſſim affirmativæ. Nam ex æquatione $\frac{\dot{y}}{\dot{x}} = 3 + 2y - \frac{yy}{x}$, abſque termini $\frac{yy}{x}$ reductione præſcriptâ, emerget $y = 3x - \frac{3}{2}x^2 - 4x^3$ &c.

56. Et

PROB. II.

	$-\frac{1}{xx}+\frac{1}{x}$
$-y$	$* \quad -\frac{1}{x}$
Summa	$-\frac{1}{xx} \quad 0$
$y=\frac{1}{x}$	

56. Et ex $\frac{\dot{y}}{\dot{x}} = y + \frac{1}{x} - \frac{1}{xx}$, opere ad modum annexi ſpeciminis inſtituto, emerget $y=\frac{1}{x}$.

57. Ubi obiter nota, quòd inter modos infinitos quibus quælibet æquatio reſolvi poteſt, ſæpenumero contingit, aliquos eſſe, qui ad finitum valorem quantitatis eliciendæ, ſicut in allato ſpecimine, finiuntur; et quos haud difficile eſt invenire, ſi pro primo valoris termino ſymbolum aliquod aſſumatur: et, reſolutione peractâ, conſulatur de ſymboli illius quantitate; quâ valor elicitus evadat finitus.

58. (III) Porro ſi valor y ex æquatione $\frac{\dot{y}}{\dot{x}} = \frac{y}{2x} + 1 - 2x + \frac{1}{2}x^2$ eliciendus ſit, id ſine aliquâ reductione termini $\frac{y}{2x}$ non incommodè fiet; fingendo, pro more analytico, datum eſſe quod quæritur. Utpote pro primo termino valoris ejus effingo $2ex$, aſſumendo $2e$ pro numerali coefficiente, quæ nondum innoteſcit. Et hunc $2ex$ pro y in termino marginali $\frac{y}{2x}$ ſubſtituens, prodit e quem ſcribo

	$1-2x+\frac{1}{2}xx$
$\frac{y}{2x}$	$e+fx+gxx+hx^3$
Summa	$+1-2x+\frac{1}{2}x^2+hx^3$ $+e+fx+gxx$
Hypotheticè $y=2ex+2fx^2+2gx+2hx^4$ &c.	
Conſequenter $y=+x-x^2+\frac{1}{6}x^3+\frac{1}{4}hx^4$ $+ex+\frac{1}{2}fx^2+\frac{1}{3}gx^3$	
Revera $y=2x-\frac{1}{4}x^2+\frac{1}{5}x^3$	

ad dextram; et ſumma $1+e$ dabit $x+ex$ pro eodem primo termino valoris y, quem priùs deſignaveram termino $2ex$. Pono itaque $2ex=x+ex$ et inde elicio $e=1$. Adeoque valoris y primus terminus ($2ex$) eſt $2x$. Ad eundem modum, pro ſecundo termino deſignando, effictum $2fx^2$ uſurpo; et inde tandem eruo $-\frac{2}{3}$ pro valore f, adeoque $-\frac{4}{3}xx$ pro ſecundo termino. Et ſic effictus g in tertio termino valebit $\frac{1}{10}$, at h in quarto valebit 0: et proinde, cùm nullos præterea terminos ſupereſſe video, concludo opus finitum eſſe, et y valere $2x-\frac{4}{3}x^2+\frac{1}{5}x^3$ præciſè.

59. Ad eundem fere modum ſi eſſet $\frac{\dot{y}}{\dot{x}} = \frac{3y}{4x}$, effinge $y=ex^s$, ubi e ignotam coefficientem, et s numerum dimenſionum ſimiliter ignotum denotet. Et ex^s pro y ſubſtituto, prodibit $\frac{\dot{y}}{\dot{x}} = \frac{3ex^{s-1}}{4}$,

$\frac{3ex^{s-1}}{4}$, et inde rurſus $y=\frac{3ex^{s}}{4s}$. Conferantur jam valores y, et videbis eſſe $\frac{3e}{4s}=e$, adeoque $s=\frac{3}{4}$ et e indefinitum. Quare aſſumpto utcunque e erit $y=ex^{\frac{3}{4}}$. INVENTIO FLUENTIUM.

60. (IV) Ad hæc nonnunquam opus ab altiſſimâ dimenſione æquabilis quantitatis inchoari poteſt, et ad depreſſiores continuò pergere. Veluti ſi detur $\frac{\dot{y}}{\dot{x}}=\frac{y}{xx}+\frac{1}{xx}+3+2x-\frac{4}{x}$; et ab altiſſimo termino $2x$ opus inchoetur; diſponendo capitalem ſeriem in ordine præcedentibus contrario, emerget tandem $y=xx+4x-\frac{1}{x}$ &c. prout in appoſitâ operandi formâ videre eſt.

	$+2x+3-\frac{4}{x}+\frac{1}{x^2}$
$+\frac{y}{xx}$	$*+1+\frac{4}{x}*-\frac{1}{x^3}+\frac{1}{2x^4}$ &c.
Summa	$2x+4-*+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{2x^4}$
	$y=x^2+4x*-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}$ &c.

Et hîc in tranſitu notari poteſt, quòd inter operandum potuit inter terminos, $4x$ et $-\frac{1}{x}$, pro intermedio deficienti termino, quælibet data quantitas inſeri, et ſic valor y modis infinitis extrahi.

61. (V) Siqui præterea ſint fracti dimenſionum Relatæ quantitatis indices, ad integros reduci poſſunt, fingendo quantitatem illam, ſuâ fractâ dimenſione affectam, eſſe alii cuilibet tertiæ fluenti quantitati æqualem; et ſubſtituendo tum illam quantitatem, tum fluxionem ejus, ab illâ fictâ æquatione oriundam, pro Relatâ quantitate et ejus fluxione.

62. Quemadmodum ſi exponitur æquatio $\frac{\dot{y}}{\dot{x}}=3xy^{\frac{2}{3}}+y$, ubi Relata quantitas fracto dimenſionis indice $\frac{2}{3}$ afficitur; aſſumptâ ad arbitrium fluenti quantitate z, fingo eſſe $y^{\frac{1}{3}}=z$, ſive $y=z^3$; et fluxionum relatio juxta Prob. 1. erit $\dot{y}=3\dot{z}zz$. Quare ſubſtituto $3z^2\dot{z}$ pro $\dot{y}$, ut et z^3 pro y, ac z^2 pro $y^{\frac{2}{3}}$, emerget $\frac{3\dot{z}z^2}{\dot{x}}=3xz^2+z^3$, ſive $\frac{\dot{z}}{\dot{x}}=x+\frac{1}{3}z$; ubi z ſupplet vices Relatæ quantitatis. Poſtquam verò valor z eo nomine eruitur, utpote $z=\frac{1}{2}x^2+\frac{1}{18}x^3+\frac{1}{216}x^4+\frac{1}{3240}x^5$ &c. pro z reſtitue $y^{\frac{1}{3}}$, et habebis deſideratam relationem inter x et y: nempe $y^{\frac{1}{3}}=\frac{1}{2}x^2+\frac{1}{18}x^3+\frac{1}{216}x^4$ &c. Et cubis partium utrobique poſitis erit $y=\frac{1}{8}x^6+\frac{1}{24}x^7+\frac{7}{864}x^8$ &c (h).

(h) Emendavi calculis ſuadentibus.

Prob. II.

63. Pari ratione ſi detur $\frac{\dot{y}}{\dot{x}} = \sqrt{4y} + \sqrt{xy}$ ſive $= 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$; fingo $z = y^{\frac{1}{2}}$ ſive $z^2 = y$, et inde per Prob. 1. elicio $2z\dot{z} = \dot{y}$: et conſequenter $\frac{2\dot{z}z}{\dot{x}} = 2z + x^{\frac{1}{2}}z$, ſive $\frac{\dot{z}}{\dot{x}} = 1 + \frac{1}{2}x^{\frac{1}{2}}$. Adeoque per caſum priorem hujus eſt z, $(y^{\frac{1}{2}}) = x + \frac{1}{3}x^{\frac{3}{2}}$, et partibus quadratis $y = x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. Sin valorem y modis infinitis deſideres, fac $z = c + x + \frac{1}{3}x^{\frac{3}{2}}$, aſſumpto utcunque initiali termino c; et erit y (z^2) æqualis $c^2 + 2cx + \frac{2}{3}cx^{\frac{3}{2}} + x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. Aſt hæc nimis officioſè tractare videor, ſiquidem rariſſimè uſui eſſe poſſunt.

SOLUTIONIS CASUS III.

64. Problematis, ubi tres vel plures quantitatum fluxiones æquatio complectitur, reſolutio brevi abſolvitur: ſcilicet inter duas quaſlibet iſtarum quantitatum relatio (ubi ex ſtatu quæſtionis non determinatur) quælibet effingi debet, et earum fluxionum exinde quæri; eò ut alterutra, unà cum ejus fluxione, ex æquatione expoſitâ exterminari poſſit. Quâ de causâ, ſi trium inſunt quantitatum fluxiones, unica effingenda eſt æquatio; ac duæ ſi inſunt quatuor, et ſic porro; ut expoſita æquatio in aliam tandem æquationem transformetur, cui non inſint plures duabus: et hâc deinde ut ſuprà reſolutâ, reliquarum quantitatum relationes eruentur.

65. Sic æquatione $2\dot{x} - \dot{z} + \dot{y}x = 0$ expoſitâ; quo quantitatum x, y, et z (quarum fluxiones $\dot{x}$, $\dot{y}$ et $\dot{z}$ æquatio complectitur) relationes inter ſe obtineam, relationem inter duas quaſlibet ut x et y pro lubitu effingo: puta quòd ſit $x = y$, vel $2y = a + z$, vel $x = yy$ &c. Sit autem $x = yy$, et inde erit $\dot{x} = 2\dot{y}y$; quare ſcriptis $2\dot{y}y$ pro $\dot{x}$, et yy pro x, expoſita æquatio transformabitur in $4\dot{y}y - \dot{z} + \dot{y}y^2 = 0$. Et inde relatio inter y & z emerget $2yy + \frac{1}{3}y^3 = z$. Ubi ſi x pro yy, et $x^{\frac{3}{2}}$ pro y^3 viciſſim ſcribatur, prodibit etiam $2x + \frac{1}{3}x^{\frac{3}{2}} = z$.

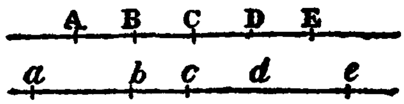
Adeoque inter modos infinitos quibus x, y et z ad invicem referuntur, unus his æquationibus, $x = yy$, $2y^2 + \frac{1}{3}y^3 = z$, et $2x + \frac{1}{3}x^{\frac{3}{2}} = z$, deſignatus inveſtigatur.

Demonſtratio.

Demonſtratio.

66. Problema tandem confecimus, ſed demonſtratio ſupereſt. Inventio Fluentium. Et in tantâ rerum copiâ ne per nimias ambages è propriis fundamentis ſyntheticè derivetur, ſufficiat per analyſin ſic breviter indicare. Scilicet æquatione quâlibet expoſitâ, poſtquam opus ad finem perduxeris, experiri eſt, quòd ex elicitâ æquatione expoſita viciſſim (per Prob. I.) eruetur. Et proinde quantitatum relatio in elicitâ æquatione exigit relationem fluxionum in expoſitâ, et contrâ: ſicut oſtendendum erat. Sic æquatione $\dot{y}=x$ expoſitâ, elicietur $y=\frac{1}{2}x^2$, et inde viciſſim (per Prob. I.) $\dot{y}=\dot{x}x$, ſive $=x$, quandoquidem $\dot{x}$ ſupponitur eſſe I. Et ſic ex $\dot{y}=1-3x+y+x^2+xy$ provenit $y=x-x^2+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5-\frac{1}{45}x^6$ &c ([i]). Et inde viciſſim (per Prob. I) $\dot{y}=1-2x+x^2-\frac{2}{3}x^3+\frac{1}{6}x^4-\frac{2}{15}x^5$ &c. Qui duo valores ipſius $\dot{y}$ conveniunt, ut patet, ſubſtituendo $x-x^2+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5$ &c. pro y in priori.

67. Cæterum in æquationum reductione adhibui operationem, de quâ præterea rationem reddere oportet: eſtque tranſmutatio fluentis quantitatis per connexionem cum quantitate datâ. Sunto AE et *ae* lineæ utrinque infinitæ, per quas mobilia duo è longinquo trajiciantur, ſimul attingentia locos A et *a*, B et *b*, C et *c*, D



et *d*, &c. Et ſit B punctum à cujus diſtantiâ rei mobilis in AE motus æſtimetur, ita ut $-$BA, BC, BD, BE ſucceſſivè ſint fluentes quantitates, quando mobile ſit in locis A, C, D, E. Sitque *b* conſimile punctum in alterâ lineâ: et erunt $-$BA ac $-$*ba* contemporaneæ fluentes quantitates, ut et BC ac *bc*, BD ac *bd*, BE ac *be*, &c. Quòd ſi vice punctorum B & *b*, ſubſtituantur A & *c*, ad quæ tanquam quieſcentia motus referantur, tunc o & $-$*ca*, AB & $-$*bc*, AC & o, AD & *cd*, AE & *ce*, &c. erunt contemporaneæ fluentes quantitates. Mutantur itaque fluentes quantitates additione et ſubtractione datarum AB & *bc*, ſed non mutantur quoad motûs celeritatem et fluxionis mutuum reſpectum: nam ejuſdem longitudinis ſunt partes contemporaneæ, AB & *ab*, BC et *bc*, CD et *cd*, DE et *de*, in utroque caſu. Et ſic in æquationibus, quibus hæ quantitates deſignantur, partes contemporaneæ

([i]) Per § 46. hujus Capitis, poſitâ $\dot{x}=1$.

quan-

PROB. II. quantitatum non ideo mutantur, quòd earum abſoluta longitudo datâ aliquâ augeatur vel minuatur. Unde conſtat propoſitum: nam Problematis hujus ſcopus propriè non alius eſt, quàm contemporaneas partes, ſive abſolutarum quantitatum (v, x, y, aut z) contemporaneas differentias, datâ fluendi ratione deſcriptas determinare. Et perinde eſt cujuſnam ſint abſolutæ longitudinis quantitates illæ, dummodo contemporaneæ ſive correſpondentes earum differentiæ cum expoſitâ fluxionum relatione conveniant.

68. Poteſt et hujus rei ratio ſic algebraicè reddi. Proponatur $\dot{y}=\dot{x}xy$, et finge $x=1+z$, eritque (per Prob. 1.) $\dot{x}=\dot{z}$. Adeoque pro $\dot{y}=\dot{x}xy$ ſcribi poteſt $\dot{y}=\dot{x}y+\dot{x}zy$. Jam cùm ſit $\dot{x}=\dot{z}$, patet quantitates x et z, etſi non ſint ejuſdem longitudinis, pariter tamen fluere reſpectu ipſius y, et pares habere partes contemporaneas. Quidni itaque iiſdem ſymbolis denotem, quæ fluendi ratione conveniunt, et, ad contemporaneas differentias determinandas, vice $\dot{y}=\dot{x}xy$ uſurpem $\dot{y}=\dot{x}y+\dot{x}zy$:

69. Jam denique quo pacto partes contemporaneæ, ex æquatione quantitates involvente, inveniri poſſunt, per ſe manifeſtum eſt. E. G. Sit $y=\frac{1}{x}+x$ æquatio. Et cùm ſit $x=2$ erit $y=2\frac{1}{2}$; cùm verò ſit $x=3$, erit $y=3\frac{1}{3}$; ergo dum x fluit à 2 ad 3, y fluit à $2\frac{1}{2}$ ad $3\frac{1}{3}$. Adeoque partes in hoc tempore tranſactæ ſunt $(3-2)\ 1$, et $(3\frac{1}{3}-2\frac{1}{2})\ \frac{5}{6}$.

Jactis hiſce ſequentium fundamentis, ad Problemata magis particularia jam tranſeo.

CAPUT QUINTUM.

PROB. III.

Determinare Maximas et Minimas.

1. QUANTITAS ubi Maxima eſt vel Minima, in illo momento nec profluit nec refluit. Nam ſi profluit, id arguit minorem fuiſſe, et ſtatim majorem fore, quàm jam eſt: et contrà ſi refluit. Quamobrem fluxionem ejus per Prob. 1. quære, et pone nullam eſſe.

2. Ex-

DE MAXIMIS ET MINIMIS.

2. EXEMPL. I. Si maxima quantitas x in æquatione $x^3 - ax^2 + axy - y^3 = 3$ deſideretur; quantitatum x et y fluxiones quære, et prodibit, $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$: poſitoque $\dot{x} = 0$ reſtabit, $-3\dot{y}y^2 + a\dot{y}x = 0$, ſive $3y^2 = ax$. Cujus ope poſſis alterutram, x vel y, in æquatione primariâ exterminare, et per æquationem reſtantem determinare alteram, et utramque deinde per $3y^2 + ax = 0$.

3. Perinde eſt hæc operatio, ac ſi multiplicâſſes terminos propoſitæ æquationis per numerum dimenſionum alterius fluentis quantitatis y. Unde prodit Huddeniana notiſſima regula, quòd ad obtinendam maximam aut minimam Relatam quantitatem, æquatio juxta dimenſiones Correlatæ quantitas diſponi debet, et per quamlibet Arithmeticam progreſſionem multiplicari. Aſt cùm neque hæc regula ad æquationes ſurdis quantitatibus affectas, neque ulla alia hactenus, quòd ſciam, evulgata, abſque præviâ reductione, ſe extendat: ejus rei accipe ſequens exemplum.

4. EXEMPL. 2. Si maxima quantitas y in æquatione $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ determinanda eſt; ipſarum x et y, fluxiones quære, et emerget $3\dot{x}x^2 - 2ay\dot{y} + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a + 2ay + y^2} - \frac{4a\dot{x}xy + 6\dot{x}x^3 + a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. Et cùm ex hypotheſi ſit $\dot{y} = 0$, neglige terminos in $\dot{y}$ ductos, (id quod inter operandum ad minuendum laborem antea fieri potuit) cæteroſque per $\dot{x}x$ divide, et reſtabit $3x - \frac{2ay + 3xx}{\sqrt{ay + xx}} = 0$, factâque reductione exurget, $4ay + 3xx = 0$, cujus ope poſſis utramvis quantitatem x vel y ex æquatione primò propoſitâ exterminare, ac deinde ex æquatione reſultante, quæ cubica erit, valorem alterius elicere.

5. *Ex hoc Problemate ſequentium reſolutio petenda eſt.*

I. In dato Triangulo aut ſegmento cujuſvis Curvæ, maximum rectangulum inſcribere.

II. Maximam vel Minimam rectarum ducere, quæ inter datum punctum, et Curvam poſitione datam interjacent. Sive à dato puncto ad curvam ducere perpendiculum.

III. Maximam vel Minimam rectarum ducere, quæ per datum

punctum

punctum tranſeuntes interjacent aliis duabus, ſive rectis ſive curvis lineis.

IV. A puncto intra Parabolam dato rectam ducere, quæ parabolam omnium obliquiſſimè ſecabit. Et idem in aliis curvis facere.

V. Curvarum Vertices, Maximas aut Minimas latitudines, Puncta in quibus partes circumactæ ſe decuſſant, determinare.

VI. Curvarum puncta invenire, ubi maximè aut minimè curvantur.

VII. Invenire minimum angulorum in quibus rectæ ad diametros ſuas in datâ Ellipſi ordinatim applicantur.

VIII. Ellipſium, per data quatuor puncta tranſeuntium, vel minimam definire, vel eam quæ ad formam circularem maximè accedit.

IX. Amplitudinem ſphæricæ ſuperficiei determinare, quam lux è longinquò fluens, poſtquam ab anteriori hemiſphærio refracta fluit, illuſtrat in poſteriori.

Et hujuſmodi alia permulta faciliùs excogitari poſſunt, quàm (propter computandi faſtidium) reſolvi.

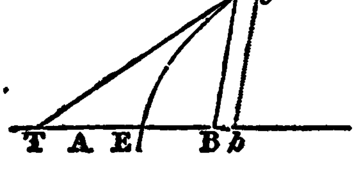
CAPUT SEXTUM.

PROB. IV.

Curvarum Tangentes ducere.

MODUS I.

1. TANGENTES pro variis relationibus curvarum ad rectas variè ducuntur. Et imprimis eſto BD recta in dato angulo ad aliam rectam, AB tanquam baſin ordinata, et ad curvam ED terminata. Et moveatur hæc ordinata, per infinitè parvum ſpatium, ad locum *bd*; ita ut momento *cd* augeatur, dum AB augetur momento B*b*, cui D*c* æqualis eſt. Jam producatur D*d*, donec cum AB in T conveniat, et hæc tanget curvam in D vel *d*:



eruntque

eruntque triangula *dc*D, DBT ſimilia. Adeoque TB : BD : : DC : *cd*. DE TANGENTIBUS DUCENDIS.

2. Cùm itaque relatio BD ad AB, in æquatione quâlibet pro curvâ determinandâ, exponitur, quære relationem fluxionum (per Prob. I.) et cape TB ad BD in ratione fluxionis AB ad fluxionem BD, ac TD tanget curvam in D.

3. EXEMPL. I. Nominatâ AB, x; et BD y; eſto earum relatio $x^3-ax^2+axy-y^3=0$. Et fluxionum relatio erit $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$. Adeoque $\dot{y}:\dot{x}::3xx-2ax+ay:3yy-ax::$ BD(y) : BT. Ergo BT $=\frac{3y^3-axy}{3x^2-2ax+ay}$. Dato itaque puncto D, et inde DB et AB, ſive y et x; dabitur longitudo BT, quâ tangens DT determinatur.

4. Poteſt autem hæc operandi methodus ſic concinnari. Æquationis expoſitæ terminos fac eſſe nihilo æquales: per proprium numerum dimenſionum ordinatæ quantitatis multiplica, et exitum colloca in numeratore: dein terminos ejuſdem æquationis per proprium numerum dimenſionum baſis multiplica, et exitum, per baſin diviſum, colloca in denominatore valoris BT. Et illam BT cape ad partes adverſus A, ſi valor ejus ſit affirmativus, aut verſus A ſi ſit negativus.

$$\begin{matrix} & 0 & . & 0 & . & 1 & . & 3 \end{matrix}$$

5. Sic æquatio $x^3-ax^2+axy-y^3=0$, per ſuperiores numeros

$$\begin{matrix} & 3 & . & 2 & . & 1 & . & 0 \end{matrix}$$

multiplicata dat $axy-3y^3$ pro numeratore; et per inferiores multiplicata, ac diviſa per x, dat $3x^2-2ax+ay$ pro denominatore valoris BT.

6. Sic æquatio $y^3-by^2-cdy+bcd+dxy=0$ (quæ deſignat parabolam ſecundi generis, cujus beneficio Des Cartes conſtruxit æquationes ſex dimenſionum) primâ fronte dat $\frac{3y^3-2by^2-cdy+dxy}{dy}$ ſive $\frac{3y^2}{d}-\frac{2by}{d}-c+x=$ BT.

7. Et ſic $a^2-\frac{r}{q}x^2-y^2=0$ (quæ deſignat Ellipſin cujus centrum A) dat $\frac{-2yy}{-\frac{2r}{q}x}$ ſive $\frac{qyy}{rx}=$ BT. Et ſic in aliis.

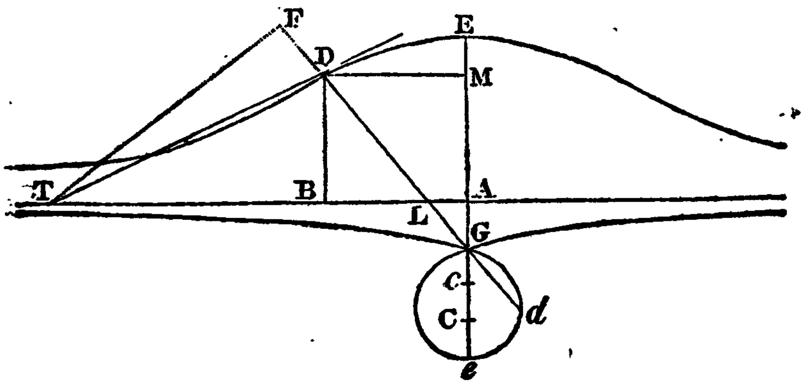
8. Et nota, quòd nihil intereſt cujuſnam quantitatis ſit angulus ordinationis ABD.

Aſt hæc regula ſe ad æquationes ſurdis quantitatibus affectas, curvaſque

PROB. IV. curvaſque mechanicas non extendit. In iſtis caſibus ad fundamentalem methodum recurrendum eſt (k).

8. EXEMPL. 2. Eſto $x^3 - ay^2 + b\frac{y^3}{a+y} - x^2\sqrt{ay+x^2} = 0$ æquatio deſignans relationem inter AB & BD; et per Prob. I. relatio fluxionum erit $3\dot{x}x^2 - 2a\dot{y}y + \frac{3a^2\dot{y}y^2 + 2b\dot{y}y^3}{a^2 + 2ay + y^2} - \frac{4a\dot{x}xy + 6\dot{x}x^3 + a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. Atque adeo eſt $3x^2 - \frac{4axy + 6x^3}{2\sqrt{ay+x^2}} : 2ay + \frac{3aby^2 + 2by^3}{aa + 2ay + y^2} - \frac{axx}{2\sqrt{ay+x^2}}$ $(:: \dot{y} : \dot{x}) ::$ BD : BT.

9. EXEMPL. 3. Sit ED Conchoides Nichomedea, polo G, aſymptoto AT, et intervallo LD deſcripta. Sitque GA $= b$, LD $= c$, AB $= x$, et BD $= y$. Et propter ſimilia triangula DBL et DMG, erit LB : BD :: DM : MG. $\sqrt{cc-yy} : y :: x : b+y$. Adeoque $\overline{b+y} \times \sqrt{cc-yy} = yx$. Nactus hanc æquationem fingo $\sqrt{cc-yy} = z$. Et ſic duas æquationes, $bz + yz = yx$, et $zz = cc - yy$ habeo: quarum ope fluxiones quantitatum x, y et z (per Prob. I.) quæro. Et è primâ prodit $b\dot{z} + y\dot{z} + \dot{y}z = \dot{y}x + \dot{x}y$; ac è ſecundâ $2\dot{z}z = -2\dot{y}y$, ſive $\dot{z}z + \dot{y}y = 0$. E quibus exterminato $\dot{z}$, oritur $-\frac{b\dot{y}y}{z} - \frac{\dot{y}y^2}{z} + \dot{y}z = \dot{y}x + \dot{x}y$. Quâ reſolutâ (l) fit $y : z - \frac{by}{z} - \frac{y^2}{z} - x$ $(:: \dot{y} : \dot{x}) ::$ BD : BT. Cùm ergo BD ſit y, erit BT $= z - x - \frac{by + y^2}{z}$. Hoc eſt, − BT = AL + $\frac{\text{BD} \times \text{GM}}{\text{BL}}$. Ubi ſignum − ipſi BT præfixum denotat punctum T ad partes *adverſus* A capiendum eſſe.



SCHOLIUM.

10. Et hinc obiter inventio puncti diſterminantis concavam et convexam partem Conchoidis prodit. Nempe cum AT ſit omnium minima, erit D ejuſmodi punctum. Eſto itaque AT $= v$, et

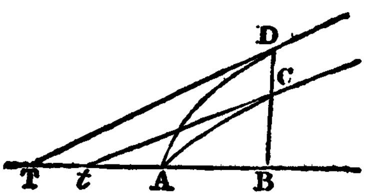
(k) § 1 & 2. traditam.
(l) In Analogiam ſcilicet.

cùm

cùm ſit BT $=-z+x+\frac{by+y^2}{z}$, erit $v=-z+2x+\frac{by+y^2}{z}$. Ubi, ad opus abbreviandum, pro x ſubſtitue $\frac{bz+yz}{y}$ valorem è ſuperioribus erutum, et fiet $\frac{2bz}{y}+z+\frac{by+y^2}{z}=v$. Unde per Prob. I. fluxionibus $\dot{v}$, $\dot{y}$ et $\dot{z}$ quæſitis, et per Prob. III. ſuppoſitâ $\dot{v}=0$, emerget $\frac{2b\dot{z}}{y}-\frac{2b\dot{y}z}{yy}+\dot{z}+\frac{b\dot{y}+2\dot{y}y}{z}-\frac{b\dot{z}y+\dot{z}yy}{zz}=\dot{v}=0$. In hâc denique ſubſtitue $-\frac{\dot{y}y}{z}$ pro $\dot{z}$, et $cc-yy$ pro zz, valores $\dot{z}$ et zz è ſuperioribus petendos, et, factâ reductione, obtinebitur $y^3+3by^2-2bc^2=0$. Cujus æquationis conſtructione dabitur y, ſive AM; et per M acta MD ipſi AB parallela incidet in punctum flexûs contrarii, D. DE TANGENTIBUS DUCENDIS.

11. Præterea ſi Curva Mechanica eſt, cujus tangentem ducere oportet, quantitatum fluxiones, ut in Exemp. 5. Prob. I. quærendæ ſunt, cæteraque ut in præcedentibus peragenda.

12. EXEMPL. 4. Sunto AC et AD duæ Curvæ, quibus recta BCD, ad baſin AB in dato angulo applicata, occurrit in C et D; et (per Prob. I. præparat. ad Exemplum 5) erit $\dot{z}=\dot{x}\times$ BC. Jam



ſit AC Circulus, aut curva quævis nota, et ad alteram curvam, AD, definiendam, exponatur quævis æquatio, cui z intexta eſt; veluti $zz+axz=y^4$. Et, per Prob. I. erit $2\dot{z}z+ax\dot{z}+a\dot{x}z=4\dot{y}y^3$. Et ſcripto $\dot{x}\times$ BC pro $\dot{z}$, fiet $2\dot{x}z\times$ BC $+a\dot{x}x\times$ BC $+a\dot{x}z=4\dot{y}y^3$. Adeoque $2z\times$ BC $+ax\times$ BC $+az:4y^3$ $(::\dot{y}:\dot{x})$ :: BD : BT. Quamobrem ſi ex naturâ curvæ AC detur ordinata BC, et area ACB, ſive z, dabitur punctum T, per quod tangens DT tranſibit.

13. Ad eundem modum ſi $3z=2y$ ſit æquatio ad curvam AD, erit $3\dot{z}$ (ſive $3\dot{x}\times$ BC) $=2\dot{y}$. Adeoque 3CB : 2 $(::\dot{y}:\dot{x})$ :: BD : BT. Et ſic in aliis.

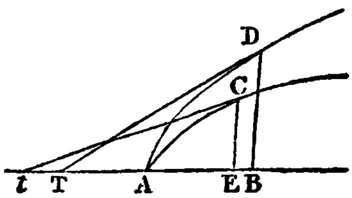
14. EXEMPL. 5. Sit AB $=x$, BD $=y$, ut antè; et Curvæ cujuſvis AC longitudo ſit z; ductâque ad eam tangente, Ct, erit Bt:Ct:: $\dot{x}:\dot{z}$ ſive $\dot{z}=\frac{\dot{x}\times Ct}{Bt}$.

15. Jam ad aliam Curvam AD, cujus tangens ducenda eſt, detur quælibet æquatio, in quâ z involvitur; puta ſi $z=y$, erit $\dot{z}=\dot{y}$.

CAPUT VI. $\dot{z}=\dot{y}$. Adeoque Ct : Bt (: : $\dot{y}:\dot{x}$) : : BD : BT. Invento autem T age DT tangentem.

16. Sic poſito $xz=yy$, erit $\dot{x}z+\dot{z}x=2\dot{y}y$, et pro $\dot{z}$ ſcripto $\frac{\dot{x}\times Ct}{Bt}$, emerget $\dot{x}z+\frac{\dot{x}x\times Ct}{Bt}=2\dot{y}y$; quare eſt $z+\frac{x\times Ct}{Bt}:2y::$ BD : BT.

17. EXEMPL. 6. Sit AC Circulus aut alia quævis nota curva, quam tangat Ct; et ſit AD alia curva cujus tangentem DT ducere oportet, et quæ definitur aſſumendo AB = arcui AC, et (CE ac BD in dato angulo ad AB ordinatis) referendo BD ad CE vel AE in æquatione aliquâ. Dic ergo AB vel AC$=x$, BD$=y$, AE$=z$, & CE$=v$, et patet $\dot{v}$, $\dot{x}$ & $\dot{z}$ fluxiones ipſarum CE, AC & AE, eſſe inter ſe ut ſunt CE, Ct & Et. Adeoque $\dot{x}\times\frac{CE}{Ct}=\dot{v}$, & $\dot{x}\times\frac{Et}{Ct}=\dot{z}$.

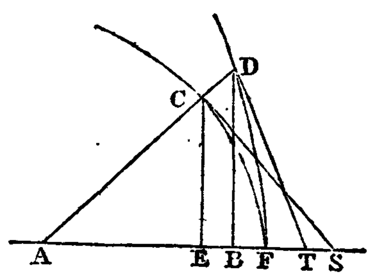


Detur jam quælibet æquatio ad definiendam curvam AD, veluti $y=z$, et erit $\dot{y}=\dot{z}$. Adeoque Et : Ct (: : $\dot{y}:\dot{x}$) : : BD : BT.

Vel detur $y=z+v-x$, et erit $\dot{y}=(\dot{z}+\dot{v}-\dot{x})=\frac{\dot{x}\times CE+Et-Ct}{Ct}$. Adeoque CE$+Et-Ct$: Ct (: : $\dot{y}:\dot{x}$) : : BD : BT.

Vel denique detur $ayy=v^3$, et erit $2a\dot{y}y\ (=3\dot{v}v^2)=3\dot{x}\times v^2\frac{CE}{Ct}$. Adeoque $3vv\times$ CE : $2ay\times$ Ct : : BD : BT.

18. EXEMPL. 7. Sit FC Circulus quem tangat CS; ſitque FD curva, quæ definitur aſſumendo quamvis relationem applicatæ DB ad FC, arcum quem DA, ad centrum ducta, intercipit. Et demiſſâ CE in circulo applicatâ, dic AC vel AF$=1$, AB$=x$, BD$=y$, AE$=z$, CE$=v$, CF$=t$: et erit $\dot{t}z=(\dot{t}\times\frac{CE}{CS})=\dot{v}$ (m), et $-\dot{t}v\ (=\dot{t}\times\frac{-ES}{CS})=\dot{z}$ (n). Ubi pono $\dot{z}$ negativè, quòd AE diminuitur dum EC augetur. Eſt inſuper AE : EC : : AB : BD, adeoque $zy=vx$, et inde per Prob. r. $\dot{z}y+\dot{y}z=\dot{v}x+\dot{x}v$. Et hæc, exterminatis $\dot{v}$, $\dot{z}$ & v, faciunt $\dot{y}x-\dot{t}y^2-\dot{t}x^2=\dot{x}y$.



19. Definiatur jam Curva DF æquatione quâvis, à quâ valor $\dot{t}$, hîc ſubſtituendus, deduci poſſit: puta ſit $t=y$ (æquatio ad Primam Quadraticem) et per Prob. I. erit $\dot{t}=\dot{y}$. Adeoque $\dot{y}x-\dot{y}y^2-\dot{y}x^2=\dot{x}y$. Unde

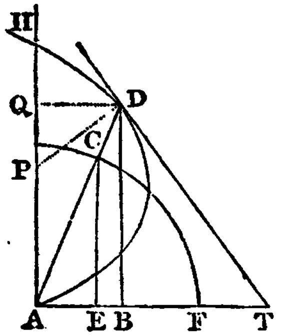
(r) Nempe cùm ſit fluxio arcûs ad fluxionem ſinùs ut radius ad ſinum complementi (Excerpt. IV. ex Epiſt. not. (') Cor.) cùmque propter angulos ad C, E rectos ſit CS : CE $=$ AC : AE $=$ I : z.

$y:xx$

$y : xx+yy-x$ $(:: \dot{y} : -\dot{x}) ::$ BD (y) : BT. Quare BT $= x^2+y^2-x$. Et AT $= xx+y^2 = \frac{\overline{AD}|^2}{AF}$. De Tangentibus ducendis.

20. Ad eundem modum, ſi ſit $tt=by$, proveniet $2\dot{t}t=b\dot{y}$; et inde AT $= \frac{b}{2t} \times \frac{\overline{AD}|^2}{AF}$. Et ſic in aliis.

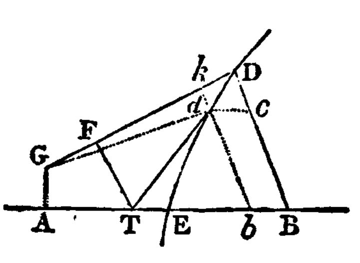
21. Exempl. 8. Quòd ſi AD ſumatur æqualis arcui FC, exiſtente ADH Spirali Archimedeâ; tum, ſtantibus jam poſitis linearum nominibus, eſt (propter angulum ABD rectum) $xx+yy=tt$. Et inde (per Prob. I.) $\dot{x}x+\dot{y}y=\dot{t}t$. Eſt etiam AD : AC :: BD : CE, adeoque $tv=y$, et inde per Prob. I. $\dot{t}v+\dot{v}t=\dot{y}$. Denique eſt fluxio arcûs FC ad fluxionem rectæ CE, ut AC ad AE, ſive ut AD ad AB, hoc eſt $\dot{t} : \dot{v} :: t : x$, et inde $\dot{t}x=\dot{v}t$. Confer jam inventas æquationes, et videbis eſſe $\dot{t}v+\dot{t}x=\dot{y}$, et inde $\dot{x}x+\dot{y}y\,(=\dot{t}t)=\frac{\dot{y}t}{v+x}$. Atque adeo, completo parallelogrammo ABDQ, ſi fiat QD : QP $(::$ BD : BT $:: \dot{y} : -\dot{x}) :: x : y-\frac{t}{v+x}$, hoc eſt, ſi capiatur AP $= \frac{t}{v+x}$, erit PD ad Spiralem perpendicularis.



22. Ex his opinor ſatis manifeſtum eſt, quo pacto Curvarum omnium Tangentes ducendæ ſunt. Attamen non abs re erit, ſi præterea confectionem problematis, ubi Curvæ aliis quibuſcunque modis ad rectas referuntur, oſtenderem; ut è pluribus methodis facillima et ſimpliciſſimâ ſemper poſſit adhiberi.

MODUS II.

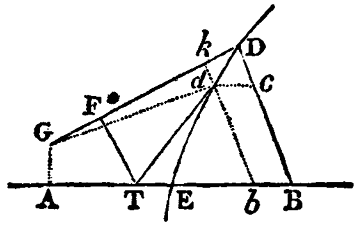
23. Sit D punctum in Curvâ, à quo ſubtenſa DG ducitur ad datum punctum G, ac DB, in dato quovis angulo, ordinatur ad baſin AB. Punctum verò D per infinitè parvum intervallum, Dd, in curvâ fluat, inque GD, ſumatur Gk æqualis Gd, et compleatur parallelogrammum dbBC. Et erunt Dk ac Dc contemporanea momenta ipſarum GD et BD, quibus nempe diminuuntur, dum D transfertur ad d. Jam Dd rectâ producatur, donec



(ⁿ) Nempe cùm ſit fluxio arcûs ad fluxionem ſinus complementi ut radius ad ſinum, (Excerpt. ex Epiſt. not. (ᵏ) § 5.) et propter angulos ad C, E rectos ſit CS : SE = AC : CE. I : v.

 cum

CAPUT VI.



cum AB conveniat in T; et ab iſto T ad ſubtenſam GD demittatur perpendiculum TF, et erunt trapezia D*cdk* ac DBTF ſimilia, adeoque DB : DF : : DC : D*k*.

24. Cùm itaque relatio BD ad GD in æquatione quâlibet, pro curvâ definiendâ, exponitur, quære relationem fluxionum, et cape FD ad DB in ratione fluxionis GD ad fluxionem BD; dein ab F erige perpendiculum FT, quod cum AB concurrat in T, et acta TD curvam tanget in D. Cape autem DF verſus G ſi ſit affirmativa; ſin ſecus, cape ad contrarias partes.

25. EXEMPL. I. Dic GD$=x$ et BD$=y$, et eſto earum relatio $x^3-ax^2+axy-y^3=0$. Eritque fluxionum ratio $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y+a\dot{y}x-3\dot{y}y^2=0$. Atque adeo $3x^2-2ax+ay : 3y^2-ax\,(::\dot{y}:\dot{x})::$ DB(y) : DF. Eſt ergo DF $=\frac{3y^3-axy}{3x^2-2ax+ay}$. Adeoque dato quolibet in curvâ puncto D, et inde BD et GD, ſive y et x, dabitur punctum F: unde ſi normalem FT erigas, ad ejus concurſum cum baſi AB ducta DT curvam tanget.

26. Et hinc patet Regulam perinde ac in priori caſu concinnari poſſe. Scilicet æquationis expoſitæ terminos omnes ad eaſdem partes diſpone, et ſigillatim per dimenſiones ordinatæ y multiplica, et exitum colloca in numeratore: dein terminos ejus ſigillatim per dimenſiones ſubtenſæ x multiplica, et exitum, per ſubtenſam illam x diviſum, colloca in denominatore valoris DF. Illamque DF cape ad partes contra G ſi ſit affirmativa, ſin ſecus, cape ad eaſdem partes. Et nota, quòd nihil interſit quanto intervallo punctum G diſtat à baſi AB, ſi fortè diſtat, neque quinam ſit angulus ordinationis ABD.

27. Sic æquatio ſuperior $x^3-ax^2+axy-y^3=0$, primâ fronte dat $axy-3y^3$ pro numeratore, et $3x^2-2ax+ay$ pro denominatore valoris DF.

28. Sic etiam $a+\frac{b}{a}x-y=0$ (quæ æquatio eſt ad conicam ſectionem (°) dat $-y$ pro numeratore et $\frac{b}{a}$ pro denominatore valoris DF, quæ ideo erit $=-\frac{ay}{b}$.

(°) Nimirum quæ umbilicum habeat punctum G; directricem, rectam quandam poſitione datam cum

29. Et

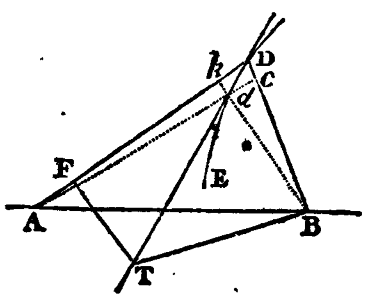
29. Et ſic in Conchoide (ubi res expeditiùs abſolvitur quàm antè) poſito GA $=b$, LD $=c$, GD $=x$ et BD $=y$, (vid. Fig. Ex. 3. Mod. I.) De Tangentibus Ducendis.
erit $\begin{matrix} BD : DL :: GA : GL \\ y : c :: b : x-c \end{matrix}$. Adeoque $xy-cy=cb$, ſive $xy-cy-cb=0$.
Quæ æquatio juxta regulam dat $\frac{xy-cy}{y}$, hoc eſt $x-c=$ DF. Produc ergo GD ad F, ut ſit DF $=$ LG, et ad F erige normalem FT occurrentem aſymptoto AB in T, et acta DT Conchoidem tanget.

30. Siquando compoſitæ quantitates in æquatione reperiantur, ad methodum generalem recurrendum eſt, niſi ubi malueris æquationem reducere.

31. Exempl. 2. Si detur æquatio $\overline{b+y}\times\sqrt{cc-yy}=yx$ pro relatione inter GD et BD (fig. præced.) determinandâ, fluxionum relationem juxta Prob. I. quære. Utpote ficto $\sqrt{cc-yy}=z$, æquationes $bz+yz=yx$, et $cc-yy=zz$ habebis; et inde fluxionum $\dot{x}$, $\dot{y}$ et $\dot{z}$, relationes $b\dot{z}+y\dot{z}+\dot{y}z=\dot{y}x+y\dot{x}$, et $-2\dot{y}y=2\dot{z}z$. Et exterminatis $\dot{z}$ et z, orietur $\dot{y}\sqrt{cc-yy}-\frac{b\dot{y}y+\dot{y}y^2}{\sqrt{cc-yy}}-\dot{y}x=\dot{x}y$. Eſt ergo $y:\sqrt{cc-yy}-\frac{by+yy}{\sqrt{cc-yy}}-x\,(::\dot{y}:\dot{x})::$ BD (y) : DF.

MODUS III.

32. Præterea ſi Curva ad duas ſubtenſas AD et BD referatur, quæ à datis punctis A ac B ductæ ad curvam conveniunt: concipe punctum illud D per infinitè parvum ſpatium Dd in curvâ profluere. Et in AD et BD, cape A$k=$ Ad et B$c=$ Bd. Et erunt kD & cD contemporanea momenta linearum AD et BD. Cape jam DF ad BD in ratione momenti Dk ad momentum Dc (i. e. in ratione fluxionis lineæ AD ad fluxionem lineæ BD) et erige perpendicula BT, FT, concurrentia in T; eruntque trapezia DFTB ac Dkdc ſimilia, et perinde diagonalis DT curvam tanget.



33. Per æquationem itaque, quâ relatio inter AD et BD definitur, quære relationem fluxionum ope Prob. I. Et cape FD ad BD in eâdem ratione.

cum rectâ AB, poſitione datâ, parallelam. Hamilton. Con. Lib. II. Prop. II.

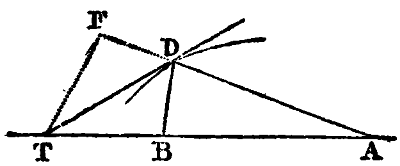
34. Ex-

CAPUT VI.

34. EXEMPL. Poſito AD$=x$ et BD$=y$, ſit earum relatio $a+\frac{ex}{d}-y=0$, quæ æquatio eſt ad Ellipſes ſecundi generis, quarum proprietates ad lucem refringendam Des-Cartes in Lib. II. Geometriæ docuit. Et fluxionum relatio erit $\frac{e\dot{x}}{d}-\dot{y}=0$. Eſt itaque $e:d\,(::\dot{y}:\dot{x})$ BD : DF.

35. Et pari ratione ſi $a-\frac{ex}{d}-y=0$, erit $e:-d::$ BD : DF. In priori caſu, cape DF verſus A, et ad contrarias partes in poſteriori.

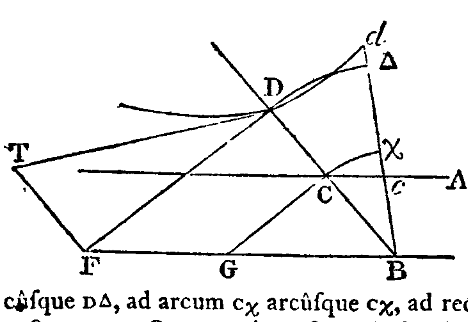
36. COROL. I. Hinc ſi $d=e$ (quo caſu curva evadit conica ſectio) erit DF$=$DB. Et inde triangula DFT, DBT, æqualia, angulusque FTB à tangenti biſecabitur.



37. COROL. 2. Hinc etiam quæ Des-Cartes de his Curvis circa refractiones, haud abſque circuitu, demonſtravit, per ſe manifeſta ſunt: ſiquidem DF ac DB (quæ ſunt in datâ ratione d ad e) reſpectu ſinûs totius DT ſint ſinus angulorum DTF ac DTB, id eſt, incidentiæ radii AD in ſuperficiem curvæ, et reflectionis vel refractionis ejus DB. Eſtque par ratio de refractionibus conicarum ſectionum, ſi modò punctorum A vel B alterutrum infinitè diſtare concipiatur.

38. Perfacile eſt hanc Regulam pro more præcedentium concinnare, et pluribus exemplis donare; quinimo ubi Curvæ aliis quibuſcunque modis ad rectas referuntur, et ad præcedentes formas haud commodè reduci poſſunt, perfacile eſt alias Regulas, ad harum exemplar, pro re natâ excogitare.

MODUS

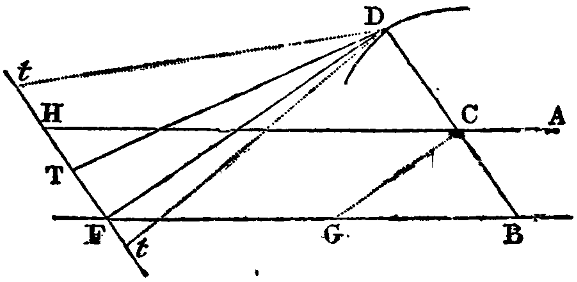


(F) PUTA rectam mobilem BCD in locum Bcd perveniſſe. Centro B radiis BC, BD ſcribantur arcus circulares Cχ, DΔ qui rectæ Bd in punctis χ, Δ occurrant. Per D ductam puta rectam DT, quæ curvam illam, ad quam terminari intelligitur recta mobilis BD, in puncto D contingat; & per C ducatur CG cum rectâ DF parallela. Fluxio rectæ BD ad fluxionem rectæ BC rationem habet eam, quæ prima eſt naſcentis Δd ad naſcentem $c\chi$. Sed ratio Δd ad $c\chi$ componitur è rationibus illius Δd ad arcum DΔ arcûſque DΔ, ad arcum Cχ arcûſque Cχ, ad rectam χc. Sed arcus DΔ eſt ad arcum Cχ ut recta BD ad rectam BC. Quare ratio rectæ Δd ad rectam $c\chi$ componitur è rationibus illius Δd ad arcum DΔ, rectæque DB, ad CB, arcûſque Cχ ad rectam χc. Naſcentis autem Δd ad arcum naſcentem DΔ prima ratio eadem eſt quæ rectæ FT ad rectam FD. Et arcûs naſcentis Cχ ad naſcentem rectam χc prima ratio eadem eſt quæ rectæ GC ad rectam CB, ſive rectæ FD ad DB. Naſcentis igitur $d\Delta$ ad naſcentem χc ratio prima, ſive ratio fluxionis rectæ BD ad fluxionem rectæ BC, componitur è rationibus

MODUS IV.

De Tangentibus Ducendis.

39. Quemadmodum ſi rectæ BD circa datum punctum B revolventis, punctum D ſit ad Curvam aliquam, et C ſit interſectio ejus cum rectâ AC poſitione datâ, habeaturque relatio inter BC et BD quâcunque æquatione deſignata; age BF parallelam AC, eique occurrat DF normalis ad BD. Et ad DF itidem erige normalem FT, et cape in ratione ad BC, quam habet fluxio ipſius BD ad fluxionem ipſius BC. Acta DT curvam tanget (p).



MODUS V.

40 Sin dato puncto A, æquatio relationem inter AC ac BD deſignat, duc CG parallelam DF, et cape FT in ratione ad BG, quam habet fluxio BD ad fluxionem AC (q).

MODUS VI.

41. Vel denique ſi æquatio relationem inter AC et CD definit: conveniant AC et FT in H, et cape HT in ratione ad BG quam habet fluxio CD ad fluxionem AC (r). Et ſic in aliis.

tionibus rectæ FT ad rectam FD, rectæque BD ad rectam BC, rectæque FD ad rectam BD; hoc eſt è rationibus rectæ FT ad rectam FD, rectæque FD ad rectam BD, rectæque BD ad rectam BC. Verum ex iiſdem componitur rectarum FT, BC inter ipſas ratio. Quare $\dot{\overline{BD}} : \dot{\overline{BC}} = FT : BC$. Atque ex his ſatis patet veritas conſtructionis à Newtono præſcriptæ.

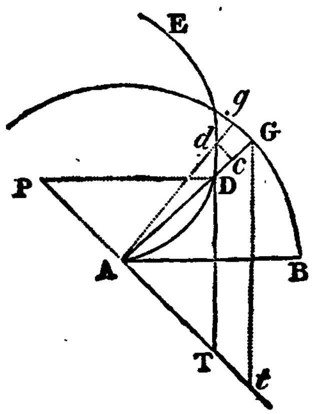
(q) Manentibus quæ modò conſtructa ſunt, fluxio rectæ BC ad fluxionem rectæ AC rationem habebit eam quæ prima eſt naſcentis χc ad naſcentem cc: ſive eam quam recta BC habet ad BG. Jam cùm ex iis quæ ſuprà ſunt oſtenſa, ſit $\dot{\overline{BD}} : \dot{\overline{BC}} = FT : BC$, et ex mox oſtenſis ſit $\dot{\overline{BC}} : \dot{\overline{AC}} = BC : BG$, erit ex æquo $\dot{\overline{BD}} : \dot{\overline{AC}} = FT : BG$.

(r) Cum ſit $FT : BC = \dot{\overline{BD}} : \dot{\overline{BC}}$ (not. p) et æquales ſint FH, BC, erit $FT : FH = \dot{\overline{BD}} : \dot{\overline{BC}}$. Invertendo et convertendo $FH : HT = \dot{\overline{BC}} : \dot{\overline{BC}} - \dot{\overline{BD}}$. Sed $\dot{\overline{BC}} - \dot{\overline{BD}} = \dot{\overline{CD}}$. (Geometr. Flux. Th. v.) Quare $FH : HT = \dot{\overline{BC}} : \dot{\overline{CD}}$. Invertendo $HT : FH = \dot{\overline{CD}} : \dot{\overline{BC}}$. Sed $FH : FT = \dot{\overline{BC}} : \dot{\overline{BD}}$ (not. p) et $FT : BG = \dot{\overline{BD}} : \dot{\overline{AC}}$ (not. q). Quare ex æquo $HT : BG = \dot{\overline{CD}} : \dot{\overline{AC}}$.

CAPUT VI.

MODUS VII. DE SPIRALIBUS.

42. Haud ſecus abſolvitur Problema, ubi Curvæ non ad rectas, ſed ad alias Curvas lineas, uti ſolent Mechanicæ, referuntur. Sit BG circuli peripheria, in cujus ſemidiametro AG, dum circa centrum A convolvitur, moveatur utcunque punctum D, et Spiralem ADE deſcribat. Et concipe D*d* eſſe partem curvæ infinitè parvam, per quam D fluit; et in AD cape AC=A*d*, et erunt *c*D ac G*g* contemporanea momenta rectæ AD et peripheriæ BG. Duc ergo A*t* parallelam rectæ *cd*, id eſt perpendicularem rectæ AD, et cum eâ tangens DT conveniat in T. Eritque *c*D : *cd* : : AD : AT. Sit inſuper G*t* parallela tangenti, et erit *cd* : G*g* (: : A*d* vel AD : AG) : : AT : A*t*.



43. Quare expoſitâ quâcunque æquatione, quâ relatio BG ad AD definitur, quære rationem fluxionum per Prob. 1; et cape A*t* in illâ ratione ad AD : eritque G*t* tangenti parallela.

44. EXEMPLUM 1. Dictis BG=x, et AD=y, ſit earum relatio $x^3-ax^2+axy-y^3=0$. Et ope Prob. 1. emerget $3x^2-2ax+ay:3y^2-ax$ $(::\dot{y}:\dot{x})$: : AD : A*t* : : AP : AG. Puncto *t* ſic invento, duc G*t* eique parallelam DT, et illa curvam tanget.

45. EXEMPL. 2. Si ſit $\frac{ax}{b}=y$ (quæ æquatio eſt ad Spiralem Archimedeam) erit $\frac{a\dot{x}}{b}=\dot{y}$. Adeoque $a:b(::\dot{y}:\dot{x})$: : AD : A*t*. Unde obiter ſi TA producatur ad P, ut ſit AP : AB : : $a:b$, PD ad curvam recta erit (f).

46. EXEMPL. 3. Si $xx=by$, erit $2\dot{x}x=b\dot{y}$, adeoque $2x:b$: : AD : A*t*. Et ſic Tangentes ad quaſcunque Spirales nullo negotio determinari poſſunt.

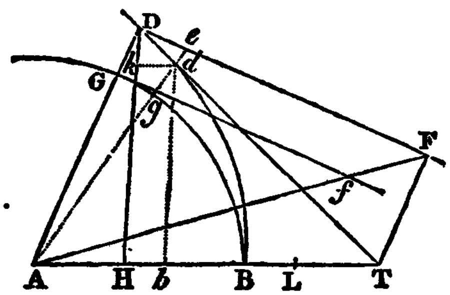
MODUS

(f) PUTA enim eductam eſſe DP ad angulos cum helice, hoc eſt, cum tangente ejus DT rectos. Et propter angulos ad A rectos, erit PA ad AD ut AD ad AT. (El. VI. 8.) Ratio autem AD ad AT componitur è rationibus illius AD ad A*t* et A*t* ad AT ; ſive è rationibus a ad b et AG vel AB ad AD. Quare PA ad AD rationem habet eam, quæ componitur è rationibus a ad b et AB ad AD. Et ex iiſdem componitur ratio rectanguli $a\times$AB ad rectangulum $b\times$AD. (El. VI. 23.) Erit igitur PA ad AD ut rectangulum $a\times$AB ad rectangulum $b\times$AD. Sed PA eſt ad AD ut rectangulum $b\times$PA ad rectangulum $b\times$AD. (El. VI. 1.) Rectangulum igitur $b\times$PA erit ad rectangulum $b\times$AD ut rectangulum

MODUS VIII. De Quadraticibus.

De Tangentibus Ducendis.

47. Ad hæc ſi Curva ſit ejuſmodi, ut per centrum A ductâ utcunque AGD, quæ circulo in G, curvæque in D occurrat; relatio inter arcum BG et rectam DH, quæ in dato angulo ad baſin AB ordinata eſt, æquatione quâvis definiatur: concipe punctum D per infinitè parvum intervallum ad d in curvâ moveri, et completo parallelogrammo dhHk, productâque Ad ad e ut ſit Ae=AD: erunt Gg arcûs BG, et Dk ordinatæ DH contemporanea momenta.



Produc jam Dd rectâ ad T, ubi cum AB conveniat ([t]), et demitte TF in De ([u]) perpendicularem; eruntque trapezia, Dkde, DHTF, ſimilia: atque adeo Dk : De : : DH : DF. Et præterea ſi Gf ad AG normalis erigatur, quæ cum AF concurrat in f, propter parallelas DF, Gf, erit De : Gg : : DF : Gf. Quamobrem ex æquo eſt DH : Gf : : Dk : Gg, hoc eſt, ut momenta ſive fluxiones linearum DH et BG.

48. Per æquationem itaque quâ relatio BG ad DH definitur, quære rationem fluxionum (per Prob. I.) et in eâ ratione cape Gf (tangentem circuli BG) ad DH. Age DF parallelam Gf, quæ cum Af productâ conveniat in F. Et ad F erige normalem FT occurrentem AB in T, et acta DT Quadratricem tanget.

49. Exempl. I. Nominatis BG=x, ac DH=y, eſto $xx=by$, et (per Prob. I.) erit $2\dot{x}x=b\dot{y}$. Adeoque $2x : b : : (\dot{y} : \dot{x}) : :$ DH : Gf. Et invento f, cætera, ut præſcriptum, eſt determinabis.

50. Cæterùm hæc Regula fortè ſic elegantior evadet; fac $\dot{x} : \dot{y} : :$ AB : AL. Dein AL : AD : : AD : AT, et DT curvam tanget. Nam propter æqualia triangula AFD, ADT eſt AD×DF=AT×DH. Adeoque AT : AD (: : DF : DH, ſive $\frac{\dot{y}}{\dot{x}}$ Gf) : : AD : $\frac{\dot{y}}{\dot{x}}$ AG, ſive AL ([x]).

51. Ex-

rectangulum a×AB ad rectangulum b×AD. (El. V. 11.) Rectangula igitur $\overline{b\times \text{PA}}$, $\overline{a\times \text{AB}}$ ſunt inter ſe æqualia. (El. V. 9.) Quare PA erit ad AB ut a ad b. (El. VI. 16.)

(t) Hoc eſt, produci intelligatur recta, quæ curvam in D contingit, uſque dum cum AB conveniat. Concurſus eſto T.

(u) Hoc eſt, in tangentem arcûs circularis centro A radio AD ſcripti.

(x) Hæc ratiocinatio magis geometricè ad hunc modum concinnari poteſt. " Propter rectangula

CAPUT VI.

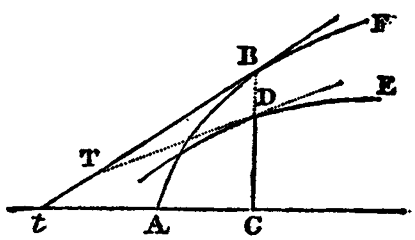
51. EXEMPL. 2. Eſto $x=y$ (quæ æquatio eſt ad Veterum Quadraticem) et per Prob. I. $\dot{x}=\dot{y}$. Adeoque AB : AD :: AD : AT.

52. EXEMPL. 3. Eſto $axx=y^3$, et erit $2a\dot{x}x=3\dot{y}y^2$. Fac ergo $3y^2 : 2ax (:: \dot{x} : \dot{y}) ::$ AB : AL. Dein AL : AD :: AD : AT.

53. Atque ita Tangentes aliarum Quadraticum, utcunque compoſitarum, poſſis expeditè determinare.

MODUS IX.

54. Si denique ABF ſit Curva quævis data, quam tangat recta Bt, et rectæ BC, in dato angulo ad baſin AC applicatæ, pars BD, inter hanc et aliam curvam DE intercepta, relationem ad curvæ portionem AB in æquatione quâcunque definitam habeat: alterius curvæ tangentem DT duces, capiendo, in hujus tangente, BT in eâ relatione ad BD, quam habet fluxio curvæ AB ad fluxionem rectæ BD.



55. EXEMPL. I. Dictis AB$=x$, et BD$=y$. Eſto $ax=yy$, et per Prob. I. erit $a\dot{x}=2\dot{y}y$. Adeoque $a : 2y (:: \dot{y} : \dot{x}) ::$ BD : BT.

56. EXEMPL. 2. Sit $\frac{a}{b} x=y$ (æquatio ad Trochoidem ſi modò ABF ſit Circulus) et erit $\frac{a}{b} \dot{x}=\dot{y}$. Adeoque $a : b ::$ BD : BT.

57. Et nihilo difficiliùs Tangentes, ubi ipſius BD ad AC vel ad BC relatio in æquatione quâvis exprimitur, vel ubi Curvæ aliis quibuſcunque modis ad rectas, aliaſve curvas, referuntur, poſſis ducere.

58. SUNT etiam alia non pauca Problemata, quorum ſolutiones ex hiſce fluent. Cujuſmodi ſunt,

I. Invenire punctum Curvæ, ubi Tangens eſt ad baſin, vel quamvis poſitione datam rectam, parallela, vel perpendicularis, vel in alio quovis angulo inclinata.

II. Invenire punctum, ubi Tangens maximè minimève ad baſin,

aut

gula AD × DF, AT × DH inter ſe æqualia, erit AT ad AD ut DF ad DH. Præterea cùm AL ſit ad AB ut $\dot{y}$ ad $\dot{x}$, ita enim factum eſt, cùm ſit etiam $\dot{y}$ ad $\dot{x}$ ut DH ad Gf (id enim oſtenſum § 47.) erit AL ad AB ut DH ad Gf. Sed cùm AT ſit ad AD ut DF ad DH, et AL ad AB ut DH ad Gf, ratio rectæ DF ad rectam Gf, quæ componitur e rationibus illius DF ad DH, et DH ad Gf, componetur illa quidem è rationibus rectæ AT ad rectam AD rectæque AL ad AB. Ex eiſdem autem componitur ratio rectanguli

AT × AL.

aut aliam poſitione datam rectam, inclinatur; hoc eſt invenire confinium flexûs contrarii. Hujus autem ſpecimen in Conchoide jam antè exhibui.

III. A dato quovis, extra Curvæ perimetrum, puncto rectam ducere, quæ cum perimetro aut angulum contactûs, aut rectum angulum, aut alium quemvis datum conficiet: hoc eſt, Tangentes vel Perpendiculares, vel aliter ad Curvam inclinatas rectas à dato quovis puncto ducere.

IV. A dato quovis intra Parabolam puncto rectam ducere, quæ maximum minimumve, quem poteſt, angulum cum perimetro ejus conficiet. Et idem de aliis curvis intellige.

V. Rectam ducere, quæ duas poſitione datas Curvas, vel eandem Curvam (ſi poteſt) in duobus punctis tangat.

VI. Curvam quamvis ſub datis conditionibus ducere, quæ aliam poſitione datam Curvam in dato puncto tanget.

VII. Luce in quamlibet curvam ſuperficiem incidente, cujuſvis radii refractionem determinare.

Horum et ſimilium Problematum confectiones, ubi non obſtat computandi tædium, non ſunt ita difficiles ut iis explicandis immorari opus ſit. Et Geometris, credo, magis gratum erit ſic tantùm recenſuiſſe.

CAPUT SEPTIMUM.

PROB. V.

Curvæ alicujus ad datum punctum Curvaturam invenire.

1. PROBLEMA cum primis elegans videtur, et ad Curvarum ſcientiam utile. In ejus autem conſtructionem generalia quædam præmittere convenit.

I. Ejuſdem Circuli eadem eſtque undique Curvatura, et inæqualium Circulorum Curvaturæ ſunt reciprocè proportionales diametris. Si alicujus diameter diametro alterius duplo minor eſt, ejus peripheriæ Curvatura erit duplo major; ſi diameter triplo minor eſt, Curvatura erit triplo major, &c.

AT × AL ad rectangulum AD × AB. (El. VI. 23.) Rectangulum igitur AT × AL erit ad rectangulum AD × AB ut DF ad Gf, ſive ut AD ad AG vel AB; hoc eſt ut quadratum ex AD, ad rectangulum AD × AB. Rectangulum igitur AT × AL quadrato ex AD æquale eſt. (El. V. 9.) Quare AT:AD = AD:AL. (El. VI. 17.)

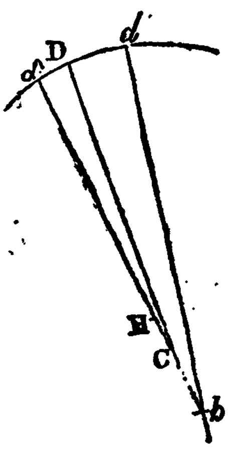
CAPUT VII. II. Si Circulus curvam aliquam ad partem concavam in dato punćto tangat, ſitque talis magnitudinis, ut alius contingens circulus in angulis contaćtûs proximè punćtum iſtud interſcribi nequeat, Circulus ille ejuſdem eſt curvitatis ac Curva in iſto punćto contaćtûs. Nam circulus, qui inter curvam et alium circulum juxta punćtum contaćtûs interjacet, minus deflećtit à curvâ, ejuſque curvaturam magis appropinquat, quàm ille alius circulus: et proinde curvaturam ejus maximè appropinquat, inter quem et curvam non alius quiſquam poteſt intercedere.

III. Itaque Centrum Curvaminis, ad aliquod curvæ punćtum, eſt centrum tangentis circuli æqualiter incurvati: et ſic Radius vel ſemidiameter Curvaminis eſt pars perpendiculi ad iſtud centrum terminata.

IV. Et proportio curvaminis, ad diverſa ejus punćta, è proportione curvaminis circulorum æquè curvorum, ſive è reciprocâ proportione Radiorum curvaminis innoteſcit.

2. PROBLEMA itaque ad hunc locum redit, ut Radius vel Centrum Curvaminis inveniatur.

3. Concipe ergo, quòd ad tria curvæ punćta δ, D ac d, ducantur perpendicula; quarum quæ ſunt ad, D et δ, conveniant in H; et quæ ad D et d, conveniant in b. Et punćto D exiſtente medio, ſi major eſt curvitas à parte Dδ quàm Dd, erit δH minor quàm db. Sed quo perpendicula δH ac db propiora ſunt intermedio perpendiculo, eo minùs diſtabunt punćta H et b, et convenientibus tandem perpendiculis, coaleſcent. Coaleſcant autem in punćto C, et erit illud C centrum curvaminis, ad curvæ punćtum D, cui perpendicula inſiſtunt. Id quod per ſe manifeſtum eſt.



4. Hujus autem C varia ſunt ſymptomata; quæ ad ejus determinationem inſervire poſſunt. Quemadmodum.

I. Quòd ſit concurſus perpendiculorum hinc et inde à DC infinitè parùm diſtantium.

II. Quòd perpe[illegible]culorum finitè parum diſtantium interſećtiones hinc et inde ultim[illegible] [illegible]terminat. [illegible]

quæ ſunt à parte curviori Dδ citeriùs, ad H, conveniant; et quæ ſunt ex alterâ minùs curvâ parte, Dd, remotiùs conveniant, ad b. De Radio Curvaturæ.

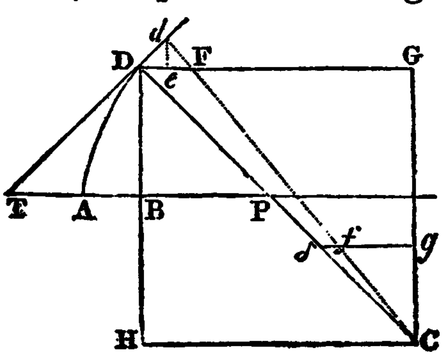
III. Si DC, dum curvæ perpendiculariter inſiſtit, moveri concipiatur, illud ejus punctum c (ſi demas motum accedendi vel recedendi à puncto inſiſtentiæ D) minimè movebitur, ſed centri motionis rationem habebit.

IV. Si centro c, intervallo DC, circulus deſcribatur, non poteſt alius deſcribi circulus qui juxta contactum interjacebit.

V. Denique ſi alterius alicujus tangentis circuli centrum, ut H, vel b, paulatim ad hujus centrum c accedat, donec tandem conveniat, tunc aliquod è punctis, in quibus circulus ille curvam ſecavit, ſimul conveniet punctum contactûs D.

Et unumquodque horum ſymptomatum anſam præbet diverſimodè reſolvendi Problema. Nos autem primum tanquam ſimpliciſſimum eligemus.

5. Ad quodlibet curvæ punctum D, eſto DT tangens, DC perpendiculum, et c centrum curvaminis, ut antè. Sitque AB baſis, ad quam DB in angulo recto applicatur, et cui DC occurrit in P. Age DG parallelam AB, et CG perpendiculum; inque eo cape Cg cujuſlibet datæ magnitudinis; et age $g\delta$ perpendiculum, quod occurrat DC in δ. Eritque Cg : $g\delta$: : (TB : BD : :) fiuxio baſis : fluxionem applicatæ. Concipe præterea punctum D per infinitè parvum intervallum Dd in curvâ promoveri.

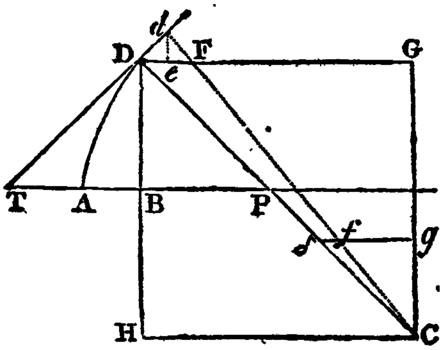


Et actis de ad DG et Cd ad curvam normalibus, quarum Cd occurrit DG in F et Cg in f: erit De momentum baſis, de momentum applicatæ, ac δf contemporaneum momentum rectæ $g\delta$. Eſtque DF = De + $\frac{de \times de}{De}$. Habitis itaque horum momentorum, ſive quod perinde eſt fluxionum generantium, rationibus, habebitur ratio GC ad datam gc (quippe quæ eſt DF ad δf) et inde punctum c determinabitur.

6. Sit ergo AB $= x$, BD $= y$, C$g = 1$. Et $g\delta = z$, et erit $1 : z : : \dot{x} : \dot{y}$ ſeu $z = \frac{\dot{y}}{\dot{x}}$. Hujus autem z momentum δf dic $\dot{z} \times o$, factum nempe

ex:

CAPUT VII.



ex velocitate et infinite parvâ quantitate. Eritque momentum $De = \dot{x} \times o$, $de = \dot{y} \times o$ et inde $DF = \dot{x}o + \frac{\dot{y}\dot{y}o}{\dot{x}}$. Eſt ergo $cg\,(1) : CG :: (\delta f : DF) :: \dot{z}o : \dot{x}o + \frac{\dot{y}\dot{y}o}{\dot{x}}$. Adeoque $CG = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{\dot{x}\dot{z}}$.

7. Cùm inſuper baſis fluxioni $\dot{x}$ (ad quam tanquam uniformem fluxionem cæteras referre convenit) liberum ſit quamcunque velocitatem tribuere: dic eſſe 1. Et erit $\dot{y} = z$, et $CG = \frac{1+zz}{\dot{z}}$. Et inde $DG = \frac{z+z^3}{\dot{z}}$, ac $DC = \frac{\overline{1+zz} \times \sqrt{1+zz}}{\dot{z}}$ (y).

8. Expoſitâ itaque quâvis æquatione, quâ relatio BD ad AB pro curvâ definiendâ deſignetur, imprimis quære relationem inter $\dot{x}$ et $\dot{y}$ per Prob. 1. et interea ſubſtitue 1 pro $\dot{x}$ et z pro $\dot{y}$. Dein ex æquatione reſultante per idem Prob. 1. quære relationem inter $\dot{x}$, $\dot{y}$ et $\dot{z}$, et interea ſubſtitue 1 pro $\dot{x}$, et z pro $\dot{y}$ ut antè. Atque ita per priorem operationem obtinebis valorem z, et per poſteriorem obtinebis $\dot{z}$: quibus habitis, produc DB ad H verſus concavam partem curvæ ut ſit $DH = \frac{1+zz}{\dot{z}}$; et age HC parallelam AB et perpendiculo CD occurrentem in C, eritque C centrum curvaturæ ad curvæ punctum D. Vel cùm ſit $1 + zz = \frac{PT}{BT}$ (z), fac $DH = \frac{PT}{\dot{z} \times TB}$, vel $DC = \frac{\overline{DP}^3}{\dot{z} \times \overline{DB}^3}$ (aa).

9. EXEMPL. 1. Sic expoſitâ $ax + bx^2 - y^2 = 0$ (æquatione ad Hyperbolam cujus latus rectum eſt a, ac tranſverſum $\frac{a}{b}$): emerget (per Prob. 1) $a + 2bx - 2zy = 0$ (ſcriptis nempe 1 pro $\dot{x}$ et z pro $\dot{y}$ in æquatione reſultante, quæ ſecus foret $a\dot{x} + 2b\dot{x}x - 2y\dot{y} = 0$). Et hinc denuo prodit $2b - 2zz - 2\dot{z}y = 0$ ſcriptis iterum 1 pro $\dot{x}$ et z pro

(y) $DC = \sqrt{CG^2 + DG^2}$. $CG = \frac{1+zz}{\dot{z}}$. Et $DG = CG \times z = \frac{\overline{1+zz} \times z}{\dot{z}}$.

Quare $CG^2 = \frac{\overline{1+zz}^2}{\dot{z}\dot{z}}$. Et $DG^2 = \frac{\overline{1+zz}^2 \times zz}{\dot{z}\dot{z}}$. Ergo $CG^2 + DG^2 = \frac{\overline{1+zz}^2 \times \overline{1+zz}}{\dot{z}\dot{z}}$.

Ergo $\sqrt{CG^2 + DG^2} = \frac{\overline{1+zz} \times \sqrt{1+zz}}{\dot{z}} = DC$.

(z) $PB : BT = DB^2 : BT^2 = \dot{y}\dot{y} : \dot{x}\dot{x} = zz : 1$.

Ergo

pro $\dot{y}$. Per priorem eſt $z=\frac{a+2bx}{2y}$, et per poſteriorem $\dot{z}=\frac{b-zz}{y}$. *De Radio Curvaturæ.*
Dato itaque quovis curvæ puncto D, et per conſequentiam x et y, ex his dabuntur z et $\dot{z}$; quibus cognitis, fac $\frac{1+zz}{\dot{z}}=$ GC, vel DH, et age HC.

10. Quemadmodum ſi definitè ſit $a=3$, et $b=1$, adeoque $3x+xx=yy$, hyperbolæ conditio: et ſi aſſumatur $x=1$, erit $y=2$, $z=\frac{5}{4}$, $\dot{z}=-\frac{9}{32}$, et DH $=-9\frac{1}{9}$. Invento H erige HC occurrentem perpendiculo DC priùs ducto. Vel quod perinde eſt, fac DH:HC $(::1:z)::1:\frac{5}{4}$, et age DC curvaminis radium.

11. Siquando computationem non admodum perplexam fore cenſeas, poſſis indefinitos valores ipſorum $\dot{z}$ et z, in $\frac{1+zz}{\dot{z}}$ valore co ſubſtituere. Et ſic in hoc exemplo, per debitam reductionem, obtinebis DH $=y+\frac{4y^3+4by^3}{aa}$, cujus tamen DH valor per calculum negativus prodit, ſicut in exemplo numerali videre eſt. At hoc tantùm arguit DH ad partes verſus B capiendam eſſe. Nam ſi fuiſſet affirmativus, ad contrarias partes duxiſſe oporteret.

12. Corol. Hinc ſi ſignum ſymbolo $+b$ præfixum mutetur; ut fiat $ax-bxx-yy=0$, æquatio ad Ellipſin, erit DH $=y+\frac{4y^3-4by^3}{aa}$. At poſito $b=0$, ut æquatio fiat $ax-y^2=0$ ad Parabolam, erit DH $=y+\frac{4y^3}{a^2}$, indeque DG $=\frac{1}{2}a+2x$.

13. Ex hiſce facilè colligitur Radium curvaturæ cujuſvis conicæ ſectionis valere $\frac{4\overline{DP}^3}{aa}$.

14. Exempl. 2. Si $x^3=ay^2-xy^2$ (æquatio ad Ciſſoidem Dioclis) exponatur, per Prob. 1. imprimis obtinebitur $3x^2=2azy-2xzy-yy$; ac deinde $6x=2a\dot{z}y+2az\dot{z}-2\dot{z}y-2x\dot{z}y-2xz\dot{z}-2\dot{z}y$: adeoque $z=\frac{3xx+yy}{2ay-2xy}$, et $\dot{z}=\frac{3x-az^2+2zy+xz^2}{ay-xy}$. Dato itaque quolibet Ciſſoidis punc-

Ergo PT : BT $=1+zz:1$. Ergo $\frac{PT}{BT}=1+zz$.

(r) DC $=\frac{1+zz\times\sqrt{1+zz}}{\dot{z}}=\frac{PT}{BT\times\dot{z}}\times\frac{\sqrt{PT}}{\sqrt{BT}}=\frac{\sqrt{PT^3}}{\dot{z}\sqrt{BT^3}}$. Sed PT : TB $=PT^2:TD^2=PD^2:DB^2$.

Ergo $\frac{PT}{TB}=\frac{PD^2}{DB^2}$. Ergo $\frac{PT^3}{BT^3}=\frac{PD^6}{DB^6}$. Et $\sqrt{\frac{PT^3}{BT^3}}=\frac{PD^3}{DB^3}$. Quare DC $=\frac{PD^3}{DB^3\times\dot{z}}$

3 to,

 to, et inde x et y, dabuntur z & $\dot{z}$. Quibus cognitis fac $\frac{1+zz}{\dot{z}}=$ CG.

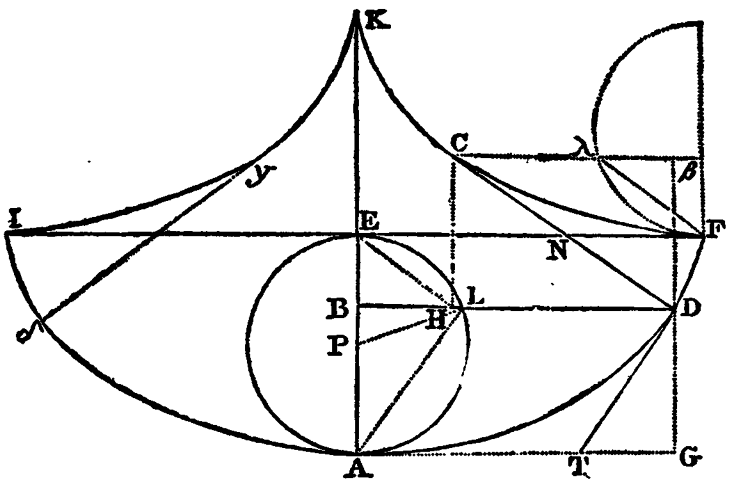
15. Exempl. 3. Si detur $\overline{b+y}\sqrt{cc-yy}=xy$ æquatio ad Conchoidem, ut ſuprà; finge $\sqrt{cc-yy}=v$, et emerget $bv+yv=xy$. Jam harum prior (viz. $cc-yy=vv$) per Prob. 1. dat $-2yz=2\dot{v}v$ (ſcripto z pro $\dot{y}$) et poſterior dat $b\dot{v}+y\dot{v}+zv=y+zx$. Et ex his æquationibus ritè diſpoſitis determinantur $\dot{v}$ et z. Ut autem $\dot{z}$ præterea determinetur, è noviſſimâ æquatione extermina fluxionem $\dot{v}$, ſubſtituendo $\frac{-yz}{v}$; et emerget $-\frac{byz}{v}-\frac{yyz}{v}+zv=y+xz$. Æquatio quæ fluentes quantitates ſine aliquibus earum fluxionibus, prout exigit reſolutio Prob. 1.) complectitur. Hinc itaque per Prob. 1. elicies $-\frac{bz^2}{v}-\frac{by\dot{z}}{v}+\frac{byz\dot{v}}{vv}-\frac{2yzz}{v}-\frac{yy\dot{z}}{v}+\frac{yyz\dot{v}}{vv}+z\dot{v}+\dot{z}v=2z+x\dot{z}$; quâ æquatione in ordinem reductâ et concinnatâ, dabitur $\dot{z}$. Inventis autem z et $\dot{z}$ fac $\frac{1+zz}{\dot{z}}=$ CG.

16. Si penultimam æquationem per z diviſiſſes, exinde poſtmodum, per Prob. 1. obtinuiſſes $-\frac{bz}{v}+\frac{by\dot{v}}{vv}-\frac{2yz}{v}+\frac{yy\dot{v}}{vv}+\dot{v}=2-\frac{y\dot{z}}{zz}$, æquationem priori ſimpliciorem pro determinando $\dot{z}$.

17. Dedi quidem hoc exemplum, ut modus operandi in ſurdis æquationibus conſtaret. At Conchoidis curvatura ſic breviùs inveniri potuit. Æquationis, $\overline{b+y}\sqrt{cc-yy}=xy$, partibus quadratis et per yy diviſis, exurgit $\frac{bbcc}{yy}+\frac{2bcc}{y}+\begin{matrix}cc\\-bb\end{matrix}-2by-y^2=xx$. Et inde per Prob. 1. exoritur $-\frac{2b^2c^2z}{y^3}-\frac{2bc^2z}{y^2}-2bz-2y\dot{z}=2x$; ſive $-\frac{b^2c^2}{y^3}-\frac{bc^2}{y^2}-b-y=\frac{x}{z}$. Et hinc denuo per Prob. 1. exoritur $\frac{3b^2c^2z}{y^4}+\frac{2bc^2z}{y^3}-z=\frac{1}{z}-\frac{x\dot{z}}{zz}$. Per priorem exitum determinatur z et per poſteriorem $\dot{z}$.

18. Exempl. 4. Sit ADF Trochois ad circulum ALE, cujus diameter eſt AE, accommodata; et ordinatâ BD ſecante circulum in L, dic AE$=a$, AB$=x$, BD$=y$, BL$=v$, et arcum AL$=t$, Et imprimis, ducto PL ſemidiametro, erit fluxio baſis AB ad fluxionem arcûs AL ut BL ad PL, hoc eſt, $\dot{x}$ ſive $1 : \dot{t} :: v : \frac{1}{2}a$. Atque adeo $\frac{a}{2v}=\dot{t}$.

19. Porro ex naturâ circuli eſt $ax-xx=vv$, et inde per Prob. 1. $a-2x=2v\dot{v}$ ſive $\frac{a-2x}{2v}=\dot{v}$. DE RADIO CURVATURÆ.



20. Ad hæc ex naturâ Trochoidis eſt LD= arc. AL; adeoque $v+t=y$, et inde per Prob. 1. $\dot{v}+\dot{t}=z$.

21. Denique pro fluxionibus $\dot{v}$ & $\dot{t}$, valores hi ſubſtituantur, et emerget $\frac{a-x}{v}=z$. Unde per Prob. 1. deducitur $\frac{-a\dot{v}}{vv}+\frac{x\dot{v}}{vv}-\frac{1}{v}=\dot{z}$. Et his inventis, fac $\frac{1+zz}{\dot{z}}=-$DH, et erige HC.

22. COROL. Cæterùm ex his conſectatur,

I. Quòd ſit DH=2BL, & CH=2BE, ſive quòd EF in N biſecat CD radium curvaminis. Et hoc patebit ſubſtituendo valores $\dot{z}$ et z jam inventos in æquatione $\frac{1+zz}{\dot{z}}=$DH, et exitum probè reducendo.

II. Hinc Curva FCK, in quâ centrum curvaminis indefinitè verſatur, eſt alia huic æqualis Trochois, cujus vertices ad I et F adjacent hujus cuſpidibus. Nam circulus Fλ, æqualis ALE et ſimiliter poſitus, deſcribatur, et agatur Cβ parallela EF, circuloque occurrens in λ; et erit arcus Fλ (= arc. EL=NF) = Cλ.

III. CD quæ recta eſt ad trochoidem IAF contingit trochoidem IKF in C.

IV. Hinc, inverſis Trochoidibus, ſi ſuperioris trochoidis cuſpidi, K, pondus ad diſtantiam KA, ſive 2EA, filo appenſum innitatur, et, undulante pondere, filum ſe applicet ad trochoidis partes KF et KI, hinc inde obſiſtentes, ne in rectam diſtendatur, et cogentes, ut ad earum normam, dum digreditur à perpendiculo, paulatim deſuper inflectatur, parte CD ſub infimo contactûs puncto manente rectâ; pondus in inferioris trochoidis perimetro movebitur, utpote cui filum CD ſemper perpendiculare eſt.

 V. Eſt

 V. Eſt itaque tota fili longitudo KA æqualis perimetro trochoidis KCF, ejuſque pars CD æqualis parti perimetri CF.

VI. Cùm filum circa mobile punctum c, tanquam centrum, undulando convolvitur; ſuperficies, per quam tota CD continuò trajicitur, erit ad ſuperficiem, per quam pars CN ſupra rectam IF ſimul trajicitur, ut $\overline{CD}^2$ ad $\overline{CN}^2$, hoc eſt ut 4 ad 1. Eſt itaque area CFN quarta pars areæ CFD, et area KCNE quarta pars areæ AKCD.

VII. Quinimo cùm ſubtenſa EL ſit æqualis et parallela CN, et circa immobile centrum, E, perinde ac CN circa mobile centrum C circumagitur, æquales erunt ſuperficies per quas ſimul trajiciuntur; nempe area CFN et circuli ſegmentum EL, et inde area NFD tripla erit ſegmenti iſtius, ac tota EADF tripla ſemicirculi.

VIII. Denique cùm pondus D attingit punctum F, totum filum circum Trochoidis perimetrum KCF flectetur, radio curvaminis CD manente nullo. Et proinde Trochois IAF ad ejus cuſpidem F curvior eſt quàm quilibet circulus, et cum tangente ƒF productâ conſtituit angulum contactûs infinitè majorem, quàm circulus cum rectâ poteſt conſtituere.

23. Sunt etiam anguli contactûs Trochoidalibus infinitè majores. Et illis deinceps alii infinitè majores, et ſic in infinitum, et tamen maximi ſunt infinitè minores rectilineis. Sic $xx=ay$, $x^3=by^2$, $x^4=cy^3$, $x^5=dy^4$, &c. denotant ſeriem Curvarum, quarum quælibet poſterior cum baſi conſtituit angulum contactûs infinitè majorem, quàm prior cum eâdem baſi poteſt conſtituere; eſtque angulus contactûs, quem prima $xx=ay$ conſtituit, ejuſdem generis cum circularibus; et ille, quem ſecunda $x^3=by^2$ conſtituit, ejuſdem generis cum Trochoidalibus. Et quamvis ſubſequentium anguli angulos præcedentium perpetim infinitè ſuperant, tamen anguli rectilinei magnitudinem nunquam poſſunt aſſequi.

24. Ad eundem modum $x=y$, $x^2=ay$, $x^3=b^2y$, $x^4=c^3y$, &c. denotant ſeriem linearum, quarum ſubſequentium anguli ad vertices, cum baſibus confecti, ſunt angulis præcedentium perpetim infinitè minores. Quinetiam inter angulos contactûs duorum quorumlibet ex his generibus, poſſunt alia angulorum, ſe infinitè ſuperantium, intercedentia genera in infinitum excogitari.

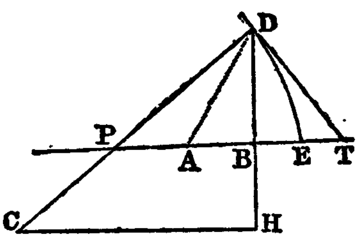
25. Angulorum verò contactûs unum genus eſſe infinitè majus

jus alio conſtat, cùm unius generis Curva, utcunque magna, inter rectam tangentem et alterius generis Curvam, quantumvis parvam, juxta punctum contactûs non poteſt interjacere: ſive cujus angulus contactûs neceſſariò continet alterius angulum contactûs, ut partem totius. Sic curvæ $x^4=cy^3$ angulus contactûs, quem cum baſi conſtituit, neceſſariò continet angulum contactûs curvæ $x^3=by^2$. Qui verò ſe mutuò ſuperare poſſunt anguli ſunt ejuſdem generis, uti de præfatis angulis Trochoidis et hujus Curvæ, $x^3=by^2$, contigit.

De Angulo Contactus.

26. Ex his patet Curvas in quibuſdam punctis poſſe infinitè rectiores eſſe, vel infinitè curviores, quolibet circulo; et tamen formam curvarum non ideo amittere. Sed hæc in tranſitu (bb).

27. EXEMPL. 5. Eſto ED Quadratrix ad circulum centro A deſcriptum pertinens, ac DB ad AE normaliter demiſſâ, dic $AB=x$, $BD=y$, & $AE=1$. Eritque $\dot{y}x-\dot{y}y^2-\dot{y}x^2=\dot{x}y$, ut ſuprà Cap. VI. § 19: quæ æquatio, ſcriptis 1 pro $\dot{x}$, et z pro $\dot{y}$, fit $zx-zy^2-zx^2=y$; et inde per Prob. I. elicitur $\dot{z}x-\dot{z}y^2-\dot{z}x^2+z\dot{x}-2z\dot{x}x-2z\dot{y}y=\dot{y}$. Factâque reductione, et ſcriptis iterum 1 pro $\dot{x}$, et z pro $\dot{y}$, exit $\dot{z}=\frac{2z^2y+2zx}{x-x^2-y}$. Inventis autem z et $\dot{z}$, fac $\frac{1+zz}{\dot{z}}=DH$; et age HC ut ſuprà.



28. Si conſtructionem concinnare placet, perbrevem invenies: nempe ad DT duc normalem DP occurrentem AT in P, et fac eſſe 2AP : AE :: PT : CH.

29. Scilicet eſt $z=\frac{y}{x-x^2-y^2}=\frac{BD}{-BT}$; et $zy=\frac{\overline{BD}^2}{-BT}=-BP$; et $zy+x=-AP$; et $\frac{2z}{x-x^2-y^2}$ in $zy+x=\frac{2BD}{AE\times\overline{BT}^2}$ in $-AP=\dot{z}$. Præterea eſt $1+zz=\frac{PT}{BT}$, (utpote $=1+\frac{\overline{BD}^2}{\overline{BT}^2}=\frac{\overline{DT}^2}{\overline{BT}^2}$,) adeoque $\frac{1+zz}{\dot{z}}=\frac{PT\times AE\times BT}{-2BD\times AP}=DH$. Denique eſt $BT:BD::DH:CH=\frac{PT\times AE}{-2AP}$. Ubi valor negativus tantùm arguit CH capiendam eſſe ad partes DH verſus AB.

30. Eâdem methodo Spiralium, et aliarum quarumvis curvarum Curvatura calculo breviſſimo determinari poteſt.

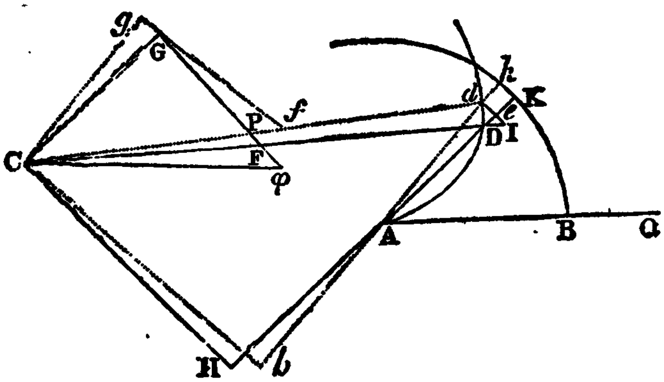
31. Ad Curvaturam præterea, cùm Curvæ aliis modis ad rectas referuntur, ſine prævià reductione determinandam, jam potuit

(bb) Idem argumentum auctor tractavit Princip. Lib. I. Sect. I. Schol.

hæc

CAPUT VII. hæc methodus applicari, perinde ut in determinando tangentes factum eſt. Sed cùm omnes Geometricæ Curvæ, ut et Mechanicæ (præſertim ubi definientes conditiones ad infinitas æquationes uti pòſt oſtendam reducantur) ad rectangulas ordinatas referri poſſint, videor ſatis præſtitiſſe. Qui plura deſiderat, haud difficulter proprio marte ſupplebit. Præſertim ſi in ejus rei illuſtrationem, ex abundanti, methodum pro Spiralibus adjecero.

32. Eſto BK circulus, A centrum ejus, B punctum in circumferentiâ datum, ADd Spiralis, DC perpendiculum ejus, et C centrum Curvitatis ad punctum D. Ductâque ADK rectâ, et ei parallelâ et æquali CG, ut et normali GF occurrente CD in F: dic AB vel $AK = 1 = CG$, $BK = x$, $AD = y$, et $GF = z$. Præterea concipe punctum D per infinitè parvum ſpatium Dd in ſpirali moveri, et perinde per d agi ſemidiametrum AK, eique parallelam et æqualem Cg, et normalem gf occurrentem Cd in f. Cui etiam GF occurrit in P: produc GF ad ϕ ut ſit $G\phi = gf$, et ad AK demitte normalem de, et produc, donec cum CD conveniat ad I: et ipſarum BK, AD ac $G\phi$ contemporanea momenta erunt Kk, De et Fϕ; quæ proinde dicentur $\dot{x} \times o$, $\dot{y} \times o$, et $\dot{z} \times o$.



33. Jam eſt AK:Ae(AD)::kK:$de = oy$, ubi aſſumo $\dot{x} = 1$ ut ſuprà. Item CG:GF::de:eD $= yoz$, adeoque $yz = \dot{y}$. Præterea, CG:CF:: de:dD$(= oy \times CF)$::dD:dI $= oy \times \overline{CF}^2$. Ad hæc propter ang. PC$\phi$ $(= \angle GCg) = \angle DAd$, et $\angle CP\phi$ $(= \angle Cd I = \angle ed D + \text{rect.}) = \angle ADd$, triangula CP$\phi$ et ADd ſunt ſimilia: et inde AD:Dd::CP (CF):$P\phi = o \times \overline{CF}^2$, unde aufer F$\phi$ et reſtabit $PF = o \times \overline{CF}^2 - o \times \dot{z}$. Denique demiſſâ CH normali ad AD, eſt PF:dI::CG:eH vel $DH = \frac{y \times \overline{CF}^2}{\overline{CF}^2 - \dot{z}}$; vel ſubſtituto $1 + zz$ pro $\overline{CF}^2$, erit $HD = \frac{y + yzz}{1 + zz - \dot{z}}$. 34. Et

(cc) NIMIRUM erit DA : DH $= 1 + zz - \dot{z} : 1 + zz$ (per § 33.) Verùm ex eo quòd ſit $\dot{y}z + y\dot{z} = o$ (§ 36) veniet $\dot{z} = \frac{-\dot{y}z}{y} = -z^2$. Unde $1 + zz - \dot{z} = 1 + 2z^2$, et DA : DH $= 1 + 2z^2 : 1 + z^2$. Rurſum

34. Et nota, quòd in hujuſmodi computationibus quantitates (ut AD et Ae) pro æqualibus habeo, quarum ratio à ratione æqualitatis non niſi infinitè parum differt. De Radio Curvaturæ.

35. Ex his autem prodit hujuſmodi Regula. Relatione inter x et y per quamlibet æquationem definitâ, quære relationem fluxionum $\dot{x}$ et $\dot{y}$, ope Prob. 1. et ſubſtitue 1 pro $\dot{x}$ et yz pro $\dot{y}$. Deinde ex æquatione prodeunte quære denuò per Prob. 1. relationem inter $\dot{x}$, $\dot{y}$ et $\dot{z}$, et iterum ſubſtitue 1 pro $\dot{x}$. Prior exitus, per debitam reductionem, dabit $\dot{y}$ et z, et poſterior dabit $\dot{z}$: quibus cognitis fac $\frac{y+yzz}{1+zz-\dot{z}} =$ DH, et erige normalem HC ſpiralis perpendiculo DC, priùs ducto, occurrentem in C; et erit C centrum curvaminis. Vel quod eodem recidit, cape CH : HD :: z : 1, et age CD.

36. Exempl. 1. Si detur $ax=y$, æquatio ad Spiralem Archimedeam: erit per Prob. 1. $a\dot{x}=\dot{y}$ ſive (ſcripto 1 pro $\dot{x}$ et yz pro $\dot{y}$) $a=yz$. Et hinc denuò per Prob. 1. erit $0=\dot{y}z+y\dot{z}$. Quare ex dato quolibet ſpiralis puncto D, et inde longitudine AD ſive y, dabuntur z $(=\frac{a}{y})$ et $\dot{z}$ $(=\frac{-a\dot{y}}{y^2}$ ſive $=\frac{-az}{y})$: quibus cognitis fac $1+zz-\dot{z} : 1+zz ::$ DA (y) : DH. Et $1 : z ::$ DH : CH.

37. Et hinc facilè deducitur hujuſmodi conſtructio. Produc AB ad Q, ut ſit AB : arc. BK :: arc. BK : BQ. Et fac $\overline{\text{AB}+\text{AQ}}$: AQ :: AD : DH (cc).

38. Exempl. 2. Si $ax^2=y^3$ definit relationem inter BK et AD: obtinebis (per Prob. 1.) $2a\dot{x}x=3\dot{y}y^2$, ſive $2ax=3zy^3$. Et inde rurſus $2a\dot{x}=3\dot{z}y^3+9z\dot{y}y^2$. Eſt itaque $z=\frac{2ax}{3y^3}$, et $\dot{z}=\frac{2a-9zzy^3}{3y^3}$. Quibus cognitis fac $1+zz-\dot{z} : 1+zz ::$ DA : DH. Vel opere concinnato, fac $9xx+10 : 9xx+4 ::$ AD : DH.

39. Exempl. 3. Ad eundem modum, ſi $ax^2-bxy=y^3$, determinat relationem BK ad AD, orietur $\frac{2ax-by}{bxy+3y^3}=z$, et $\frac{2a-2bzy-bz^2xy^2-9z^2y^3}{bxy+3y^3} = \dot{z}$. Ex quibus DH, et inde punctum C determinatur ut antè.

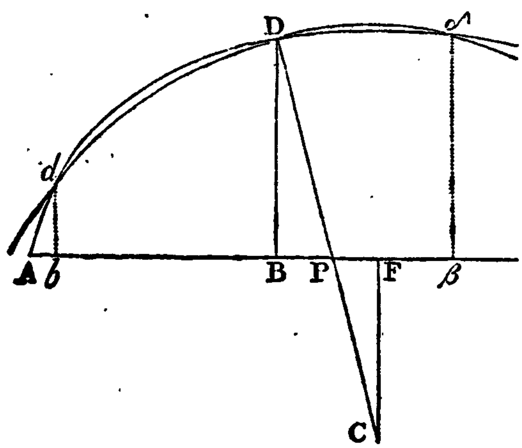
40. Et ſic aliarum quarumvis ſpiralium Curvaturam nullo negotio determinabis. Imo et ad horum exemplar Regulas pro quibuſlibet Curvarum generibus excogitare poſſis.

ſum eo quòd ſit $z=\frac{a}{y}$, et $y=ax$ (§ 36.) veniet $z=\frac{1}{x}$, et $z^2=\frac{1}{x^2}$. Undè DA : DH $= 1 + \frac{2}{x^2} : 1 + \frac{1}{x^2} = x^2+2 : x^2+1$. Scripto igitur QB × BA pro x^2, et AB² pro 1, et 2AB² pro 2, efficietur DA : DH = AQ × AB + AB² : AQ × AB = AQ + AB : AQ. Q. E. D.

41. Ab-

41. Abſolvi tandem Problema; ſed cùm methodum adhibüerim à vulgaribus operandi modis ſatis diverſam, et ipſum problema non ſit ex eorum numero, quorum contemplatio apud Geometras increbuit: in allatæ ſolutionis illuſtrationem et confirmationem non gravabor aliam ſolutionem attingere, magis obviam et uſitatis in ducendo tangentes methodis affinem. Utpote ſi centro et intervallo quovis Circulus deſcribi concipiatur, qui Curvam quamlibet in pluribus punctis ſecet, et circulus ille contrahatur vel dilatetur, donec duo interſectionum puncta conveniant, is Curvam ibidem tanget. Et præterea ſi centrum ejus accedere vel recedere à puncto contactûs fingatur, donec tertium interſectionis punctum cum prioribus in puncto contactûs conveniat, is æquè curvus ac Curva in illo puncto contactûs evadet. Quemadmodum in ultimo quinque ſymptomatum centri curvaminis ſuprà monui, è quorum ſingulis dixi Problema diverſimodè confici potuiſſe.

42. Centro itaque C et radio CD deſcribatur circulus ſecans Curvam in punctis d, D ac δ. Et demiſſis, db, DB, $\delta\beta$ et CF ad baſin AB normalibus: dic AB$=x$, BD$=y$, AF$=v$, FC$=t$ ac DC$=s$; et erit BF$=v-x$, ac DB+FC$=y+t$; quorum quadratorum aggregatum æquatur quadrato DC. Hoc eſt, $t^2+v^2-2vx+x^2+y^2+2ty=s^2$. Quam ſi placet, abbreviare poſſis, fingendo $v^2+t^2-ss=$ ſymbolo cuivis q^2, et evadet $x^2-2vx+y^2+2ty+q^2=0$. Poſtquam verò t, v, et q^2 inveneris, ſi s deſideres, fac $=\sqrt{v^2+t^2-q^2}$.



43. Proponatur jam quælibet æquatio pro Curvâ definiendâ, cujus flexuræ quantitatem invenire oportet; et ejus ope alterutram quantitatem x vel y, extermina, et emerget æquatio cujus radices (db, DB, $\delta\beta$, &c. ſi extermines x, vel Ab, AB, Aβ, &c. ſi extermines y) ſunt ad interſectionum puncta (d, D, δ, &c.) Et proinde cùm ex iſtis tres evadent æquales, circulus et Curvam continget, et erit ejuſdem curvitatis ac Curva in puncto contactûs. Æquales

les autem evadent, conferendo æquationem cum aliâ totidem di- De Radio Curvaturæ.
menſionum æquatione fictitiâ, cujus tres ſunt æquales radices, ut docuit Carteſius: vel expeditiùs multiplicando terminos ejus bis per arithmeticam progreſſionem.

44. Exempl. Sit $ax=y^2$, æquatio ad Parabolam; et exterminato x (ſubſtituendo nempe in æquatione ſuperiori valorem ejus $\frac{yy}{a}$) prodibit

$$\frac{y^4}{a^2} - \frac{2v}{a}y^2 + 2ty + q^2 = 0;$$
$$\qquad + \; y^2$$

cujus è radicibus y tres debent fieri æquales.

Et in hunc finem terminos per arithmeticam Progreſſionem bis multiplico, ut hîc videre eſt.

4.	2.	1.	0
3.	1.	0.	−1

Et erit

$$\frac{12y^4}{aa} - \frac{4v}{a}y^2 + 2y^2 \quad = 0$$

Sive $v = \frac{3y^2}{a} + \frac{1}{2}a$. Unde facilè colligitur eſſe BF $= 2x + \frac{1}{2}a$ ut ſuprà.

45. Quamobrem dato quovis Parabolæ puncto D, duc perpendiculum DP, et in axe cape PF $=$ 2AB, et erige normalem FC occurrentem DP in C, et erit C deſideratum centrum Curvitatis.

46. Idem in Ellipſi et Hyperbolâ præſtare poſſis, ſed calculo ſatis moleſto, et in aliis Curvis ut plurimùm faſtidioſiſſimo.

CAPUT VIII.

De Quæſtionibus quibuſdam Cognatis.

1. Ex hujus Problematis reſolutione conſectantur aliorum nonnullorum confectiones. Cujus modi ſunt,

I. Invenire punctum ubi linea datam habet curvaturam.

2. Sic in Parabolâ $ax=yy$, ſi punctum quæratur ad quod radius curvaturæ ſit datæ longitudinis f, è centro curvaturæ, ut priùs, invento, radium determinabis eſſe $\frac{\dot{z}+4x}{2\dot{z}}\sqrt{\dot{z}\dot{z}+4\dot{z}x}$, quem pone æqualem f. Et factâ reductione emerget $x = -\frac{1}{4}a + \sqrt[3]{\frac{1}{16}aaf}$.

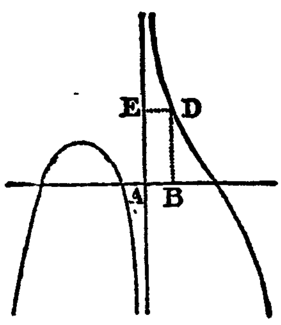
II. Invenire punctum Rectitudinis.

3. Punctum Rectitudinis voco, ad quod radius flexionis infinitus evadit, ſive centrum infinitè diſtans: quale eſt ad verticem Parabolæ $ax^3 = y^4$. Et hoc idem plerumque limes eſt flexionis contrariæ, cujus determinationem ſuprà poſui. Sed et alia haud inelegans ex hoc Problemate ſcaturit. Nempe quo longior eſt radius

Caput VIII. radius flexionis, eo minor evadit angulus DCd (fig. p. 445.) et pariter momentum $\dot{o}f$, adeoque fluxio quantitatis z unà diminuuntur, ita ut per ejus radii infinitatem prorſus evaneſcant. Quære ergo fluxionem $\dot{z}$ et ſuppone nullam eſſe.

4. Quemadmodum ſi limitem flexûs contrarii in Parabola ſecundi generis, cujus ope Carteſius conſtruxit æquationes ſex dimenſionum, determinare oportet. Ad illam curvam æquatio eſt $x^3-bx^2-cdx+bcd+dxy=0$. Et hinc per Prob. I. exit $3\dot{x}x^2-2b\dot{x}x-cd\dot{x}+d\dot{x}y+dx\dot{y}=0$. Quæ, ſcripto 1 pro $\dot{x}$ et z pro $\dot{y}$, fit $3x^2-2bx-cd+dy+dxz=0$: unde rurſus per Prob. I. exit $6\dot{x}x-2b\dot{x}+dy+d\dot{x}z+dx\dot{z}=0$. Et hæc, ſcripto iterum 1 pro $\dot{x}$, z pro $\dot{y}$, et 0, pro $\dot{z}$, fit $6x-2b+2dz=0$. Jam extermina z, ſcribendo pro dz valorem $2b-3x$ in æquatione $3x^2-2bx-cd+dy+dxz=0$, et proveniet $-cd+dy=0$, ſive $y=c$.

5. Quamobrem ad punctum A erige perpendiculum AE$=c$. Et per E duc ED parallelam AB, et punctum D, ubi Parabolæ partem convexo-concavam ſecuerit, erit in confinio flexionis contrariæ.



6. Similique methodo alia Rectitudinis puncta, quæ non interjacent partibus contrariè flexis, determinari poſſint. Veluti ſi $x^4-4ax^3+6a^2x^2-b^3y=0$ Curvam definiat, exinde per Prob. I. imprimis producetur $4x^3-12ax^2+12a^2x-b^3z=0$. Et hinc denuò $12x^2-24ax+12a^2-b^3\dot{z}=0$. Ubi ſuppone $\dot{z}=0$, et, factâ reductione, prodibit $x=a$. Quamobrem ſume AB$=a$ et BD, normaliter erecta, curvæ in deſiderato Rectitudinis puncto occurret.

III. Invenire punctum Flexûs Infiniti.

7. Quære radium curvaminis et ſuppone nullum eſſe. Sic ad Parabolam ſecundi generis, æquatione $x^3=ay^2$ definitam, erit radius ille CD$=\frac{4a+9x}{6a}\sqrt{4ax+9x^2}$; qui nullus evadit, cùm ſit $x=0$.

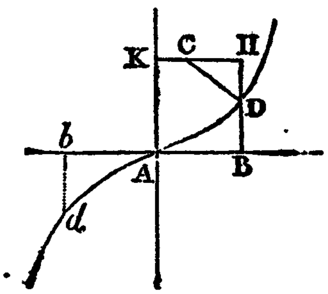
IV. Flexûs Maximi Minimive punctum determinare.

8. Ad hujuſmodi puncta radius curvaturæ aut maximus aut minimus evadit. Quare centrum Curvaturæ, ad id temporis momentum, nec verſus punctum contactûs neque ad contrarias partes movetur, ſed penitus quieſcit. Quæratur itaque fluxio radii

CD:

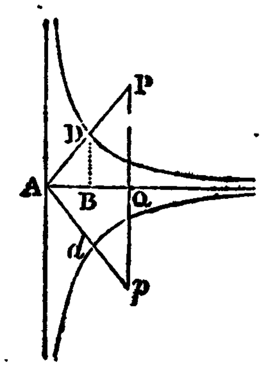
CD : vel expeditiùs quæratur fluxio alterutrius rectæ BH vel AK et ſupponantur nulla. PROBLEMATA DE CURVATURA.

9. Quemadmodum ſi de Parabolâ ſecundi generis, $x^3=a^2y$, quæſtio proponatur: imprimis ad Curvaturæ centrum determinandum invenies DH$=\frac{a^2+9xy}{6x}$, adeoque eſt BH$=\frac{a^2+15xy}{6x}$. Dic autem BH$=v$, et erit $\frac{a^2}{6x}+\frac{5}{2}y=v$; unde juxta Prob. I. educitur $-\frac{a^2\dot{x}}{6x^2}+\frac{5}{2}\dot{y}=\dot{v}$. Jam verò $\dot{v}$, ipſius BH fluxionem, ſuppone nullam eſſe: et inſuper cùm



ex hypotheſi ſit $x^3=a^2y$, et inde per Prob. I. $3\dot{x}x^2=a^2\dot{y}$, poſito $\dot{x}=1$, ſubſtitue $\frac{3x^2}{a^2}$ pro $\dot{y}$, et emerget $45x^4=a^4$. Cape ergo AB$=\sqrt[4]{\frac{a^4}{45}}$, et BD normaliter erecta occurret Curvæ in puncto Maximæ Curvaturæ. Vel quod perinde eſt fac AB : BD :: $3\sqrt{5}:1$.

10. Ad eundem modum Hyperbola ſecundi generis, per æquationem $xy^2=a^3$ deſignata, maximè flectitur in punctis D, d, quæ determinabis ſumendo AQ$=1$ in baſi, et erigendo QP$=\sqrt{5}$, eique æqualem Qp ex alterâ parte, et agendo AP et Ap quæ Curvæ occurrent in deſideratis punctis D ac d.



V. Locum centri Curvaminis determinare, ſive Curvam deſcribere in quâ centrum iſtud perpetuò verſatur.

11. Trochoidis centrum Curvaminis in aliâ Trochoide verſari oſtenſum eſt.

12. Et ſic Parabolæ centrum iſtud in aliâ ſecundi generis, quam æquatio $axx=y^3$ definit, Parabolâ verſatur, ut inito calculo facilè conſtabit.

VI. Luce in quamlibet Curvam incidente, invenire Focum, ſive concurſum radiorum, circa quodpiam ejus punctum refractorum.

13. Curvaturam ad iſtud Curvæ punctum quære, et centro radioque curvaturæ Circulum deſcribe; dein quære concurſum radiorum à circulo circa iſtud punctum refractorum. Nam idem erit concurſus refractorum à propoſitâ Curvâ.

CAPUT VIII. VII. His addi poteſt particularis inventio Curvaturæ ad vertices Curvarum ubi normaliter ſecant baſes. Nempe punctum in quo curvæ perpendiculum cum baſi conveniens, ipſam ultimò ſecuerit, eſt centrum Curvaturæ ejus.

14. Quamobrem habitâ relatione inter baſin x et rectangulam applicatam y, et inde (per Prob. 1.) relatione inter fluxiones $\dot{x}$ et $\dot{y}$, valor $\dot{y}y$, ſi in eo ſcribas 1 pro $\dot{x}$, et fingas $y=0$, erit radius curvaturæ.

15. Sic in Ellipſi $ax - \frac{a}{b}xx = yy$, eſt $\frac{a\dot{x}}{2} - \frac{ax\dot{x}}{b} = \dot{y}y$; qui valor $\dot{y}y$, ſi ſupponas $y=0$, et conſequenter $x=0$, et ſcribas 1 pro $\dot{x}$, evadet $\frac{1}{2}a$ radius curvaturæ. Et ſic ad vertices Hyperbolæ et Parabolæ radius curvaturæ erit etiam dimidium lateris recti.

16. Atque ita ad Conchoiden, æquatione $\frac{b^2c^2}{xx} + \frac{2bcc}{x} + cc - 2bx - bb - xx = yy$ definitam, valor $\dot{y}y$ ope Prob. 1. invenietur $-\frac{b^2c^2}{x^3} - \frac{bc^2}{x^2} - b - x$. Qui, ſupponendo $y=0$, et inde $x=c$, vel $-c$, evadet $-\frac{bb}{c} - 2b - c$, vel $\frac{bb}{c} - 2b + c$, radius curvaturæ. Fac ergo AE : EG :: EG : EC (vid. fig. p. 432) et A*e* : *e*G :: *e*G : *ec*, et habes curvaturæ centra, C et *c*, ad vertices conjugatarum Conchoidum, E et *e*.

PROB. VI.

Curvaturæ ad datum Curvæ alicujus punctum Qualitatem determinare.

17. Per *Qualitatem* curvaturæ intelligo formam ejus, quatenus eſt plus vel minùs inæquabilis; ſive quatenus plus vel minùs variatur in proceſſu per diverſas partes curvæ.

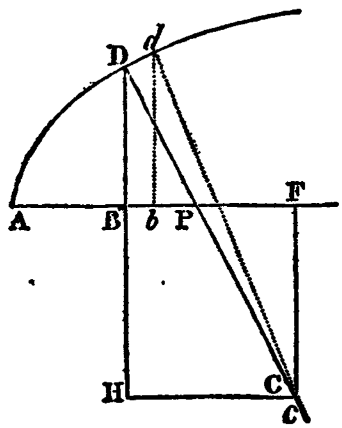
18. Sic interroganti qualis ſit Circuli curvatura, reſponderi poteſt, quòd ſit uniformis ſive invariata; et interroganti qualis ſit curvatura Spiralis, (dd) quæ deſcribitur per motum puncti D, cum acceleratâ celeritate, AD, in rectâ AK, uniformiter circa centrum A gyrante, progredientis ab A, adeo ut recta AD ad arcum BK, dato puncto, K, deſcriptum, rationem habeat numeri ad logarithmum ejus, reſponderi poteſt, quòd ſit uniformiter varia-

(dd) Vide fig. p. 430.

ta;

ta; ſive quòd ſit æquabiliter inæquabilis. Et ſic aliæ Curvæ, in ſingulis earum punctis, *aliquales* pro Curvaturæ variatione denominari poſſunt. De Qualitate Curvaturæ.

19. Quæritur itaque Curvaturæ, circa aliquod Curvæ punctum, inæquabilitas ſive variatio. Quâ de causâ animadvertendum eſt, I. Quòd ad puncta in ſimilibus curvis ſimiliter poſita ſimilis eſt inæquabilitas, ſive variatio curvaturæ. II. Et quòd momenta radiorum curvaturæ ad illa puncta ſunt proportionalia contemporaneis momentis curvarum, et fluxiones fluxionibus. III. Atque adeo quòd ubi fluxiones illæ non ſunt proportionales, diſſimilis erit inæquabilitas curvaturæ. Utpote major erit inæquabilitas, ubi major eſt ratio fluxionis radii curvaturæ ad fluxionem curvæ. Adeoque fluxionum ratio illa non immeritò dici poteſt *Index inæquabilitatis ſive variationis curvaturæ.*



20. Ad Curvæ alicujus, AD, puncta D ac d infinitè parùm diſtantia, ſunto radii curvaturæ DC, ac, dc; et exiſtente Dd momento curvæ erit Cc contemporaneum momentum radii curvaturæ, et $\frac{Cc}{Dd}$ Index inæquabilitatis curvaturæ: nempe tanta dicetur inæquabilitas illa, quantam eſſe indicat rationis illius $\frac{Cc}{Dd}$ quantitas; ſive curvatura dicetur tanto diſſimilior curvaturæ circuli.

21. Demiſſis jam ad quamlibet, AB, occurrentem DC in P, rectangulis applicatis DB ac db, dic AB$=x$; BD$=y$; DP$=t$; DC$=v$; et inde B$b=\dot{x}\times o$. Eritque C$c=\dot{v}\times o$. Et BD : DP = Bb : D$d = \frac{t\dot{x}o}{y}$, ac $\frac{Cc}{Dd}=\frac{\dot{v}y}{t\dot{x}}$, ſive $=\frac{\dot{v}y}{t}$ ſuppoſito $\dot{x}=1$. Quamobrem relatione inter x et y per quamlibet æquationem definitâ, et inde, juxta Prob. IV. et V. invento perpendiculo DP ſive t, et radio curvaturæ v, ejuſque radii fluxione $\dot{v}$ per Prob. I. dabitur Index inæquabilitatis curvaturæ $\frac{\dot{v}y}{t}$.

22. Exempl. I. Sit $2ax=yy$ æquatio ad Parabolam. Et per Prob. IV. erit BP$=a$, adeoque DP $=\sqrt{aa+yy}=t$. Item per Prob. V. (ee) HC $=a+2x$. Et PB : DP = HC : DC $=\frac{at+2tx}{a}=v$. Jam æ-

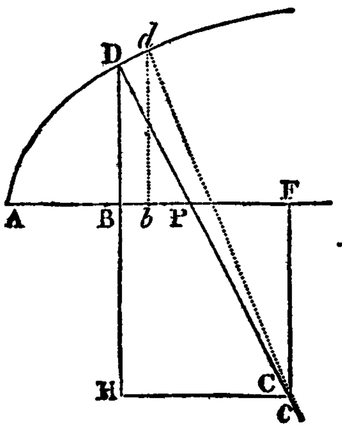
(ee) Cap. VII. § 12.

 quationes

Caput VIII. quationes $2ax=yy$, et $aa+yy=tt$, et $\frac{at+2tx}{a}=v$, per Prob. I. dant $2a\dot{x}=2\dot{y}y=2\dot{t}t$: et $\frac{a\dot{t}+2\dot{t}x+2t\dot{x}}{a}=\dot{v}$. Quibus ordinatis, et poſito $\dot{x}=1$, orientur $\dot{y}=\frac{a}{y}$; $\dot{t}=\frac{y\dot{y}}{t}=\frac{a}{t}$. Et $\dot{v}=\frac{a\dot{t}+2\dot{t}x+2t}{a}$. Et ſic inventis $\dot{y}$, $\dot{t}$, et $\dot{v}$, habebitur $\frac{\dot{v}y}{t}$ Index inæquabilitatis curvaturæ.

23. Quemadmodum ſi in numeris definiatur $a=1$, ſive $2x=yy$, et $x=\frac{1}{2}$, erit $y=\sqrt{2x}=1$; $\dot{y}=\frac{a}{y}=1$; $t=\sqrt{aa+yy}=\sqrt{2}$; $\dot{t}=\frac{a}{t}\sqrt{\frac{1}{2}}$; et $\dot{v}=\frac{a\dot{t}+2x\dot{t}+2t}{a}=3\sqrt{2}$ (ff). Adeoque $\frac{\dot{v}y}{t}=3$ indici inæquabilitis.

24. Sin autem definiatur $x=2$, erit $y=2$; $\dot{y}=\frac{1}{2}$; $t=\sqrt{5}$; $\dot{t}=\sqrt{\frac{1}{5}}$; et $\dot{v}=3\sqrt{5}$. Adeoque $\frac{\dot{v}y}{t}=6$ Index inæquabilitatis. Quamobrem inæquabilitas curvaturæ, ad punctum à quo ad axem demiſſa ordinatim applicata æquatur lateri recto Parabolæ, dupla eſt ejus ad punctum, à quo demiſſa ordinatim applicata æquatur dimidio ejuſdem lateris recti: hoc eſt, curvatura in priori caſu duplo diſſimilior eſt curvaturæ Circuli, quàm [illegible] poſteriori.



25. Exempl. 2. Sit $2ax-bxx=yy$, et, per Prob. IV. erit $a-bx=$ BP, et inde $aa-2abx+bbxx+yy=tt$: ſive $aa-byy+yy=tt$. Item, per Prob. V. erit DH $=y+\frac{y^3-by^3}{aa}$ (gg). Ubi ſi pro $yy-byy$ ſubſtituas $tt-aa$, evadet DH $=\frac{tty}{aa}$. Et eſt BD : DP = DH : DC $=\frac{t^3}{aa}=\dot{v}$. Jam, per Prob. I. æquationes $2ax-bxx=yy$; et $aa-byy+yy=tt$; et $\frac{t^3}{aa}=v$; dant $a-bx=y\dot{y}$ (hh); et $y\dot{y}-by\dot{y}=t\dot{t}$; et $\frac{3t^2\dot{t}}{aa}=\dot{v}$. Et ſic invento $\dot{v}$ dabitur $\frac{\dot{v}y}{t}$ Index inæquabilitatis curvaturæ.

26. Sic ad Ellipſin $2x-3xx=yy$, ubi eſt $a=1$, et $b=3$, ſi ſupponatur $x=\frac{1}{2}$, erit $y=\frac{1}{2}$; $\dot{y}=-1$; $t=\sqrt{\frac{1}{2}}$; $\dot{t}=\sqrt{2}$; $\dot{v}=3\sqrt{\frac{1}{2}}$ et $\frac{\dot{v}y}{t}=\frac{3}{2}$ Indici inæquabilitatis curvaturæ. Unde patet curvaturam hujus Ellipſis, ad hîc definitum punctum D, eſſe duplo minùs inæ-

(ff) Nimirum $\frac{a\dot{t}+2x\dot{t}+2t}{a}=t+\dot{t}+2t=2\times\overline{\dot{t}+t}=2\sqrt{\frac{1}{2}}+\sqrt{2}=2\frac{\sqrt{1}+\sqrt{4}}{\sqrt{2}}=2\times\frac{3}{\sqrt{2}}=\sqrt{2}\times 3$.

quabilem

quabilem (ſive duplo ſimiliorem curvaturæ Circuli) quàm curvatura Parabolæ, ad illud ejus punctum, à quo ad axem demiſſa ordinatim applicata æquatur dimidio ejus lateris recti.

De Qualitate Curvaturæ.

27. Si concluſiones in his exemplis concinnare patet, ad Parabolam $2ax=yy$, exibit $\frac{\dot{r}y}{t}=\frac{3y}{a}$ Index inæquabilitatis. Et ad Ellipſin, $2ax-bxx=yy$, exibit Index $\frac{\dot{r}y}{t}=\frac{3y-3by}{aa}\times$ BP. Et ſic ad Hyperbolam $2ax+bxx=yy$, obſervatâ analogiâ, erit index $\frac{\dot{r}y}{t}=\frac{3y+3by}{aa}\times$ BP. Unde patet, quòd ad diverſa puncta cujuſvis conicæ ſectionis, ſeorſim ſpectatæ, curvaminis inæquabilitas eſt ut rectangulum BD $\times$ BP. Et quòd ad diverſa puncta Parabolæ eſt ut ordinatim applicata BD.

28. Cæterùm cùm Parabola ſit ſimpliciſſima linearum inæquabili curvaturâ flexarum, ejuſque curvaturæ inæquabilitas tam levi negotio determinatur, utpote cujus index ſit $\frac{6\times\text{ordin. applicat.}}{\text{lat. rect.}}$; aliarum Curvarum curvaturæ ad curvaturam hujus non incommodè referri poſſunt. Quemadmodum ſi quæratur qualis ſit Ellipſeos $2x-3xx=yy$ curvatura, ad illud ejus punctum, quod definitur aſſumendo $x=\frac{1}{2}$: quoniam index ejus (ut ſuprà) ſit $\frac{3}{2}$, reſponderi poteſt, eſſe ſimilem curvaturæ Parabolæ $6x=yy$, ad illud ejus punctum, inter quod et axem recta $=\frac{1}{2}$ ordinatim applicatur.

29. Sic cùm lineæ Spiralis ADE jam ante deſcriptæ (ii) fluxio ſit ad fluxionem ſubtenſæ AD in datâ quâdam ratione, puta d ad e: verſus partes concavas ejus erige ad AD normalem AP $=\frac{e}{\sqrt{dd-ee}}\times$ AD, et erit P centrum curvaturæ, et $\frac{\text{AP}}{\text{AD}}$, ſive $\frac{\sqrt{dd-ee}}{e}$, Index inæquabilitatis ejus. Quare Spiralis hæcce curvaturam habet ubique ſimiliter inæquabilem, ac Parabola $6x=yy$ habet in illo ejus puncto, à quo demittitur ad axem ordinatim applicata $=\frac{\sqrt{dd-ee}}{e}$.

30. Et ſic Index inæquabilitatis ad quodvis Trochoidis punctum D (kk) invenietur eſſe $\frac{\text{AB}}{\text{BL}}$. Quare curvatura ejus ad idem D tam inæquabilis eſt, ſive tam diſſimilis curvaturæ Circuli, quàm

(gg) Cap. VII. § 12.
(hh) Nimirum poſito $\dot{x}=1$.
(ii) Vide fig. p. 440.
(kk) Vide fig. p. 449.

curvatura

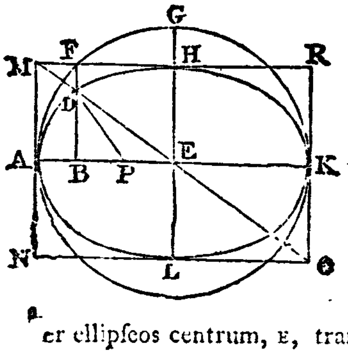
CAPUT VIII. curvatura Parabolæ cujuſvis $ax=yy$, ad illud ejus punctum ubi ordinatim applicata æquatur $\frac{1}{6}a \times \frac{AB}{BL}$.

31. Ex his credo ſenſus Problematis ſatis eluceſcet; quo benè perſpecto, non difficile erit animadvertenti ſeriem rerum ſuprà traditarum plura exempla de proprio ſuppeditare, et hujuſmodi complures alias operandi methodos, prout res exiget, concinnare. Quinetiam cognata Problemata ubi, perplexâ computatione non conteritur et fatigatur, haud majori difficultate tranſiget: Cujuſmodi ſunt,

I. Invenire punctum Curvæ alicujus, ubi vel Nullam, vel Infinitam, vel Maximam, vel Minimam, vel datam quamvis habeat Inæquabilitatem curvaturæ. Sic ad vertices conicarum ſectionum nulla eſt inæquabilitas curvaturæ; ad cuſpidem Trochoidis infinita eſt; et ad puncta Ellipſeos maxima eſt, ubi rectangulum BD × BP ſit maximum; hoc eſt, ubi lineæ diagonales rectanguli parallelogrammi circumſcripti Ellipſin ſecant, cujus latera tangunt illam in principalibus verticibus (ll).

II. Curvam alicujus definitæ ſpeciei, puta conicam ſectionem, determinare, cujus Curvatura, ad aliquod punctum, et æqualis ſit et ſimilis curvaturæ alterius alicujus Curvæ ad datum punctum ejus.

III. Conicam ſectionem determinare, ad cujus punctum aliquod Curvatura et lineæ tangentis, reſpectu axis, poſitio ſit ſimilis Curvaturæ ac Tangentis poſitioni alterius alicujus Curvæ, ad aſſignatum punctum ejus. Et hujus Problematis uſus eſt, ut vice Ellipſium ſecundi generis, quarum refringendi proprietates Carteſius



(ll) ELLIPSIS ADL, cujus axes, tranſverſus quidem AK, ſecundus HL, rectangulo MNOR circumſcribatur, cujus utique latera cum axibus ellipſeos erunt parallela. Sit D punctum Ellipſeos, à quo ſi deducatur DB in axem AK ordinatim, aliaque DP ad perpendiculum cum rectâ quæ curvam in D contingat, rectangulum DB × BP maximum ſit. Dico punctum D eſſe ad alteram è diagoniis parallelogrammi MO. Sit enim E centrum ellipſeos. Centro E radio EA ſcribatur circulus, cui axis ellipſeos EH in G occurrat, rectaque BD in F. Sit M angulus ille parallelogrammi circumſcripti, quem quadrans ellipſeos, AH, in quo poſitum eſt punctum D, convexus ſpectat. Jungaturque EM. Quoniam diagoniæ parallelogrammi MNOR utrinque ellipſeos centrum, E, tranſeunt, erit utique EM altera è diagoniis illis. Dico punctum D eſſe

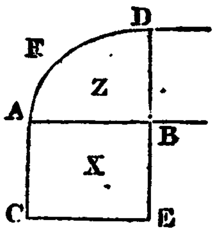
Carteſius in Geometriâ demonſtravit, conicæ ſectiones, idem in refractionibus quàm proximè præſtantes, ſubrogari poſſint. Atque idem de aliis Curvis intellige. DE QUALITATE CURVATURÆ.

CAPUT IX.

PROB. VII.

1. *Curvas pro arbitrio multas invenire, quarum Areæ per finitas Æquationes deſignari poſſunt.*

SIT AB baſis Curvæ, ad cujus initium A erigatur normalis AC = 1, et agatur CE parallela AB, ſit etiam BD rectangula applicata concurrens rectæ CE in E, et Curvæ AD in D. Et concipe has areas ACEB et ADB à rectis BE et BD, per AB delatis, generari. Et earum incrementa, ſive fluxiones, perpetim erunt ut lineæ deſcribentes BE et BD (mm). Quare parallelogrammum ACEB ſive AB × 1, dic x, et Curvæ aream, ADC dic z: et fluxiones $\dot{x}$ et $\dot{z}$ erunt ut BE et BD, adeoque poſito $\dot{x} = 1 = $ BE, erit $\dot{z} = $ BD.



Si jam ad arbitrium aſſumatur æquatio quævis, pro definiendâ relatione z ad x, exinde per Prob. 1. elicietur $\dot{z}$. Atque ita duæ habebuntur æquationes, quarum poſterior Curvam definiet, et prior Aream ejus.

3. EXEMPL. Aſſumatur $xx = z$ et inde per Prob. 1. elicietur $2\dot{x}x = \dot{z}$, ſive $2x = \dot{z}$ ſiquidem eſt $\dot{x} = 1$.

Aſſumatur $\frac{x^3}{a} = z$, et inde prodibit $\frac{3x^2}{a} = \dot{z}$, æquatio ad Parabolam.

eſſe ad rectam EM. Namque rectæ BP ad BE ratio data eſt. (Hamilton. Conic. Lib. 2. Prop. XXIV.) Quare rectanguli BD × BP ad rectangulum BD × BE ratio data. Et propter rationem rectæ BD ad BF datam, rectanguli EB × BD ad rectangulum EB × BF ratio data. Rectangula igitur DB × BP, FB × BE, quorum utrumque ad tertium DB × BE rationem datam habet, hæc datam inter ſe rationem gerent. (Euclid. dat. 8.) Maximo igitur exiſtente illo DB × BP, alterum FB × BE maximum etiam erit. Maximo autem exiſtente rectangulo FB × BE, erunt EB, BF inter ſe æquales. Id enim vulgo notum eſt. Sed æqualium EB, BF ad tertiam BD eadem erit ratio. (El. V. 7.) Et è naturis ellipſeos circulique, quibus axis AK eſt communis, erit FB ad BD, ut EG vel EA ad EH vel AM. Erit igitur EB ad BD ut EA ad AM. (El. V. 11.) Punctum igitur D erit ad rectam LM. (Elem. VI. 32.) Q. E. D.

(mm) Hoc eſt incrementorum ſimul naſcentium inter ipſa prima ratio, rectarum BE, BD ratio erit. Quæ proinde, *perpetim*, id eſt, in omni rectæ BD ſitu, fluxionum inter ipſas ratio erit.

Aſſumatur.

CAPUT IX. Aſſumatur $ax^3=z^2$, ſive $a^{\frac{1}{2}}x^{\frac{3}{2}}=z$; et emerget $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}=\dot{z}$, ſive $\frac{9}{4}ax=\dot{z}\dot{z}$, æquatio iterum ad Parabolam.

Aſſumatur præterea $a^3x=z^2$, ſive $a^{\frac{3}{2}}x^{\frac{1}{2}}=z$; et elicietur $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}=\dot{z}$, ſive $a^3=4x\dot{z}\dot{z}$.

Item aſſumatur $\frac{a^3}{x}=z$, ſive $a^3x^{-1}=z$, et elicietur $-a^3x^{-2}=\dot{z}$ ſive $a^3+\dot{z}xx=0$: ubi negativus valor ipſius $\dot{z}$ tantùm denotat BD capiendam eſſe ad partes contra BE [nn].

Ad hæc ſi aſſumas $c^2a^2+c^2x^2=z^2$ elicies $2c^2x=2\dot{z}z$; et exterminato z, proveniet $\frac{cx}{\sqrt{a^2+x^2}}=\dot{z}$.

Vel ſi aſſumas $\frac{a^2+x^2}{b}\sqrt{a^2+x^2}=z$, dic $\sqrt{a^2+x^2}=v$; et erit $\frac{v^3}{b}=z$, et inde (per Prob. I.) $\frac{3\dot{v}v^2}{b}=\dot{z}$. Item æquatio $a^2+x^2=v^2$ per Prob. I. dat $2x=2v\dot{v}$, cujus ope ſi extermines $\dot{v}$, fiet $\frac{3vx}{b}=\dot{z}=\frac{3x}{b}\sqrt{a^2+x^2}$.

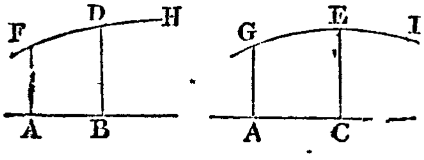
Si denique aſſumas $8-3xz+\frac{2}{5}z=z^2$, elicies $-3z-3x\dot{z}+\frac{2}{5}\dot{z}=2\dot{z}z$. Quare per aſſumptam æquationem imprimis quære Aream z; ac deinde Applicatam $\dot{z}$, per elicitam.

Atque ex Areis, qualeſcunque effingas, ſemper poſſis Applicatas determinare.

PROB. VIII.

3. *Curvas pro arbitrio multas invenire, quarum Areæ ad Aream datæ alicujus Curvæ relationem habent per finitas æquationes deſignabilem.*

Sit FDH data Curva, ac GEI quæſita, et earum Applicatas, DB et EC, concipe ſuper baſibus, AB et AC, erectas incedere.



Et Arearum, quas ita tranſigunt, incrementa [oo], ſive fluxiones, erunt ut Applicatæ illæ ductæ in earum velocitates incedendi; hoc eſt, in fluxiones baſium. Sit ergo AB$=x$, BD$=v$, AC$=z$, ac CE$=y$. Area AFDB$=s$, et area AGEC$=t$; eritque $\dot{x}v:\dot{z}y::\dot{s}:\dot{t}$. Quare ſi ſupponatur $\dot{x}=1$ et $v=\dot{s}$, ut ſuprà, erit $\dot{z}y=\dot{t}$, et inde $\frac{\dot{t}}{\dot{z}}=y$.

[nn] Immo, eam potius ſignificat Curvæ ſpeciem, cujus area, z, contigua eſt baſi, AB, ultra ordinatam productæ.

[oo] Naſcentia utique.

Aſſumantur itaque duæ quævis æquationes, quarum una defi- niat relationem Arearum s ac t, et altera relationem Baſium x et z, et inde (per Prob. 1.) quærantur fluxiones $\dot{t}$ et $\dot{z}$, et ſubſtituatur $\frac{\dot{t}}{\dot{z}}=y$. De Curvarum Comparatione.

4. Exempl. 1. Data Curva AFD ſit Circulus, æquatione $ax-x^2=vv$, deſignatus; et quærantur aliæ Curvæ, quarum areæ adæquant aream ejus. Ex hypotheſi ergo eſt $s=t$, et inde $\dot{s}=\dot{t}=v$. Et $y=(\frac{\dot{t}}{\dot{z}})=\frac{v}{\dot{z}}$. Supereſt ut $\dot{z}$ determinetur, aſſumendo relationem aliquam inter baſes x et z.

Veluti ſi fingas $ax=zz$, erit per Prob. 1. $a=2\dot{z}z$: quare ſubſtitue $\frac{a}{2z}$ pro $\dot{z}$, et fiet $y=(\frac{v}{\dot{z}})=\frac{2vz}{a}$. Eſt autem $v=(\sqrt{ax-xx}=)$ $\frac{z}{a}\sqrt{aa-zz}$, adeoque $\frac{2zz}{aa}\sqrt{aa-zz}=y$, æquatio ad Curvam cujus area æquatur areæ Circuli.

Ad eundem modum ſi fingas $x^2=z$, proveniet $2x=\dot{z}$, et inde $y=(\frac{v}{\dot{z}})=\frac{v}{2x}$; et exterminato v et x, fiet $y=\frac{\sqrt{az^{\frac{1}{2}}-z}}{2z^{\frac{1}{2}}}$.

Vel ſi fingas $cc=xz$, proveniet $0=z+x\dot{z}$, et inde $-\frac{vx}{z}=y=$ $-\frac{c^3}{z^3}\sqrt{az-cc}$.

Atque ita ſi fingas $ax+\frac{s}{1}=z$, ope Prob. 1. obtinebitur $a+\dot{s}=\dot{z}$ et inde $\frac{v}{a+\dot{s}}=y=\frac{v}{a+v}$, quæ Curvam Mechanicam deſignat.

5. Exempl. 2. Detur iterum Circulus $ax-x^2=vv$, et quærantur Curvæ, quarum areæ ad aream ejus habeant aliam quamlibet aſſumptam relationem. Veluti ſi aſſumas $cx+s=t$, et præterea fingas $ax=zz$, mediante Prob. 1. elicies $c+\dot{s}=\dot{t}$, et $a=2\dot{z}z$. Quare eſt $y=(\frac{\dot{t}}{\dot{z}})=\frac{2cz+2\dot{s}z}{a}$; et ſubſtituto $\sqrt{ax-xx}$ pro $\dot{s}$, et $\frac{z^2}{a}$ pro x, fit $y=\frac{2cz}{a}+\frac{2z^2}{a^2}\times\sqrt{a^2-z^2}$.

Quòd ſi aſſumas $s-\frac{2v^3}{3a}=t$ et $x=z$, invenies ope Prob. 1. $\dot{s}-\frac{2\dot{v}v^2}{a}=\dot{t}$, et $1=\dot{z}$. Adeoque $y=(\frac{\dot{t}}{\dot{z}})=\dot{s}-\frac{2\dot{v}v^2}{a}$, ſive $=v-\frac{2\dot{v}v^2}{a}$. Jam verò pro exterminando $\dot{v}$, æquatio $ax-x^2=vv$, per Prob. 1. dat $a-2x=2\dot{v}v$; et proinde eſt $y=\frac{2vx}{a}$, ubi ſi ſupprimas v et x, ſubſtituendo valores $\sqrt{ax-x^2}$, et z, emerget $y=\frac{2z}{a}\sqrt{az-zz}$.

CAPUT IX.

Sin aſſumas $ss=t$, et $x=z^2$, emerget $2\dot{s}=\dot{t}$ et $1=2\dot{z}z$, atque adeo $y=(\frac{\dot{t}}{\dot{z}})=4\dot{s}sz$; et pro $\dot{s}$ et x ſubſtitutis $\sqrt{ax-x^2}$ et zz, fiat $y=4sz^2\sqrt{a-zz}$, æquatio ad Curvam Mechanicam.

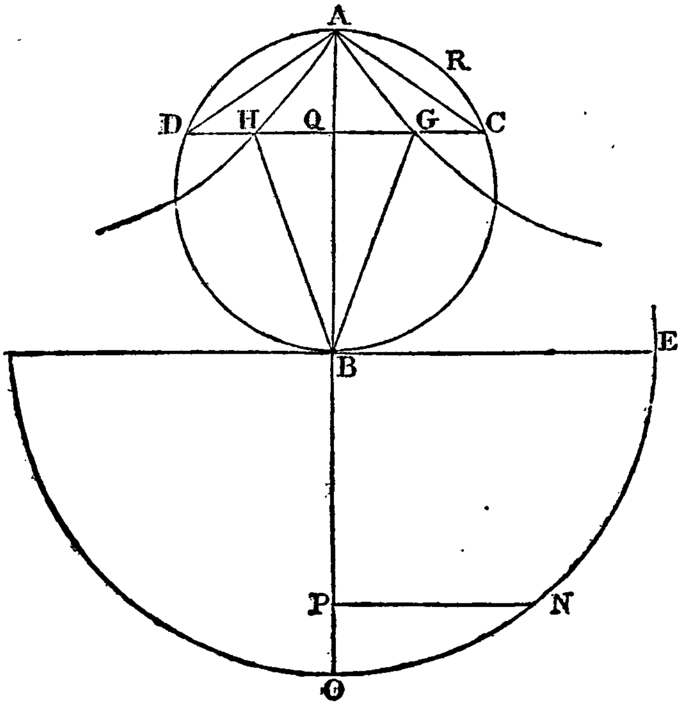
6. EXEMPL. 3. Ad eundem modum figuræ, aſſumptam relationem ad aliam quamvis datam figuram habentes, inveniuntur. Sic datâ Hyperbolâ $cc+xx=vv$, ſi aſſumas $s=t$, et $xx=cz$, elicies per Prob. 1. $\dot{s}=\dot{t}$, et $2x=c\dot{z}$, et inde $y=(\frac{\dot{t}}{\dot{z}})=\frac{c\dot{s}}{2x}$, et ſubſtitutis $\sqrt{cc+x^2}$ pro $\dot{s}$, et $c^{\frac{1}{2}}z^{\frac{1}{2}}$ pro x proveniet $y=\frac{c}{2z}\sqrt{cz+zz}$.

Atque ita ſi aſſumas $xv-s=t$, et $xx=cz$, elicies $v+\dot{v}x-\dot{s}=\dot{t}$, et $2x=c\dot{z}$. Eſt autem $v=\dot{s}$, et inde $\dot{v}x=\dot{t}$; quare $y=(\frac{\dot{t}}{\dot{z}})=\frac{c\dot{v}}{2}$. Jam verò $cc+xx=vv$, ope Prob. 1. dat $x=\dot{v}v$. Adeoque eſt $y=\frac{cx}{2v}$, et ſubſtitutis $\sqrt{cc+xx}$ pro v, et $c^{\frac{1}{2}}z^{\frac{1}{2}}$ pro x, fit $y=\frac{cz}{2\sqrt{cz+zz}}$.

7. Ex-

COMPARATIO

(pp) Nempe poſito $\frac{x}{3}\sqrt{ax-x^2}=b$, fiet $\frac{ax^3-x^4}{9}=bb$. Et ex hâc æquatione, ope Prob. 1. elicietur fluxionum æquatio, $\frac{3ax^2-4x^3}{9}\dot{x}=2b\dot{b}$. Hoc eſt, poſito $\dot{x}=1$, $\frac{3ax^2-4x^3}{9}=2b\dot{b}$.



(qq) EA eſt hujuſmodi. Sit Circulus ACBD ad quem pertineat Ciſſoides GAH, cujus axis ſit AB circuli diameter, Aſymptota recta BE, quæ circulum ACBD in puncto B contingit. A puncto quovis Q, in axe Ciſſoidis, AB, pro arbitrio ſumendo, eductam puta ad perpendiculum rectam DC; quæ Circulo in punctis, D, C, Ciſſoidi, in illis, H, G, occurrat. Jungantur BH, BG. Centro BA, radio BE, quæ rectæ AB ſit æqualis, ſcribatur circuli quadrans ENO. Jungatur BC. Acceptâque BP æquali rectæ BC, à puncto P ad perpendiculum educatur PN, quæ arcui quadrantis in N occurrat. Sector ciſſoidis AHBG ſpatii circularis OPN triplus erit.

Nam ſi AB dicatur a, AQ, x; erit QC vel QD recta, quæ ſymbolo $\sqrt{ax-xx}$ deſignata eſt (propter circulum). Et QG vel QH erit recta quæ ſymbolo $\frac{xx}{\sqrt{ax-xx}}$ deſignatur

7. EXEMPL. 4. Adhæc ſi detur Ciſſoides, $\frac{xx}{\sqrt{ax-x^2}}=v$, ad quam relatæ aliæ figuræ ſunt inveniendæ, et eâ de cauſâ, aſſumatur $\frac{x}{3}\sqrt{ax-x^2}+\frac{2}{3}s=t$; finge $\frac{x}{3}\sqrt{ax-x^2}=b$, et erit $b+\frac{2}{3}s=t$, et inde per Prob. 1. $\dot{b}+\frac{2}{3}\dot{s}=\dot{t}$. Æquatio autem $\frac{ax^3-x^4}{9}=bb$, per Prob. 1. dat $\frac{3ax^2-4x^3}{9}=2\dot{b}b$ (PP); ubi ſi extermines b, fiet $\dot{b}=\frac{3ax-4xx}{6\sqrt{ax-xx}}$. Quare cùm præterea ſit $\frac{2}{3}\dot{s}=(\frac{2}{3}v)=\frac{4xx}{6\sqrt{ax-xx}}$, erit $\frac{av}{2\sqrt{ax-xx}}=\dot{t}$. Porro ad determinandum z et $\dot{z}$, aſſumatur $\sqrt{aa-ax}=z$, et ope Prob. 1. emerget $-a\dot{x}=2\dot{z}z$, ſive $\dot{z}=\frac{-a\dot{x}}{2z}$; quare eſt $y=(\frac{\dot{t}}{\dot{z}})=\frac{-zx}{\sqrt{ax-x^2}}=\sqrt{\frac{zzx}{a-x}}=\sqrt{ax})=\sqrt{aa-zz}$. Quæ æquatio cùm ſit ad Circulum, habebitur ratio arearum Circuli et Ciſſoidis (qq).

DE CURVARUM COMPARATIONE.

Atque ita ſi aſſumpſiſſes $\frac{2x}{3}\sqrt{ax-x^2}+\frac{1}{3}s=t$, & $x=z$, prodiiſſet $y=\sqrt{az-zz}$. Æquatio denuò ad Circulum.

8. Haud ſecus ſi detur Curva aliqua Mechanica, poſſunt aliæ ad eam relatæ Curvæ Mechanicæ inveniri; ſed ad eliciendum Geometricas convenit, ut è rectis, ab invicem geometricè dependentibus, aliqua pro baſi adhibeatur; et ut Area ad parallelogram-

natur (propter ciſſoidem). Erit igitur rectangulum AQ × QC, ſive illi æquale triangulum DAC, ſpatium ſymbolo $x\sqrt{ax-xx}$ ſignificatum; et ſpatium Ciſſoidis, AHG, illud erit quod ſymbolum $2s$ ſignificat. Præterea, propter circulum ACB, recta BC, vel illi æqualis BP, ea erit quam litera z deſignat. Erit igitur PN $=y$, & ſpatium OPN $=t$. Poſito enim $x=0$, hoc eſt puncto Q in A tranſlato, ut ſpatium $\frac{x}{3}\sqrt{ax-x^2}+\frac{2}{3}s$ in nihilum abeat, recta BC, vel BP, ipſi BA, vel BO æqualis evaſerit, et area OPN in nihilum abierit. Unde areæ illæ, $\frac{x}{3}\sqrt{ax-x^2}+\frac{2}{3}s$, et OPN, cùm ſimul à nihilo generari incipiant, et æqualibus fluxionibus uſque creſcant, erunt ſemper inter ſe æquales. Quare OPN $=t$.

CISSOIDIS CIRCULIQUE.

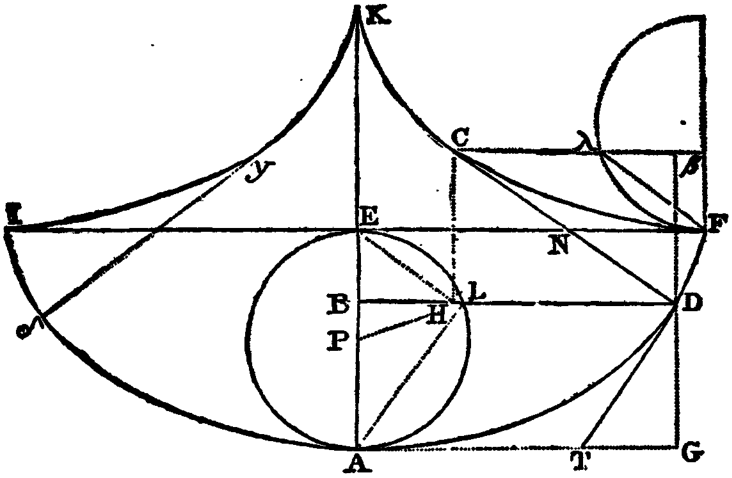
Jam propter æquationem à Newtono poſitam, $\frac{x}{3}\sqrt{ax-xx}+\frac{2}{3}s=t$, erit $x\sqrt{ax-xx}+2s=3t$. Hoc eſt triangulum DAG, ſpatio HAG auctum triplum erit ſpatii OPN. Sed propter Ciſſoidem, BQ : QA = CQ : QG = DC : HG. Triangula igitur DAC, BHG inter ſe ſunt æqualia. Triangulum igitur DAC, ſpatio HAG auctum, æquale erit triangulo BHG acceſſione ejuſdem ſpatii HAG aucto; hoc eſt, ſectori Ciſſoidis BHAG. Quare ſector ciſſoidis BHAG areæ circularis OPN triplus erit. Et hæc eſt relatio ciſſoidis circulique, quam calculi auctoris indicant. Quæ quidem longè elegantiùs rationibus geometricis oſtendi poſſet.

2. Poſito autem $\frac{2x}{3}\sqrt{ax-x^2}+\frac{1}{3}s=t$, & $x=z$; efficietur $y=\sqrt{az-zz}$. Sive eam eſſe Ciſſoidis Circulique relationem, ut ſpatium ciſſoidis, ABG, ſegmenti ARC triplum ſit.

Atque hinc efficitur ſpatium illud omne, quod aſymptotæ BE, et brachiis ciſſoidis, AG, AH, infinitè protenſis adjacet, circuli ACBD, ad quem ciſſoides pertinet, triplum eſſe.

 mum

CAPUT IX. mum complementalis quæratur, ſupponendo fluxionem ejus valere baſin ductam in fluxionem ordinatim applicatæ.



9. EXEMPL. 5. Sic Trochoide ADF propoſitâ, refero ad baſin AB; et completo parallelogrammo, ABDG, quæro complementalem ſuperficiem ADG, concipiendo deſcriptam eſſe per motum rectæ GD; et proinde fluxionem ejus valere illam GD in celeritatem progrediendi ductam, hoc eſt, $x \times \dot{v}$. Jam cùm AL ſit parallela tangenti DT, erit AB ad BL ut fluxio ejuſdem AB ad fluxionem applicatæ BD, hoc eſt ut 1 ad $\dot{v}$ (rr). Quare eſt $\dot{v} = \frac{BL}{AB}$, adeoque $x\dot{v} = BL$. Et proinde area ADG deſcribitur fluxione BL: atque adeo cùm area circularis ABL eâdem fluxione deſcribatur æquales erunt.

Pari ratione ſi concipias ADF eſſe figuram arcuum ſive ſinuum verſorum, hoc eſt, cujus applicata BD æquatur arcui AL: cùm fluxio arcûs AL ſit ad fluxionem baſis AB, ut PL ad BL, hoc eſt $\dot{v} : 1 :: \frac{1}{2}a : \sqrt{ax - x^2}$ erit $\dot{v} = \frac{a}{2\sqrt{ax - x^2}}$. Adeoque $\dot{v}x$, fluxio areæ ADG, erit $\frac{ax}{2\sqrt{ax - x^2}}$. Quare ſi ad ipſius AB punctum B recta

(r) NIMIRUM $DG : GT = \dot{DG} : \dot{GA} = \dot{x} : \dot{v}$ (Introduct. ad Quad. Curv. § 4.) Sed $DG : GT = AB : BL$ (propter triangula ABL, DGT inter ſe ſimilia et æqualia). Quare $\dot{x} : \dot{v} = AB : BL$. Quare $BL \times \dot{x} = AB \times \dot{v} = DG \times \dot{v}$. Areæ igitur ADG, ABL, cûm ſimul à nihilo generari inceperint, inter ſe æquales erunt. (De Quadrat. Curv. Prop. IX.)

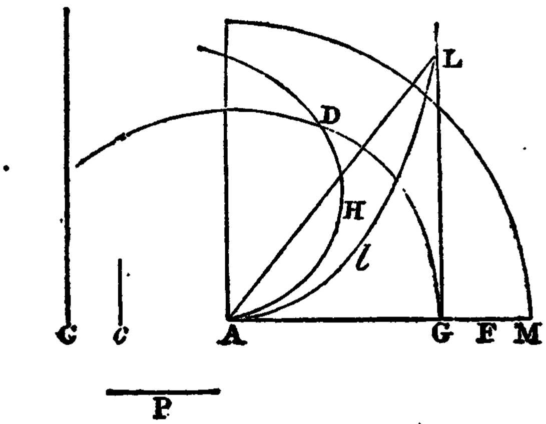
(rr) HÆC et alia hujuſmodi apertiora, credo, fuiſſent, ſi auctor notam $\dot{x}$ retinuiſſet. Illa enim unitatis ſuppoſitio, licet formulis Algebraicis aliquam inducat ſimplicitatis ſpeciem, perſpicuitati tamen ſignificationis Geometricæ adeo nihil confert, ut, me judice, vel potiùs officiat. Neque profecto nihil eſt, quod magnus ille olim Vieta præcipiebat; in formulis Algebraicis condendis, quoties illud ageretur, ut ex illis Geometricæ conſtructiones inferrentur, “*omnino ſervari oportere homogeneorum legem. Cui non attendiſſe cauſam fuiſſe* dicit *multæ caliginis & cæcutiei veterum analyſtarum.*” (Iſagog. Cap. III. De Lege Homogeneorum, & Cap. V. De Legibus Zeteticès.)

Nimirum

recta æqualis $\frac{ax}{2\sqrt{ax-x^2}}$, in angulo recto, applicari concipiatur, illa De Curvarum Comparatione.
ad Curvam quandam Geometricam terminabitur, cujus area, baſi AB adjacens, æquatur areæ ADG (ss).

10. Et ſic aliis figuris, per arcuum Circuli, Hyperbolæ vel cujuſvis Curvæ ad arcuum iſtorum ſinus rectos vel verſos aut alias quaſvis geometricè determinabiles rectas lineas, in datis angulis applicationem, conſtitutis, æquales Geometricæ Figuræ inveniri poſſunt.

11. Circa Spiralium areas leviſſimum eſt negotium. Utpote centro convolutionis A, radio quovis AG deſcripto arcu DG occurrente AF in G, et ſpirali in D: cùm arcus ille, ad inſtar lineæ ſuper baſi AG incedentis, deſcribit Spiralis aream AHDG, ita ut ejus areæ fluxio ſit ad fluxionem rectanguli $1 \times AG$ ut arc. GD ad 1; ſi rectam GL, arcui iſti æqualem, erigas, illa, ſimiliter incedendo ſuper eâdem AG, deſcribet aream ALG æqualem areæ ſpiralis AHDG; Curvâ A*l*L exiſtente Geometricâ. Et præterea, ſi ſubtenſa AL ducatur, triangulum ALG $(=\frac{1}{2}AG \times GL = \frac{1}{2}AG \times GD) =$ ſectori AGD (tt); adeoque complementalia ſegmenta AL*l* et ADH erunt etiam æqualia. Et hæc non tantùm Spirali Archimedeæ, ubi A*l*L evadit Parabola Apolloniana, (uu) ſed et aliis quibuſcunque conveniunt;



(ss) Nimirum cum ſit $\dot{v} : \dot{x} = \frac{1}{2}a : \sqrt{ax-x^2}$, erit $\dot{v} = \frac{a\dot{x}}{2\sqrt{ax-x^2}}$. Ergo $\dot{v}x$, ſive fluxio areæ ADG, æqualis erit huic $\frac{ax}{2\sqrt{ax-x^2}}\dot{x}$. Vel ſi pro $\frac{ax}{2\sqrt{ax-x^2}}$, hoc eſt, ſi pro rectâ quæ naſcitur applicando rectangulum ax ad duplam rectam quæ duarum, x, $a-x$, proportione media eſt, ſi pro rectâ, inquam, quæ naſcitur ex hâc applicatione, ſcribatur z, erunt rectanguli $\dot{v}x$, $z\dot{x}$ inter ſe æqualia. Si igitur in rectâ quâdam à rectæ AB puncto B, ad perpendiculum eductâ, capiatur longitudo illi z æqualis, terminabitur illa in Curvâ quâdam Geometricâ, cujus area, baſi AB appoſita, areæ ADG æqualis erit. (Geometr. Flux. Prop. II.) Nempe cùm areæ illæ ſimul à nihilo generari inceperint, & æquales ſemper fluxiones habeant.

(tt) Duci intelligatur recta AD quæ ſculptoris incuriâ omiſſa eſt.

(uu) Sit AM longitudo illa, quam recta AD, polo A mobilis, perpetim creſcendò adepta ſit; quando

 niunt; adeo ut omnes eodem negotio in æquales Geometricas converti poſſint.

12. Poſſem plura hujus conſtruendi Problematis ſpecimina afferre; ſed hæc ſufficiant, cùm ſint adeò generalia, ut quicquid hactenus circa Curvarum Areas inventum fuerit, vel, ni fallor, inveniri poſſit, aliquo ſaltem modo complectantur; et ut plurimùm leviori curâ ſine ſolitis ambagibus determinant.

13. Præcipuus autem hujus et præcedentis Problematis uſus eſt, ut aſſumptis conicis ſectionibus, vel quibuſlibet notæ magnitudinis Curvis; aliæ Curvæ, quæ cum his conferri poſſunt, inveſtigentur; et earum definientes æquationes in catalogum ordinatim diſponantur. Et conſtructo ejuſmodi catalogo, cùm Curvæ alicujus Area quæritur, ſi æquatio ejus definiens vel immediatè in catalogo reperiatur, vel in aliam, quam catalogus complectitur, transformari poteſt; exinde cognoſces aream ejus. Quinetiam catalogus ille determinandis Curvarum Longitudinibus, Centris Gravitatum, Solidis per convolutionem generatis, ſolidorum Superficiebus, et cuilibet fluenti quantitati, per analogam fluxionem generatæ, inſervire poteſt.

PROB. IX.

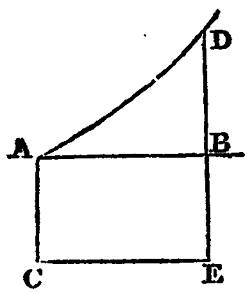
14. *Propoſitæ alicujus Curvæ Aream Determinare.*

Problematis reſolutio in eo fundatur, ut quantitatum fluentium relatio ex relatione fluxionum (per Prob. II.) eliciatur.

Et

Cognatio do ſemel circumacta in locum illum reverſa ſit, unde moveri inceperat. Centro A radio AM ſcribatur Circulus cujus peripheriæ æqualis ſit recta C. Peripheriæ autem circuli, centro A radio AD ſcripti, æqualis ſit recta *c*. E naturâ Helicis Archimedei, recta AD erit ad AM ut angulus MAD, ad quatuor rectos: hoc eſt, ut arcus GD ad circuitum circuli GDO, ſive ut recta GL ad rectam *c*. Sed recta GL ad rectam *c* proportionem habet compoſitam ex proportionibus rectæ GL ad rectam C rectæque C ad *c*. Et recta C ad *c* proportionem habet eam quam AM ad AG. Quare proportio rectæ GL ad rectam *c* compoſita eſt ex proportionibus rectæ GL ad C rectæque AM ad AG. Sed rectanguli GL × AM ad rectangulum C × AG ex eiſdem eſt compoſita proportio. Erit igitur AD, ſive AG, ad AM ut GL × AM ad C × AG. Quadratum igitur ex AG erit ad rectangulum AM × AG ut GL × AM ad C × AG. Eſt autem rectangulum AM × AG ad quadratum ex AM ut C × AG ad C × AM. Erit igitur ex æquo quadratum ex AG ad quadratum ex AM ut GL × AM ad C × AM. Duabus C, AM ſit P proportione tertia. Erit igitur quadratum ex AM rectangulo C × P æquale.

Et



Et imprimis ſi recta BD, cujus motu quæſita area ADB deſcribitur, ſuper baſi AB, poſitione datâ, erectè incedat; concipe, ut ſuprà, parallelogrammum ABEC à BE, unitatem æquante, interea deſcribi. Et poſitâ BE fluxione parallelogrammi, erit BD fluxio areæ quæſitæ. CURVARUM AREÆ.

Dic ergo AB $=x$, et erit etiam ABEC $=(1\times x)=x$, et BE $=\dot{x}$: dic inſuper aream ADB $=z$, et erit BD $=\dot{z}$, ut et $=\frac{\dot{z}}{\dot{x}}$, eo quòd ſit $\dot{x}=1$. Et proin per æquationem definientem BD ſimul definitur fluxionum ratio $\frac{\dot{z}}{\dot{x}}$, et exinde (per Prob. II. Caſ. I.) elicietur relatio fluentium quantitatum x et z.

15. EXEMPLA PRIMA.

Ubi BD, *ſive* $\dot{z}$, *valet ſimplicem aliquam quantitatem.*

Detur $\frac{x^2}{a}=\dot{z}$, vel $=\frac{\dot{z}}{\dot{x}}$, æquatio nempe ad Parabolam, et (per Prob. II.) emerget $\frac{x^3}{3a}=z$. Eſt ergo $\frac{x^3}{3a}$ ſive (xx) $\frac{1}{3}$ AB $\times$ BD = areæ parabolicæ ADB.

Detur $\frac{x^3}{a^2}=\dot{z}$, æquatio ad Parabolam ſecundi generis, et (per Prob. II.) emerget $\frac{x^4}{4a^2}=z$, hoc eſt (yy) $\frac{1}{4}$ AB $\times$ BD = areæ ADB.

Detur $\frac{a^3}{x^2}=\dot{z}$, ſive $a^3x^{-2}=\dot{z}$, æquatio ad Hyperbolam ſecundi generis; et emerget $-a^3x^{-1}=z$ (zz) ſive $\frac{-a^3}{x}=z$; hoc eſt, AB $\times$ BD =

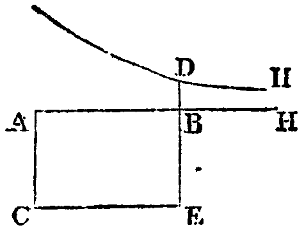
Et quadratum ex AG erit ad rectangulum C $\times$ P ut GL $\times$ AM ad C $\times$ AM, hoc eſt ut GL ad C, ſive ut GL $\times$ P ad C $\times$ P. Quadratum igitur ex AG rectangulo GL $\times$ P æquale erit. Recta autem P magnitudine data eſt, propter duas C, AM datas. Quare punctum L erit ad Parabolam poſitione datam, illam ſcilicet cujus vertex A, axis rectæ GL parallelus, parameter datæ P æqualis. Q.E D. HELICIS PARABOLÆQUE.
Eſt hæc eſt cognatio illa Helicis Archimedei cum Parabolâ Apollonianâ, cujus jam olim meminere Cavallerius, Torricellius, Barrovius, aliique.

(xx) Sive $\frac{1}{3}x\times\frac{x^2}{a}$, ſive $\frac{1}{3}x\times\dot{z}$, ſive $\frac{1}{3}$ AB $\times$ BD.

(yy) $\frac{1}{4}x\times\frac{x^3}{a^2}$, ſive $\frac{1}{4}x\times\dot{z}$, ſive $\frac{1}{4}$ AB $\times$ BD.

(zz) $-x\times a^3x^{-2}$, ſive $-x\times\dot{z}$, ſive $-$AB $\times$ BD $=z$.

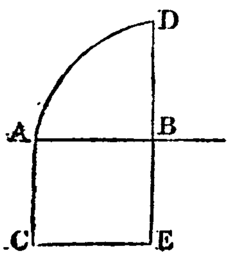
areæ



areæ infinitè longæ HDBII ex alterâ parte applicatæ BD jacentis, ut innuit valor negativus.

Atque ita ſi detur $\frac{a^4}{x^3}=\dot{z}$, emerget $\frac{-a^4}{2x^2}=z$.

Præterea ſit $ax=\dot{z}^2$, ſive $a^{\frac{1}{2}}x^{\frac{1}{2}}=\dot{z}$, æquatio iterum ad Parabolam, et proveniet $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}=z$, hoc eſt ($^{\alpha}$), $\frac{2}{3}$ AB × BD = areæ ADB.



Sit $\frac{c^3}{x}=\dot{z}^2$, et fiet $2a^{\frac{3}{2}}x^{\frac{1}{2}}=z$, ſive 2AB × BD = ADB.

Sit $\frac{a^5}{x^3}=\dot{z}^2$, et fiet $-\frac{2a^{\frac{5}{2}}}{x^{\frac{1}{2}}}=z$, ſive 2AB × BD = HDBII.

Sit $ax^2=\dot{z}^3$, et fiet $\frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}}=z$, ſive $\frac{3}{5}$ AB × BD = ADB. Et ſic in aliis.

16. EXEMPLA SECUNDA.

Ubi ż valet plures ejuſmodi connexas quantitates.

Sit $x+\frac{x^2}{a}=\dot{z}$, et fiet $\frac{x^2}{2}+\frac{x^3}{3a}=z$.

Sit $a+\frac{a^3}{x^2}=\dot{z}$, et fiet $ax-\frac{a^3}{x}=z$.

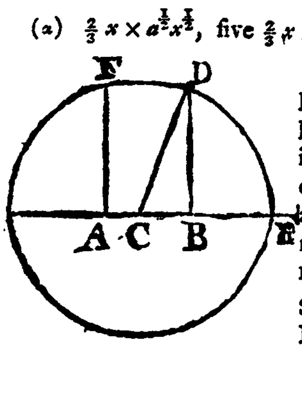
Sit $3x^{\frac{1}{2}}-\frac{5}{x^2}-\frac{2}{x^{\frac{1}{2}}}=\dot{z}$, et fiet $2x^{\frac{3}{2}}+\frac{5}{x}-4x^{\frac{1}{2}}=z$.

17. EXEMPLA TERTIA.

Ubi prævia reductio per Diviſionem requiritur.

Detur $\frac{a^2}{b+x}=\dot{z}$, æquatio ad Hyperbolam Apollonianam; et factâ in

(α) $\frac{2}{3}x\times a^{\frac{1}{2}}x^{\frac{1}{2}}$, ſive $\frac{2}{3}x\times\dot{z}$, ſive $\frac{2}{3}$ AB × BD.



(β) Nempe hoc modo. Rectâ AB poſitione datâ, datoque in illâ puncto A, ſit AB indefinitæ longitudinis, quæ dicatur x. Rectam AB ad perpendiculum inſiſtat BD, quæ dicatur $\dot{z}$. Et rectæ AB, BD eo modo inter ſe habeant atque hæc æquatio ſignificat; $\dot{z}=\sqrt{a^2+bx-x^2}$: deſignantibus, b, a longitudines quaſdam datas. Dico punctum D eſſe ad peripheriam circuli magnitudine et poſitione dati. In rectâ AB ſumatur AC $=\frac{1}{2}b$, et jungatur CD. Sumatur etiam BE = AB. Ita erit EC $=2x-\frac{1}{2}b$. Propter angulum ad B rectum, CD^2 = CB^2 + BD^2. Sed CB^2 = AB^2 − EC × CA $=x^2-\overline{2x-\frac{1}{2}b}\times\frac{1}{2}b=x^2-bx+\frac{1}{4}b^2$. Et BD$^2=a^2+bx-x^2$ (ex hypotheſi) Quare CB^2 + BD^2 ſive CD$^2=a^2+\frac{1}{4}b^2$.

in infinitum diviſione, evadet $\dot z = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \&c.$ Et inde (per Prob. II.) ut in ſecundis exemplis, obtinebitur $z = \frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$ &c. CURVARUM AREA.

Detur $\frac{1}{1+x^2} = \dot z$, et per diviſionem elicietur, $\dot z = 1 - x^2 + x^4 - x^6$ &c. Vel etiam $\dot z = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6}$ &c. Indeque (per Prob. II.) $z =$ … $\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$ &c. = AEDB; vel $z = \frac{-1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$ &c. … DBII.

Detur $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = \dot z$, et per diviſionem evadet $\dot z = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$ &c. Et inde (per Prob. II.) $z = \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3 + \frac{68}{7}x^{\frac{7}{2}}$ &c.

18. EXEMPLA QUARTA.

Ubi prævia reductio per Extractionem Radicum requiritur.

Detur $\dot z = \sqrt{a^2+x^2}$, æquatio nempe ad Hyperbolam. Et radice aduſque terminos infinitè multos extractâ, evadet $\dot z = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Atque inde, ut in præcedentibus, $z = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c.

Ad eundem modum, ſi detur $\dot z = \sqrt{aa - xx}$, æquatio ſcilicet ad Circulum, obtinebitur $z = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c.

Atque ita ſi detur $\dot z = \sqrt{x - x^2}$, æquatio iterum ad Circulum proveniet, extrahendò radicem, $\dot z = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, adeoque eſt $z = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$ &c.

Sic $\dot z = \sqrt{a^2 + bx - x^2}$, æquatio denuò ad Circulum, (6) per Extractionem Radicis, dat $\dot z = a + \frac{bx}{2a} - \frac{x^2}{2a} - \frac{b^2x^2}{8a^3}$ &c. Unde (per Prob. II.) elicitur, $z = ax + \frac{bx^2}{4a} - \frac{x^3}{6a} - \frac{b^2x^3}{24a^3}$ &c.

$a^2 + \frac{1}{4}b^2$. Datum igitur CD^2. Ergo CD magnitudine data. Sed punctum C datum, propter rectam AC magnitudine et poſitione datam, datumque punctum A. Ergo D ad peripheriam circuli poſitione dati. Scribatur circulus ille, cujus peripheriæ recta per A, cum ordinatis BD parallela, in F occurrat. Areæ circulari FABD ſeries Newtoniana $ax + \frac{bx^2}{4a} - \frac{x^3}{6a}$ &c. infinitè producta fit ultimò æqualis.

Et ſic $\sqrt{\frac{1+ax^2}{1-bx^2}} = \dot{z}$, per debitam reductionem dat $\dot{z} = 1 + \begin{array}{l}\frac{1}{2}b\\ \frac{1}{2}a\end{array} x^2 + \begin{array}{l}\frac{3}{8}b^2\\ +\frac{1}{4}ab\\ -\frac{1}{8}aa\end{array} x^4$ &c. unde (per Prob. 1.) $z = x + \begin{array}{l}\frac{1}{6}b\\ +\frac{1}{6}a\end{array} x^3 + \begin{array}{l}\frac{3}{40}b^2\\ +\frac{1}{20}ab\\ -\frac{1}{40}aa\end{array} x^5$ &c.

Sic denique $\dot{z} = \sqrt[3]{a^3+x^3}$, per extractionem Radicis Cubicæ, dat $\dot{z} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$ &c. Indeque $z = ax + \frac{x^4}{12a^2} - \frac{x^7}{63a^5} + \frac{x^{10}}{162a^8}$ &c. $=$ ADB. Vel etiam $\dot{z} = x + \frac{a^3}{3x^2} - \frac{a^6}{9x^5} + \frac{5a^9}{81x^8}$ &c. Indeque $z = \frac{x^2}{2} - \frac{a^3}{3x} + \frac{a^6}{36x^4} - \frac{5a^9}{567x^8}$ &c $=$ HDEH (γ).

19. EXEMPLA QUINTA.

Ubi prævia reductio per æquationis affectæ reſolutionem requiritur.

Si Curva per æquationem $\dot{z}^3 + a^2\dot{z} + ax\dot{z} - 2a^3 - x^3 = 0$ definiatur, extrahe radicem, et proveniet $\dot{z} = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512aa}$ (δ) &c. Unde, ut in prioribus, obtinebitur $z = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$ &c.

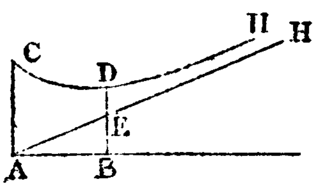
Sin $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$ ſit æquatio ad Curvam, reſolutio dabit triplicem radicem; nempe $\dot{z} = c + x - \frac{x^2}{4c} + \frac{x^3}{16c^2}$ &c. Et $\dot{z} = c - x + \frac{3x^2}{4c} - \frac{3x^3}{16c^2}$ &c. Et $\dot{z} = -c - \frac{x^2}{2c} - \frac{x^3}{2cc} + \frac{x^5}{4c^4}$ &c. Et inde trium correſpondentium arearum valores, $z = cx + \frac{1}{2}x^2 - \frac{x^3}{12c} + \frac{x^4}{64c^2}$ &c.

$z = cx - \frac{1}{2}x + \frac{x^3}{4c} - \frac{3x^4}{64c^2}$ &c.

ac $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8c^2} + \frac{x^6}{24c^4}$ &c.

20. De Curvis Mechanicis hic nihil adjicio, ſiquidem reductio ad formam Geometricam poſt oſtendetur.

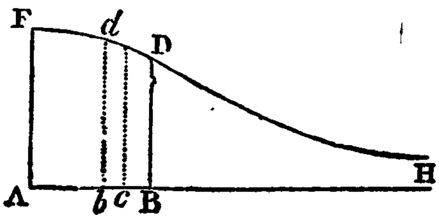
21. Cæterùm cùm ſic inventi valores z areis quandoque ad baſis finitam partem AB, quandoque ad partem BH infinitè verſus H productam, et quandoque ad utramque partem ſitis ſecundum



(γ) Sit CDH curva cujus naturam hæc æquatio indicat $\dot{z} = \overline{a^3+x^3}|^{\frac{1}{3}}$, deſignante x indefinitam AB, quæ a dato puncto A initium ſumat, $\dot{z}$ vero deſignante BD. In BD capiatur BE $=$ AB $= x$. Junctaque AE producatur. Recta AE (quæ poſitione data eſt per naturam) curvæ CDH aſymptotos erit; id enim æquatio infinita $\dot{z} = x + \frac{a^3}{3x^2} - \frac{a^6}{9x^5}$ &c. ſatis indicat. Seriei autem infinitæ $\frac{x^2}{2} - \frac{a^3}{3x}$

cundum diverſos eorum terminos competant: quo debitus areæ, ad quamlibet baſis portionem ſitæ, valor aſſignetur, area illa ſemper ponenda eſt æqualis differentiæ valorum z partibus baſis, ad initium et finem iſtius areæ terminatis, competentium. CURVARUM AREÆ.

22. E.G. Ad Curvam, quam æquatio $\frac{1}{1+xx}=\dot{z}$ definit, inventum eſt $z=x-\frac{1}{3}x^3+\frac{1}{5}x^5$ &c. Jam ut quantitatem areæ *bd*BD, ad iacentis parti baſis B*b*, determinem, à valore z qui fit ponendo AB$=x$, aufero eum qui fit ponendo A$b=x$; et (diſtinctionis gratiâ ſcriptâ X majuſculâ pro AB, et x minuſculâ pro A*b*) reſtat $\text{X}-\frac{1}{3}\text{X}^3+\frac{1}{5}\text{X}^5$ &c. $-x+\frac{1}{3}x^3-\frac{1}{5}x^5$ &c. valor areæ illius *bd*BD. Unde ſi A*b*, ſeu x, ponatur nullum, habebitur tota area AFDB$=\text{X}-\frac{1}{3}\text{X}^3+\frac{1}{5}\text{X}^5$ &c.



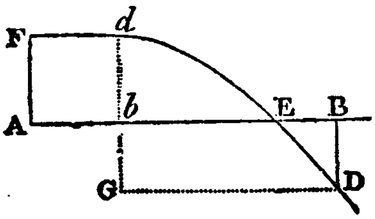
Ad eandem Curvam inventum eſt etiam $z=-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}$ &c. Unde rurſus, juxta præcedentia, erit area illa *bd*DB$=\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}$ &c. $-\frac{1}{\text{X}}+\frac{1}{3\text{X}^3}-\frac{1}{5\text{X}^5}$ &c. Adeoque ſi AB, ſeu X, ſtatuatur infinitum, area adjacens *bd*H, à parte H ſimiliter infinitè longa, valebit $\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}$ &c. Siquidem poſterior ſeries, $-\frac{1}{\text{X}}+\frac{1}{3\text{X}^3}-\frac{1}{5\text{X}^5}$ &c. propter infinitatem denominatorum, evaneſcat.

23. Ad Curvam æquatione $a+\frac{a^3}{x^2}=\dot{z}$ deſignatam, inventum eſt $ax-\frac{a^3}{x}=z$. Unde fit $a\text{X}-\frac{a^3}{\text{X}}-ax+\frac{a^3}{x}=$ areæ *bd*DB. Hæc autem evadit infinita, ſive x fingatur nulla, ſive X infinita; et proinde utraque area AFDB, et *bd*H infinitè magna eſt, ac ſolæ partes intermediæ, qualis *bd*DB, exhiberi poſſunt. Id quod ſemper evenit, ubi baſis x cum in numeratoribus aliquorum tum in denominatoribus aliorum terminorum valoris z reperitur. Ubi verò x in numeratoribus ſolummodo, ut in primo exemplo, re-

$\frac{a^3}{3x}+\frac{a^6}{36x^4}$ &c. nomen primum $\frac{x^2}{2}$ areæ trianguli rectilinearis ABE æquale eſt; nomina omnia reliqua areæ HEDH, quæ ultra ordinatam DE aſymptotæ eſt contigua. Vide Analyſin per Æquat. num. term. inf. Reg. 2. exempla tertia. Series prior $ax+\frac{x^4}{12a^2}-\frac{x^7}{63a^5}$ &c. areæ ACLD menſuram exhibet.

(δ) Vide Cap. III. § 6.

CAPUT IX. peritur; valor z competit areæ ſitæ ad AB, cis parallelè incedentem. Et ubi in denominatoribus tantùm, ut in ſecundo exemplo; valor ille mutatis omnium terminorum ſignis, competit areæ omni, ultra parallelè incedentem, infinitè productæ.



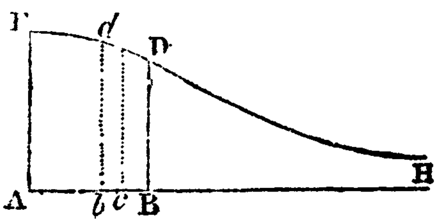
24. Si quando Curva linea ſecat baſin inter puncta b et B, puta in E, vice areæ habebitur arearum ad diverſas baſis partes differentia, bdE$-$BDE; cui ſi addatur rectangulum BDGb, obtinebitur area dEDG.

25. Præcipuè autem notandum eſt, quòd ubi in valore $\dot z$ terminus aliquis per x unius tantùm dimenſionis dividitur; area illi termino correſpondens pertinet ad Hyperbolam conicam, et proinde per infinitam feriem ſeorſim exhibenda eſt: quemadmodum in ſequentibus factum.

Sit $\frac{a^3-a^2x}{ax+x^2}=\dot z$, æquatio ad Curvam; et per diviſionem, fiet $\dot z=\frac{a^2}{x}-2a+2x-\frac{2x^2}{a}+\frac{2x^3}{a^2}$ &c. Indeque $z=\boxed{\frac{a^2}{x}}-2ax+x^2-\frac{2x^3}{3a}+\frac{x^4}{2a^2}$ &c.

Et area bdDB $=\boxed{\frac{aa}{X}}-2aX+X^2-\frac{2X^3}{3a}$ &c. $-\boxed{\frac{aa}{x}}+2ax-x^2+\frac{2x^3}{3a}$ &c.

Ubi per notas $\boxed{\frac{aa}{X}}$ et $\boxed{\frac{aa}{x}}$ deſigno areolas terminis $\frac{aa}{X}$ et $\frac{aa}{x}$ competentes.



Jam ut $\boxed{\frac{aa}{X}}$ et $\boxed{\frac{aa}{x}}$ inveſtigetur, fingo Ab ſeu x definitam eſſe; et bB indefinitam, ſeu fluentem, lineam, quam itaque ſi dicam y, erit $\boxed{\frac{aa}{x+y}}$ = areæ iſti hyperbolicæ adjacenti bB, nempe $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}$. Eſt autem, factâ diviſione $\frac{aa}{x+y}=\frac{aa}{x}-\frac{aay}{x^2}+\frac{a^2y^2}{x^3}-\frac{a^2y^3}{x^4}$ &c. Adeoque $\boxed{\frac{aa}{x+y}}$ ſeu $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}$ $=\frac{aay}{x}-\frac{a^2y^2}{2x^2}+\frac{a^2y^3}{3x^3}-\frac{a^2y^4}{4x^4}$ &c. Et proinde tota area quæſita bdBD=

aay

(ı) Nimirum ſi curva FDH ſit ipſa Hyperbola Apolloniana.

(ζ) Vide

$\frac{aay}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3}$ &c. $-2aX+X^2-\frac{2X^3}{3a}$ &c. $+2ax-x^2+\frac{2x^3}{3a}$ &c.

De Areis Hyperbolicis.

26. Ad eundem modum AB, ſeu X, pro definitâ lineâ adhiberi potuit, et ſic prodiiſſet $\left|\frac{aa}{X}\right| - \left|\frac{aa}{x}\right| = \frac{a^2y}{X} + \frac{a^2y^2}{2X^2} + \frac{a^2y^3}{3X^3} + \frac{a^2y^4}{4X^4}$ &c.

27. Quinetiam ſi biſecetur bB in C, et aſſumatur AC eſſe definitæ longitudinis, et Cb ac CB indefinitæ: tum dicto AC$=e$, et Cb vel CB$=y$, erit $bd=\frac{aa}{e-y}$ (ε) $=\frac{aa}{e}+\frac{aay}{e^2}+\frac{a^2y^2}{e^3}+\frac{a^2y^3}{e^4}+\frac{a^2y^4}{e^5}$ &c. Indeque area hyperbolica parti baſis bC adjacens $\frac{a^2y}{e}+\frac{a^2y^2}{2e^2}+\frac{a^2y^3}{3e^3}+\frac{a^2y^4}{4e^4}$ &c. Erit etiam CB $=\frac{aa}{e+y}=\frac{aa}{e}-\frac{a^2y}{e^2}+\frac{a^2y^2}{e^3}-\frac{a^2y^3}{e^4}+\frac{a^2y^4}{e^5}$ &c. Et inde area alteri baſis parti CB adjacens $\frac{a^2y}{e}-\frac{a^2y^2}{2e^2}+\frac{a^2y^3}{3e^3}-\frac{a^2y^4}{4e^4}+\frac{a^2y^5}{5e^5}$ &c. Et harum arearum ſumma $\frac{2a^2y}{e}+\frac{2a^2y^3}{3e^3}+\frac{2a^2y^5}{5e^5}$ &c. valebit $\left|\frac{aa}{X}\right| - \left|\frac{aa}{x}\right|$.

28. Sic æquatione $z^3+z^2+z-x^3=0$, ad Curvam exiſtente, ejus radix erit $z=x-\frac{1}{3}-\frac{2}{9x}+\frac{7}{81x^2}+\frac{14}{729x^4}$ (ζ) &c. Unde fit $z=\frac{1}{2}x^2-\frac{1}{3}x-\left|\frac{2}{9x}\right|-\frac{7}{81x}-\frac{14}{2187x^3}$ &c. Et area $bDBd=\frac{1}{2}X^2-\frac{1}{3}X-\left|\frac{2}{9X}\right|-\frac{7}{81X}$ &c. $-\frac{1}{2}x^2+\frac{1}{3}x+\left|\frac{2}{9x}\right|+\frac{7}{81x}$ &c. hoc eſt, $=\frac{1}{2}X^2-\frac{1}{3}X-\frac{7}{81X}$ &c. $-\frac{1}{2}x^2+\frac{1}{3}x+\frac{7}{81x}$ &c. $-\frac{4y}{9e}-\frac{4y^3}{27e^3}-\frac{4y^5}{35e^5}$ &c.

29. Poteſt autem terminus iſte hyperbolicus commodè devitari, mutando initium baſis; id eſt, augendo vel minuendo eam per datam aliquam quantitatem. Quemadmodum in exemplo priore, ubi $\frac{a^3-a^2x}{ax+x^2}=z$, erat æquatio ad Curvam; ſi faciam b eſſe initium baſis, et fingens Ab cujuſlibet eſſe determinatæ longitudinis, puta $\frac{1}{2}a$, pro baſis reſiduo, bB, jam ſcribam x: hoc eſt, ſi diminuam baſem per $\frac{1}{2}a$, ſcribendo $x+\frac{1}{2}a$ pro x; evadet $\frac{\frac{1}{2}a^3-a^2x}{\frac{3}{4}a+2ax+x^2}=z$, et per diviſionem $z=\frac{2}{3}a-\frac{28}{9}x+\frac{200x^2}{27a}$ &c. Unde fit $z=\frac{2}{3}ax-\frac{14}{9}x^2+\frac{200x^3}{81a}$ &c. = areæ bdDB.

Et ſic pro initio baſis adhibendo aliud, atque aliud ejus punctum, poteſt area cujuſvis Curvæ modis infinitis exprimi.

30. Potuit etiam æquatio $\frac{a^3-a^2x}{ax+x^2}=z$, in duas ſeries infinitas re-

(ζ) Vide Cap. III. § 15.

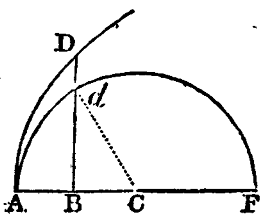
ſolvi,

Caput IX. ſolvi, prodeunte $z = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$ &c. $- a + x - \frac{x^2}{a} + \frac{x^3}{a^2}$ &c (*). Ubi terminus per x unius tantùm dimenſionis diviſus non reperitur. Sed hujuſmodi ſeries, ubi dimenſiones x in unius numeratoribus, et alterius denominatoribus, infinitè aſcendunt, minùs aptæ ſunt, ex quibus z per computum arithmeticum obtineri poſſit, cùm in ejus valore numeri pro ſpeciebus ſubſtituuntur.

31. Inſtituenti computum hujuſmodi numeroſum, poſtquam valor areæ in ſpeciebus habetur, haud aliquid difficile occurret. Tamen in præcedentem doctrinam penitiùs illuſtrandam, exemplum unum et alterum ſubjungere placuit.

Quadratura Hyperbolæ.

32. Proponatur Hyperbola AD, quam æquatio $z = \sqrt{x + x^2}$ deſignat, utpote cujus vertex eſt ad A, et uterque axis æquatur unitati.



Et è præcedentibus area ejus ADB erit $\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} + \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}$ &c. hoc eſt, $x^{\frac{1}{2}}$ in $\frac{2}{3}x + \frac{1}{5}x^2 - \frac{1}{28}x^3 + \frac{1}{72}x^4 - \frac{5}{704}x^5$ &c. Quæ ſeries infinitè producitur, multiplicando ultimum terminum continuò per ſuccedaneos terminos hujus progreſſionis; $\frac{1.3}{2.5}x$. $\frac{-1.5}{4.7}x$. $\frac{-3.7}{6.9}x$. $\frac{-5.9}{8.11}x$. $\frac{-7.11}{10.13}x$. &c. Nempe primus terminus $\frac{2}{3}x^{\frac{3}{2}} \times \frac{1.3}{2.5}x$ facit $\frac{1}{5}x^{\frac{5}{2}}$, ſecundum terminum. Hic in $\frac{-1.5}{4.7}x$ facit $-\frac{1}{28}x^{\frac{7}{2}}$ tertium terminum. Hic in $\frac{-3.7}{6.9}x$ facit $\frac{1}{72}x^{\frac{9}{2}}$ quartum terminum. Et ſic in infinitum. Sumatur jam AB cujuſlibet longitudinis, puta $\frac{1}{4}$, et hunc numerum ſcribe pro x, ejuſque radicem $\frac{1}{2}$ pro $x^{\frac{1}{2}}$; et primus terminus $\frac{2}{3}x^{\frac{3}{2}}$, ſive $\frac{2}{3} \times \frac{1}{8}$ in decimalem fractionem reductus, evadit 0,08333333 &c. Hic in $\frac{1.3}{2.5.4}$ facit 0,00625, ſecundum terminum. Hic in $\frac{-1.5}{4.7.4}$ facit $-$0,0002790178 &c. tertium terminum. Et ſic in infinitum. Terminos autem, quos ſic gradatim elicio, diſpono in duas Tabulas, affirmativos nempe in unam, et negativos in aliam, et addo, ut hîc vides.

(*) Quippe cum fractio $\frac{a^3 - a^2x}{ax + x^2} = \frac{a^3}{x^2 + ax} - \frac{a^2}{a + x}$. Horum autem prior, $\frac{a^3}{x^2 + ax} = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$ &c. poſterior $\frac{a^2}{a + x} = a - x + \frac{x^2}{a} - \frac{x^3}{a}$ &c. Cap. I. § 4.

Dein

+0,0833333333333333	−0,0002790178571429
6250000000000	3467906051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	−0,0002825719389575
1	+0,0896109885646618
+0,0896109885646618	0,0893284166257043

Dein à ſummâ affirmativarum aufero ſummam negativorum, et reſtat

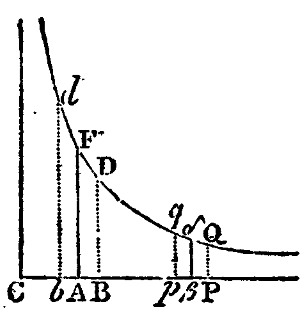
0,0893284166257043

quantitas areæ hyperbolicæ ADB, quam quærere oportuit.

33. Proponatur jam circulus ADF quam æquatio $\sqrt{x-xx}=z$ deſignat, hoc eſt, cujus diameter AF ſit unitas, et è præcedentibus area ejus A*d*B erit $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$ &c. In quâ ſerie cùm termini non differant à terminis ſeriei ſupra exprimentis aream hyperbolicam, niſi in ſignis + et −, nihil aliud agendum reſtat, quàm ut eoſdem numerales terminos cum aliis ſignis nectamus, ſubducendo nempe connexas ambarum præfatarum Tabularum ſummas 0,0898935605036193 à primo termino duplicato 0,1666666666666666, et reſiduum 0,0767731061630473, erit areæ circularis portio A*d*B; poſito ſcilicet AB quadrante diametri. Atque ita videre eſt, quòd etſi areæ Circuli et Hyperbolæ non conferantur ratione geometricâ, tamen utraque eodem computo arithmetico prodit. Quadratura Circuli.

34. Inventâ Circuli portione A*d*B, exindè tota area facilè eruitur. Nempe radio *d*C acto duc B*d*, ſeu $\frac{1}{4}\sqrt{3}$, in BC, ſeu $\frac{1}{4}$; et facti dimidium $\frac{1}{32}\sqrt{3}$ ſeu 0,0541265877365275 valebit triangulum C*d*B; quod adde areæ A*d*B, et habebitur ſector AC*d* = 0,1308996938995748; cujus ſextuplum 0,7853981633974488 eſt area tota.

35. Et hinc obiter exit peripheriæ longitudo 3,1415926535897932, dividendo nempe Aream per quadrantem Diametri.



36. Hiſce calculum areæ inter Hyperbolam *d*FD et ejus aſymptoton CA interceptæ ſubnectimus. Sit C centrum hyperbolæ, et poſitâ CA $=a$, AF $=b$, et AB $=$ A$b=x$; erit $\frac{ab}{a+x}=$ BD et $\frac{ab}{a-x}=bd$, et inde area AFDB $=bx-\frac{bx^2}{2a}+\frac{bx^3}{3a^2}-\frac{bx^4}{4a^3}$ &c. Et area AF$db=bx+\frac{bx^2}{2a}+\frac{bx^3}{3a^2}+$

I

bx^4

CAPUT. IX. $\frac{bx^4}{4a^3}$ &c. Ac earum ſumma $bd\text{DB} = 2bx + \frac{2bx^3}{3a^2} + \frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6}$ &c.

Ponamus jam CA=AF=1, et Ab vel AB$=\frac{1}{10}$, exiſtente cb=0,9, et CB=1,1; et ſubſtituendo hos numeros pro a, b, et x, primus ſeriei terminus evadet 0,2; ſecundus 0,00066666 &c. tertius 0,000004; et ſic deinceps, ut vides in hâc Tabulâ.

	0,2000000000000000	
	6666666666666	
	40000000000	
	285714286	
	2222222	
	18182	
	154	
	1	
Summa	0,2006706954621511	= areæ bdDB

37. Quòd ſi areæ hujus partes Ad et AD ſeorſim deſiderentur; ſubduc minorem DA è majori dA et reſtabit, $\frac{bx^2}{a} + \frac{bx^4}{2a^3} + \frac{bx^6}{3a^5} + \frac{bx^8}{4a^7}$ &c. Ubi ſi 1 ſcribatur pro a et b, ac $\frac{1}{10}$ pro x, termini, in decimales redacti, conficient ſequentem Tabulam.

	0,0100000000000000	
	500000000000	
	3333333333	
	25000000	
	200000	
	1667	
	14	
Summa	0,0100503358535014	= Ad − AD

38. Jam ſi hæc arearum differentia addatur et auferatur ſummæ earum priùs inventæ, aggregati dimidium 0,1053605156578263 erit major area Ad; et reſidui dimimium 0,0953101798043248, minor AD.

39. Per eaſdem Tabulas obtinentur etiam areæ illæ AD et Ad, ubi AB et Ab ponantur $\frac{1}{100}$, ſive CB=1,01 et cb=0,99 ſi modò numeri in depreſſiora loca debitè transferantur, ut hîc

	0,0200000000000000	
	6666666667	
	400000	
	28	
Summa	0,0200006667066695	= bdDB

	0,0001000000000000	
	50000000	
	3333	
Summa	0,0001000050003333	= Ad − AD

videre eſt. $\frac{1}{2}$ aggregat 0,0100503358535014=Ad; $\frac{1}{2}$ reſid. 0,0099503308531681=AD..

40. Et ſic poſitis AB et Ab $\frac{1}{1000}$, ſeu CB=1,001, et cb=0,999, obtinebitur Ad=0,0010005003335835, et AD=0,0009995003330835.

41. Ad eundem modum ſi ſtantibus CA et AF=1, ponantur AB et Ab=0,2, vel =0,02, vel =0,002, eliciantur areæ illæ.

Ad=0,2231435513142097 et AD=0,1823215567939546
vel Ad=0,0202027073175194 et AD=0,0198026272961797
vel Ad=0,002002 et AD=0,001

42. Ex

42. Ex inventis hiſce areis jam facile eſt alias, per ſolam additionem et ſubductionem, derivare. Utpote cùm ſit $\frac{1,2}{0,8}$ in $\frac{1,2}{0,9}=2$, arearum pertinentium ad rationes $\frac{1,2}{0,8}$ et $\frac{1,2}{0,9}$ (hoc eſt, inſiſtentium partibus baſis $1,2-0,8$ et $1,2-0,9$) ſumma, 0,693147180559945 3, erit area AF$\delta\beta$, exiſtente c$\beta=2$, ut notum eſt. Dein cùm ſit $\frac{1,2}{0,8}$ in $2=3$, arearum pertinentium ad $\frac{1,2}{0,8}$, et 2 ſumma, 1,0986122886681097, erit area AF$\delta\beta$, exiſtente c$\beta=3$. Pariter cùm ſit $\frac{2 \text{ in } 2}{0,8}=5$, et 2 in $5=10$, per debitam arearum additionem obtinebitur $1,6094379124341004=$ AF$\delta\beta$, exiſtente c$\beta=5$; et $2,302585092994045 7=$ AF$\delta\beta$, exiſtente c$\beta=10$. Atque ita cùm ſit $10\times10=100$, et 10 in $100=1000$, et $\sqrt{5 \text{ in } 10 \text{ in } 0,98}=7$, et 10 in $1,1=11$, et $\frac{1000\times1,001}{7\times11}=13$, et $\frac{100\times1,02}{2\times3}=17$, et $\frac{1000\times0,999}{3\times3\times3}=37$, et $100\times1,01=101$, et $\frac{1000\times1,002}{2\times3}=167$, et $\frac{1000\times0,998}{2}=499$, patet aream AF$\beta\delta$ per arearum ſuprà inventarum compoſitionem inveniri poſſe, exiſtente c$\beta=100$; 1000; 7; aut alio quolibet è recenſitis numeris, et ſtante AB$=$BF$=1$. Id quod ſignificare volui, ut methodus conſtruendo Logarithmum canoni aptiſſima pateret; quæ areas Hyperbolicas (ex quibus logarithmi facilè deducantur) tot numeris primis correſpondentes, quaſi per binas tantùm haud moleſtas operationes determinat. Cæterùm cùm canon iſte ex hoc fonte præ cæteris feliciter depromi videatur, quid ſi conſtructionem ejus, coronidis loco, perſtringam. AREÆ HYPERBOLICÆ.

43. IMPRIMIS itaque aſſumpto 0 pro logarithmo numeri 1, et 1 pro logarithmo numeri 10, ut ſolet, inveſtigandi ſunt logarithmi primorum numerorum, 2,3,5,7,11,13,17,37, dividendo inventas areas hyperbolicas per 2,302585092994045 7, aream nempe correſpondentem numero 10: vel, quod eodem recidit, multiplicando per ejus reciprocum 0,4342944819032518. Sic enim *e. g.* Si 0,69314718 &c. area correſpondens numero 2, multiplicetur per 0,43429 &c. facit 0,3010299956639812. Logarithmum numeri 2. LOGARITHMOTECHNIA.

44. Deinde logarithmi numerorum omnium in canone, qui ex horum multiplicatione fiunt, indagandi ſunt per additionem

CAPUT IX. eorum logarithmorum, ut ſolet: et loca vacua poſtmodum interpolanda, ope hujus Theorematis.

45. Sit n numerus logarithmo donandus, x differentia inter illum et proximos numeros hinc inde æqualiter diſtantes, quorum logarithmi habentur; ac d, ſemiſſis differentiæ logarithmorum: et quæſitus logarithmus numeri n obtinebitur addendo $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c. logarithmo minoris numeri. Nam ſi numeri exponantur per Cp, Cβ, & CP, et exiſtente rectangulo CBD vel C$\beta\delta = 1$ ut ſuprà, ac erectis parallelè incedentibus pq & PQ: ſi n ſcribatur pro Cβ, et x pro βp vel βP, erit area pqQP, ſive $\frac{2x}{n} + \frac{2x^3}{3n^3} + \frac{2x^5}{5n^5}$ &c. ad aream $pq\delta\beta$, ſive $\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3}$ &c. ut differentia inter logarithmos extremorum numerorum, ſive $2d$, ad differentiam inter logarithmos minoris et medii: quæ proinde erit $\dfrac{\frac{dx}{n} + \frac{dx^2}{2n^2} + \frac{dx^3}{3n^3}}{\frac{x}{n} + \frac{x^3}{3n} + \frac{x^5}{5n}}$ &c. hoc eſt, factâ diviſione, $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ (ϑ).

46. Hujus autem ſeriei duos primos terminos, $d + \frac{dx}{2n}$, pro canone conſtruendo ſat accuratos exiſtimo, etiamſi aduſque quatuordecim, vel forte quindecim figurarum loca logarithmi producerentur: ſi modò numerus logarithmo donandus non ſit minor quàm 1000 (ι). Quod ſanè calculum haud difficilem præbere poteſt, ſiquidem x ut plurimùm erit unitas, vel numerus binarius. Non opus eſt tamen omnia loca beneficio hujus regulæ interpolare. Nam logarithmi numerorum, qui prodeunt è multiplicatione

LOGARITHMO. (ϑ) $d + \frac{dx}{2n} + \frac{dx^3}{12n^3} + \frac{7dx^5}{180n^5} + \frac{181dx^7}{7560n^7} + \frac{1903dx^9}{113400n^9}$ &c.

(ι) ALIAM ſeriem acutiſſimus Halleius excogitavit, quæ in exiguis etiam numeris mirâ celeritate ad metam properat.

Sit $a + e$ numerus cujus logarithmus exquirendus eſt, datis numerorum a, $a + 2e$ logarithmis. Propter datos numerorum, a, $a + 2e$, logarithmos, data erit ſummæ illorum ſemiſſis, quæ dicatur A. Erit A numeri $\sqrt{a^2 + 2ae}$ logarithmus. Numerus autem $a + e$, numero $\sqrt{a^2 + 2ae}$ major erit; duorum ſcilicet, a, $a + 2e$, arithmeticè medius eorundem geometricè medio. Logarithmus igitur numeri $a + e$ logarithmo dato A major erit. Differentia ſit z. Erit igitur A + z numeri $a + e$ logarithmus, cujus datâ parte A, ſi pars reliqua z inventa fuerit, logarithmus totus definitus erit. Eſt autem z logarithmus rationis numeri $\sqrt{a^2 + 2ae}$ ad numerum $a + e$. Ponatur $y^2 = 2 \times a \times \overline{a + 2e} + e^2 = 2a^2 + 4ae + e^2$. Erit igitur $y^2 - e^2 = 2 \times \overline{a^2 + 2ae}$; et $y^2 + e^2 = 2 \times \overline{a + e}|^2$. Ergo $\sqrt{y^2 - e^2} : \sqrt{y^2 + e^2} = \sqrt{a^2 + 2ae} : a + e$. Quare z logarithmus erit rationis quam $\sqrt{y^2 - e^2}$ habet ad $\sqrt{y^2 + e^2}$, et $2z$ logarithmus rationis quam $y^2 - e^2$ habet ad $y^2 + e^2$. Jam ſit CA $= y^2$, AB et Ab $= e^2$.

tiplicatione vel diviſione numeri noviſſimè tranſacti, per numeros, quorum logarithmi priùs habebantur, obtineri poſſunt per additionem vel ſubductionem eorum logarithmorum. Quinetiam per differentias logarithmorum, et illarum differentiarum ſecundas differentias tertiaſque, ſi opus eſt, loca vacua expeditiùs impleri poſſunt, adhibitâ tantùm prædictâ regulâ, ubi ad obtinendum illas differentias continuatio aliquot locorum plenorum deſideratur. LOGARITHMOTECHNIA.

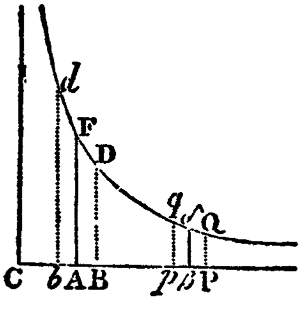
47. Eâdem methodo regulæ pro intercalatione logarithmorum inveniri poſſunt, ubi è tribus numeris dantur logarithmi minoris & medii, vel medii et majoris; idque licet numeri non ſint in arithmeticâ progreſſione.

48. Imo et hujus methodi veſtigiis inſiſtendo, regulæ, pro conſtruendis artificialium ſinuum et tangentium Tabulis, ſine adminiculo naturalium, haud difficulter depromi poſſunt. Sed hæc in tranſitu.

CAPUT DECIMUM.

1. HACTENUS Curvarum, quæ per æquationes minùs ſimplices definiuntur, quadraturam, mediante reductione in æquationes ex infinitè multis terminis ſimplicibus conſtantes, oſtendimus. Cùm vero ejuſmodi Curvæ per finitas etiam æquationes nonnunquam quadrari poſſint, vel ſaltem comparari cum aliis Curvis, quarum areæ quodammodo pro cognitis habeantur, quales ſunt ſectiones conicæ: ea propter ſequentes duos Theo-

$Ab = e^2$. Et rectangulo $CA \times AF$ exiſtente $= 1$, erit $AF = \frac{1}{y^2}$. Tum hiſce e^2, y^2, $\frac{1}{y^2}$ pro illis x, a, b in ſeriem Newtonianam ſubſtitutis, efficietur area $BDdb = \frac{2e^2}{y^2} + \frac{2e^6}{3y^6} + \frac{2e^{10}}{5y^{10}} + \frac{2e^{14}}{7y^{14}}$ &c. Et hanc ſeriem ſi in modulum ſyſtematis Briggiani multiplicaveris, logarithmum effeceris Briggianum rationis ejus quam cb habet ad CB, ſive ejus quam $y^2 - e^2$ habet ad $y^2 + e^2$, hoc eſt logarithmum $2z$. TECHNIA.



$$\text{Ergo } 2z = \frac{2e^2}{y^2} + \frac{2e^6}{3y^6} + \frac{2e^{10}}{5y^{10}} + \frac{2e^{14}}{7y^{14}} \text{ \&c. } \times 0{,}4342944 \text{ \&c.}$$

$$\text{Ergo } z = \frac{e^2}{y^2} + \frac{e^6}{3y^6} + \frac{e^{10}}{5y^{10}} + \frac{e^{14}}{7y^{14}} \text{ \&c. } \times 0{,}43429 \text{ \&c.}$$

$$\text{Et logarithmus numeri } a + e = 0{,}43429 \text{ \&c.} \times \overline{\frac{e^2}{y} + \frac{e^0}{3y^6} + \frac{e^{10}}{5y^{10}} \text{ \&c.}} + A.$$

Vel ſi e ſit unitas, quod plerumque uſuveniet, logarithmus numeri

$$a + 1 = 0{,}43429 \text{ \&c.} \times \overline{\frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^0} \text{ \&c.}} + A. \quad \text{Q. E. I.}$$

 rematum

CAPUT X. rematum Catalogos, in illum uſum ope Propoſitionis VII$^{\ae}$ et VIII$^{\ae}$, ut promiſimus, conſtructos, jam viſum eſt adjungere.

2. Horum prior exhibet areas Curvarum quæ quadrari poſſunt; et poſterior complectitur Curvas, quarum areas cum areis conicarum ſectionum conferre liceat.

3. In utriſque literæ Latinæ *d*, *e*, *f*, *g* et *h*, datas quaſvis quantitates, *x* et *z* baſes curvarum, *v* et *y* parallelè incedentes, et *s* ac *t* areas, ut ſuprà denotant. Græcæ autem, η et θ, quantitati *z* ſuffixæ denotant ejuſdem *z* dimenſionum numerum, ſive ſit integer vel fractus, ſive affirmativus aut negativus. Veluti ſi ſit $\eta=3$, erit $z^\eta=z^3$, $z^{2\eta}=z^6$, $z^{-\eta}=z^{-3}$ ſive $\frac{1}{z^3}$, $z^{\eta+1}=z^4$, et $z^{\eta-1}=z^2$.

4. Inſuper in valoribus arearum, abbreviandi causâ, ſcribitur R vice radicalis illius $\sqrt{e+fz^\eta}$, vel $\sqrt{e+fz^\eta+gz^{2\eta}}$, quâ valor incedentis *y* afficitur.

Catalogus

Catalogus Curvarum aliquot ad Rectilineas Figuras relatarum, ope Prob. VII. *Conſtructus.*

CURVARUM ORD.			AREARUM VALORES
I		$dz^{\eta-1}=y$	$\frac{d}{\eta}z^{\eta}=t$
II		$\frac{dz^{\eta-1}}{ee+2efz^{\eta}+ffz^{2\eta}}=y$	$\frac{dz^{\eta}}{\eta e^{2}+\eta efz^{\eta}}=t$, vel $\frac{-d}{\eta ef+\eta ffz^{\eta}}=t$
III	1	$dz^{\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{2d}{3\eta f}R^{3}=t$
	2	$dz^{2\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-4e+6fz^{\eta}}{15\eta f^{2}}dR^{3}=t$
	3	$dz^{3\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{16e^{2}-24efz^{\eta}+30f^{2}z^{2\eta}}{105\eta f^{3}}dR^{3}=t$
	4	$dz^{4\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-96e^{3}+144e^{2}fz^{\eta}-180ef^{2}z^{2\eta}+210f^{3}z^{3\eta}}{945\eta f^{4}}dR^{3}=t$
IV	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{2d}{\eta f}R=t$
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{-4e+2fz^{\eta}}{3\eta f^{2}}dR=t$
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{16e^{2}-8efz^{\eta}+6f^{2}z^{2\eta}}{15\eta f^{3}}dR=t$
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{-96e^{3}+48e^{2}fz^{\eta}-36ef^{2}z^{2\eta}+30f^{3}z^{3\eta}}{105\eta f^{4}}dR=t$

5. His adjiciantur ſequentia magis Generalia Theoremata, quibus via ad altiora ſternitur.

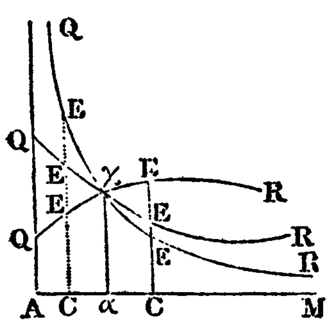
Ponatur p pro $\sqrt{h+iz^{\eta}}$

6. Poſſint

CAPUT X.

V	1	$\begin{matrix}2\theta e z^{\theta-1} + & 2\theta f\\ & +3\eta f\end{matrix} z^{\theta+\eta-1} \times \frac{1}{2}\sqrt{e+fz^{\eta}} = y$	$z^{\theta}R^{3} = t$
	2	$\begin{matrix}2\theta e z^{\theta-1} + & 2\theta f\\ & +3\eta f\end{matrix} z^{\theta+\eta-1} + \begin{matrix}2\theta g\\ +6\eta g\end{matrix} z^{\theta+2\eta-1} \text{ in } \frac{1}{2}\sqrt{e+fz^{\eta}+gz^{2\eta}} = y$	$z^{\theta}R^{3} = t$
VI	1	$\dfrac{2\theta e z^{\theta-1} + \overline{2\theta+\eta} \times f z^{\theta+\eta-1}}{2\sqrt{e+fz^{\eta}}} = y$	$z^{\theta}R = t$
	2	$\dfrac{2\theta e z^{\theta-1} + \overline{2\theta+\eta} \times f z^{\theta+\eta-1} + \overline{2\theta+2\eta} \times g z^{\theta+2\eta-1}}{2\sqrt{e+fz^{\eta}+gz^{2\eta}}} = y$	$z^{\theta}R = t$
VII	1	$\dfrac{2\theta e z^{\theta-1} + \overline{2\theta+\eta} \times f z^{\theta+\eta-1}}{e+fz^{\eta} \text{ in } 2\sqrt{e+fz^{\eta}}} = y$	$\dfrac{z^{\theta}}{R} = t$
	2	$\dfrac{2\theta e z^{\theta-1} + \overline{2\theta+\eta} \times f z^{\theta+\eta-1} + \overline{2\theta+2\eta} \times g z^{\theta+2\eta-1}}{e+fz^{\eta}+gz^{2\eta} \text{ in } 2\sqrt{e+fz^{\eta}+gz^{2\eta}}} = y$	$\dfrac{z^{\theta}}{R} = t$
VIII	1	$\dfrac{2\theta e z^{\theta-1} + \overline{2\theta+2\eta} \times f z^{\theta+\eta-1}}{ee+2efz^{\eta}+f^{2}z^{2\eta}} = 2y$	$\dfrac{z^{\theta}}{R^{2}}$ (ſive $\dfrac{z^{\theta}}{e+fz^{\eta}}$) $= t$
	2	$\dfrac{2\theta e z^{\theta-1} + \overline{2\theta+2\eta} \times f z^{\theta+\eta-1} + \overline{2\theta+4\eta} \times g z^{\theta+2\eta-1}}{e^{2}+2efz^{\eta}+\overline{f^{2}+2eg} \times z^{2\eta}+2fgz^{3\eta}+ggz^{4\eta}} = 2y$	$\dfrac{z^{\theta}}{R^{2}}$ (ſive $\dfrac{z^{\theta}}{e+fz^{\eta}+gz^{2\eta}}$) $= t$
IX		$2\theta e b z^{\theta-1} + \begin{matrix}\overline{2\theta+3\eta} \times fb\\ +\overline{2\theta+\eta} \times ei\end{matrix} z^{\theta+\eta-1} + \overline{2\theta+4\eta} \times fi z^{\theta+2\eta-1} \text{ in } \dfrac{\sqrt{e+fz^{\eta}}}{2\sqrt{b+iz^{\eta}}} = y$	$z^{\theta}R^{3}p = t$
X		$2\theta e b z^{\theta-1} + \begin{matrix}\overline{2\theta+3\eta} fb\\ +\overline{2\theta+\eta} \times ei\end{matrix} z^{\theta+\eta-1} + \overline{2\theta+2\eta} \times fi z^{\theta+2\eta-1} \text{ in } \dfrac{\sqrt{e+fz^{\eta}}}{b+iz^{\eta} \text{ in } 2\sqrt{b+iz^{\eta}}} = y$	$\dfrac{z^{\theta}R^{3}}{p} = t$

6. Poſſint et hujuſmodi alia adjici, ſed ad alterius generis Curvas, quæ cum conicis ſectionibus conferri poſſunt, jam tranſeo.



Et in hoc Catalogo expoſitam curvam lineâ QEχR deſignatam habes, cujus baſis principium ſit A; baſis AC, parallelè incedens CE, areæ principium $\alpha\chi$, et area deſcripta $\alpha\chi\varepsilon$C. Ejus autem areæ principium ſive terminus initialis (quod ut plurimùm vel baſis principio A inſiſtit, vel ad infinitam diſtantiam recedit) invenitur, quærendo baſis longitudinem, Aα, cùm areæ valor nullus eſt, et erigendo normalem $\alpha\chi$.

(x) Colſoni veriſſimam emendationem ſequor. Codices MSS. habent 868, Vitioſè. Confer Librum de Quad. Curv. not. (co).

7. A'

7. Ad eundem modum conicam ſectionem habes deſignatam lineâ SDG, (*Vide Librum, De Quad. Curv. Tab.* II. *fig.* 1, 2, 3, 4.) cujus centrum ſit A, vertex a, rectangulæ ſemidiametri Aa & AP: baſis principium A, vel a, vel α; baſis AB, vel aB, vel αB: ordinatim applicata BD, tangens TD occurrens AB in T, ſubtenſa aD, et inſcriptum vel aſcriptum rectangulum ABDO. De Tabula Curvarum Secunda.

8. Itaque retentis jam antè definitis literis, erit $AC=z$, $CE=y$, $\alpha\chi EC=t$, AB vel $aB=x$, $BD=v$ & ABDP vel $aGDB=s$. Et prætereà, ſiquando ad alicujus areæ determinationem duæ conicæ ſectiones requiruntur, poſterioris area dicetur σ, baſis ξ, et parallelè incedens Υ.

Catalogus Curvarum aliquot ad Conicas ſectiones relatarum ope Prob. VIII. *conſtructus.* Vide Tab. II. Libri de Quad. Curv.

9. Antequam Theoremata, in his Curvarum claſſibus tradita, exemplis illuſtrare pergam, juvabit obſervare

I. Quòd cùm quantitatum d, e, f, g, h & i ſigna omnia, in æquationibus curvas definientibus, affirmativa poſuerim; ſiquando contingant eſſe negativa, in ſubſequentibus baſis et incedentis lineæ ſectionis conicæ, necnon quæſitæ areæ valoribus mutari debent.

II. Numeralium η & θ, ubi negativæ ſunt, ſigna in arearum valoribus ſunt etiam mutanda. Quinetiam ipſarum ſignis mutatis Theoremata novam formam induere poſſunt. Sic in quartâ formâ poſterioris Catalogi, Theorema tertium, ſigno ipſius η mutato, evadit $\frac{d}{z^{-2\eta+1}\sqrt{c+fz^{-\eta}}}=y$, $\frac{1}{z^{-\eta}}=x$ &c. hoc eſt $\frac{dz^{3\eta-1}}{\sqrt{ez^{2\eta}+fz^{\eta}}}=y$, $z^{\eta}=x$, $\sqrt{fx+ex^2}=v$, $\frac{d}{\eta e}$ in $2xv-3s=t$, & ſic in aliis.

III. Cujuſque ordinis (ſi ſecundum prioris Catalogi demas) ſeries utrinque in infinitum continuari poteſt. Scilicet in tertii quartique ordinis ſeriebus prioris Catalogi, numeri coefficientes initialium terminorum, 2, −4, 16, −96, 768 (λ) &c. generantur multiplicando numeros −2, −4, −6, −8, −10, &c. in ſe continuò (λ); et ſubſequentium terminorum coefficientes ex initialibus in tertio ordine derivantur, multiplicando gradatim per $-\frac{3}{2}, -\frac{5}{4}, -\frac{7}{6}, -\frac{9}{8}, -\frac{11}{10}$, &c. vel in quarto ordine, multiplicando per $-\frac{1}{2}, -\frac{3}{4}, -\frac{5}{6}, -\frac{7}{8}, -\frac{9}{10}$, &c. Denominatorum verò coefficientes, 1, 3, 15, 105, &c.

(λ) Vide Librum de Quadratura Curvarum, not. (m).

ex

CAPUT X. ex ductu numerorum 1, 3, 5, 7, 9, &c. in ſe gradatim oriuntur.

In ſecundo autem Catalogo ſeries ordinum, Primi, Secundi, Quinti, Sexti, Noni & Decimi, ope ſolius diviſionis infinitè producuntur. Sic habito $\frac{dz^{4\eta-1}}{e+fz^{\eta}}=y$, ſi diviſionem ad uſque convenientem periodum inſtituas, orietur E. g. $\frac{d}{f}z^{3\eta-1}-\frac{de}{ff}z^{2\eta-1}+\frac{de^2}{f^3}z^{\eta-1}-\frac{\frac{de^3}{f^3}z^{\eta-1}}{e+fz^{\eta}}=y$. Priores tres termini ſunt primi ordinis prioris Catalogi, et quartus primæ ſpeciei hujus ordinis (μ). Unde conſtat aream valere $\frac{d}{3\eta f}z^{3\eta}-\frac{de}{2\eta f^2}z^{2\eta}+\frac{de^2}{\eta f^3}z^{\eta}-\frac{e^3}{\eta f^3}s$; poſitâ nempe s areâ ſectionis conicæ, cujus baſis x ſit $=z^{\eta}$, et incedens applicata $v=\frac{d}{e+fx}$.

Tertii autem ſeptimique ordinis ſeries, ope duorum Theorematum in quinto ordine prioris Catalogi, per debitam additionem vel ſubductionem infinitè producuntur; ut et quarti octavique ſeries, ope theorematum in ſubſequenti ſexto ordine; ac undecimi ſeries ope theorematis in decimo ordine ejuſdem prioris catalogi. E. G. Si præfati tertii ordinis ſeries ultrà producenda ſit, finge $\theta=-4\eta$; & quinti ordinis alterius Catalogi Theorema primum evadet, $-8\eta ez^{-4\eta-1}-5\eta fz^{-3\eta-1}$ in $\frac{1}{2}\sqrt{e+fz^{\eta}}=y$. $\frac{R^3}{z^{4\eta}}=t$. Eſt autem juxta quartum theorema hujus producendæ ſeriei, (ſcripto $-\frac{5\eta f}{2}$ pro d) $-\frac{5\eta f}{2}z^{-3\eta-1}\sqrt{e+fz^{\eta}}=y$. $\frac{1}{z^{\eta}}=x$. $\sqrt{fx+ex^2}=v$. & $\frac{10fv^3-15f^2s}{12e}=t$. Quare ſubductis prioribus ipſarum y, ac t, valoribus, reſtabunt $4\eta ez^{-4\eta-1}\sqrt{e+fz}=y$. et $\frac{10fv^3-15f^2s}{12e}-\frac{R^3}{z^{4\eta}}=t$. Ipſiſque in $\frac{d}{4\eta e}$ ductis, et pro $\frac{R^3}{z^{4\eta}}$ ſcripto ſi placet xv^3, emerget quintum producendæ ſeriei Theorema, $\frac{d}{z^{4\eta+1}}\sqrt{e+fz^{\eta}}=y$. $\frac{1}{z^{\eta}}=x$. $\sqrt{fx+exx}=v$, & $\frac{10dfv^3-15df^2s}{48\eta e^2}-\frac{dxv^3}{4\eta e}=t$.

IV. Horum ordinum nonnulli ex aliis etiam poſſunt aliter derivari: utpote in poſteriori Catalogo Tertius, Quartus, Septimus et Undecimus, ab Octavo, ac Nonus à Decimo. Adeo ut omiſſiſſe potuiſſem, niſi quòd uſui eſſe poſſint, quamvis non prorſus neceſſarii. Nonnullos tamen ordines omiſi, quos à primo et ſe-

(μ) Intellige, primæ ſpeciei primi ordinis poſterioris Catalogi.

cundo, nec non à nono decimoque derivaſſe potuiſſem; utpote qui denominatoribus magis compoſitis afficiuntur, & proinde vix ulli unquam uſui eſſe poſſunt. De Tabula Curvarum Secunda.

V. Si Curvæ alicujus definiens æquatio ex pluribus æquationibus diverſorum ordinum, vel diverſarum ſpecierum ejuſdem ordinis componatur, ejus aream ex areis correſpondentibus componere oportet: cavendo tamen ut ſignis + et − rectè connectantur. Nam parallelè incedentes parallelè incedentibus, et areæ correſpondentes correſpondentibus areis non ſemper ſunt ſimul addendæ, vel ſimul ſubducendæ; ſed aliquando harum ſumma, et illarum differentia ſumenda eſt pro novâ lineâ incedente, et areâ correſpondente, conſtituendâ. Et hoc fieri debet, cùm conſtituentes areæ poſitæ ſunt ad diverſam partem parallele incedentis. Ut autem hoc incommodum cauti promptiùs devitare poſſint, ſingulis arearum valoribus propria ſigna, etiamſi nonnunquam negativa, ut fit in poſterioris Catalogi tertio quartoque ordine præfixi.

VI. De arearum ſignis obſervandum eſt præterea, quod + *s* vel denotat aream conicæ ſectionis, baſi adjacentem, eſſe reliquis quantitatibus in valore *t* addendam (vide Exem. 1. ſequ.) vel aream ex alterâ parte ordinatim applicatæ eſſe ſubducendam. Et contrà, − *s* ambiguè denotat aream, baſi adjacentem, eſſe ſubducendam, vel aream, ex alterâ parte ordinatim applicatæ, eſſe addendam; prout commodum videbitur. Deinde valor ipſius *t*, ſi affirmativus prodierit, deſignat aream Curvæ propoſitæ adjacentem baſi ejus; et contrà, ſi fuerit negativus, deſignat aream ex alterâ parte ordinatim applicatæ.

VII. Cæterùm ut area illa certiùs definiatur, proſpiciendum eſt de limitibus ejus. Et quidem limitum ad baſin, parallelè incedentem, et Curvæ perimetrum, nulla poteſt eſſe incertitudo: ſed limes initialis, ſive principium à quo incipit deſcriptio ejus, varias poſitiones obtinet. In ſequentibus exemplis vel eſt ad initium baſis, vel ad infinitam diſtantiam; vel in concurſu Curvæ cum baſi ejus. Sed poteſt alibi locari. Et ubicunque ſit, invenies quærendo illam baſis longitudinem, ad quam valor ipſius *t* evadit nullus; et parallelè incedentem erigendo. Nam erecta illa linea erit limes quæſitus.

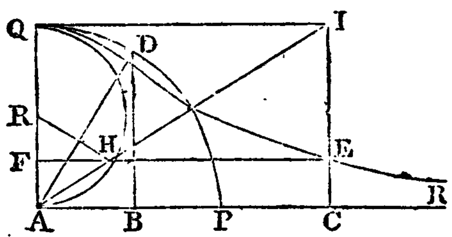
(*r*) Vide Analyſin per Æquationes numero terminorum infinitas, C. II. § 6.

CAPUT X. VIII. Siqua pars areæ infra baſin poſita ſit, t deſignabit differentiam ejus et partis ſupra baſin.

IX. Siquando dimenſiones terminorum in valoribus x, v & t nimis altæ vel nimis depreſſæ obvenerint, ad juſtum gradum liceat reducere, dividendo vel multiplicando toties per datam quamvis quantitatem, quæ vices unitatis gerere fingitur, quoties dimenſiones illæ ſint juſto altiores, vel depreſſiores.

X. Præter præcedentes Catalogos poſſunt etiam catalogi Curvarum ad alias curvas, in ſuo genere ſimpliciſſimas, (ut ad $\sqrt{e+fx^3}=v$, vel ad $x\sqrt{e+fx^3}=v$, vel ad $\sqrt{e+fx^4}=v$, &c.) relatarum conſtrui; eò ut Curvæ cujuſlibet propoſitæ aream ex origine ſimpliciſſimâ poſſimus derivare, et cum quibus Curvis affinitatem habeat cognoſcere. Cæterùm præcedentes tandem Exemplis aliquot illuſtremus.

10. EXEMPL. 1. Sit QER ejuſmodi Conchoidalis, ut, ſemicirculo QHA deſcripto, et ad diametrum AQ erecto AC perpendiculo; ſi compleatur parallelogrammum QACI, et agatur diagonalis AI ſemicirculo occurrens in H, et ab H demittatur ad IC normalis HE, punctum E incidat in Curvam; et quæratur area ACEQ.



Dic itaque $AQ=a$, $AC=z$, $CE=y$; et propter continuè proportionales AI, AQ, AH, EC, erit EC, ſive $y, = \frac{a^3}{a^2+z^2}$.

Jam ut hæc induat formam æquationum in Catalogis, finge $\eta=2$, & pro z^2 in denominatore ſcribe z^η, ac $a^3z^{\frac{1}{2}\eta-1}$ pro a^3, ſive a^3z^{1-1}, in numeratore, et emerget $y=\frac{a^3z^{\frac{1}{2}\eta-1}}{a^2+z^\eta}$, æquatio primæ ſpeciei ſecundi ordinis poſterioris Catalogi; collatiſque terminis, fiet $d=a^3$, $e=a^2$ & $f=1$. Adeoque $\sqrt{\frac{a^3}{a^2+z^2}}=x$, $\sqrt{a^3-a^2x^2}=v$, & $xv-2s=t$.

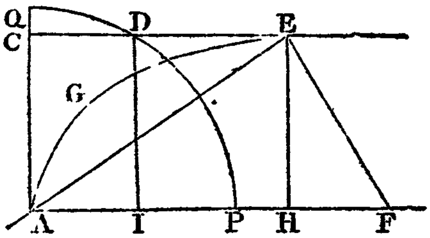
Ut autem inventi valores x & v, ad juſtum dimenſionum numerum reducantur: ſelige datam quamlibet quantitatem, velut a; per quam, tanquam unitatem, ſemel multiplicetur a^3 in valore x, et in valore v dividatur a^3 ſemel, et a^2x^2 bis; et hoc pacto obtinebis $\sqrt{\frac{a^4}{a^2+x^2}}=x$, $\sqrt{a^2-x^2}=v$, & $xv-2s=t$. Quorum conſtructio eſt ejuſmodi.

Centro

Centro A intervallo AQ, deſcribe quadrantem circuli QDP. In AC cape AB=AH. Erige normalem BD quadranti occurrentem in D, et age AD; et ſectoris ADP duplum æquabitur areæ quæſitæ ACEQ. Eſt enim $\sqrt{\frac{a^4}{a^2+z^2}}$ $(=\sqrt{AQ \times EC}=HA)=AB$ ſive x: et $\sqrt{a^2-x^2}$ $(=\sqrt{AD^2-AB^2})=BD$ ſive v; et $xv-2s=2\Delta ADB-2ABDQ$; vel etiam $=2\Delta ADB+2BDP$, hoc eſt, vel$=-2QAD$, vel$=2DAP$: quorum valorum affirmativus 2DAP competit areæ ACEQ citra EC, et negativus 2QAD competit areæ RECR, ultra EC, in infinitum protenſæ. Curvæ cum Conicis Comparandæ.

11. Solutiones problematum ſic inventæ nonnunquam concinnari poſſunt. Sic in hoc caſu, actâ RH circuli QHA ſemidiametro, propter arcus QH, DP æquales, erit ſector QRH dimidium ſectoris DAP; atque adeo pars quarta ſuperficiei ACEQ. Nam ſi à puncto D demittatur perpendicularis ad AQ erit iſta ſinus anguli QAD in quadrante QDP. Erit etiam, ob parallelas DB, QA, æqualis AB=AH chordæ ang. (ξ) Conſequenter arcus QD in quadrante = arcui AH in ſemicirculo: unde conſtat arcus QH, DP æquales eſſe.

12. Exempl. 2. Sit AGE Curva, quam normæ, AEF, punctum angulare, E, deſcribit, dum crurum alterum, AE, interminatum continuò tranſit per datum punctum A, & alterum EF, datæ longitudinis, ſuper rectâ AF, poſitione datâ, prolabitur. Demitte EH ad AF normalem, et comple parallelogrammum AHEC, ac dictis AC$=z$, CE$=y$, & EF$=a$, propter HF, HE, HA continuò proportionales, erit HA ſive $y=\frac{z^2}{\sqrt{a^2-z^2}}$.



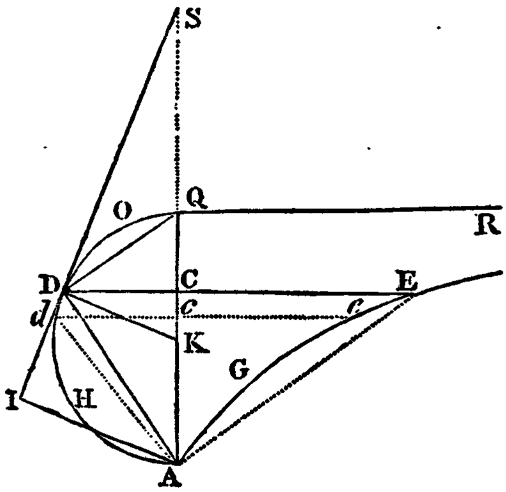
Jam ut innoteſcat area AGEC, finge $z^2=z^\eta$ ſive $2=\eta$; et inde fiet $\frac{z^{3\eta-1}}{\sqrt{a^2-z^\eta}}=y$. Ubi cùm z ſit fractæ dimenſionis in numeratore, deprime valorem y, dividendo per $z^{1\eta}$, et fiet $\frac{z^{\eta-1}}{\sqrt{a^2z^{-\eta}-1}}=y$: æquatio ſecundæ ſpeciei quarti ordinis poſterioris Catalogi. Ac terminis collatis evadet $d=1$, $e=-1$ et $f=a^2$. Adeoque z^2 $(=\frac{1}{z^{-\eta}})$ $=x^2$. $\sqrt{a^2-x^2}=v$, & $s-xv=t$. Cùm itaque x et z æquentur, et ſit $\sqrt{a^2-x^2}=v$, æquatio ad circulum cujus diameter eſt a: centro A intervallo a, ſive EF, deſcribatur circulus PDQ, cui occurrat CE

(ξ) Anguli QHA in ſemicirculo QHA.

 in

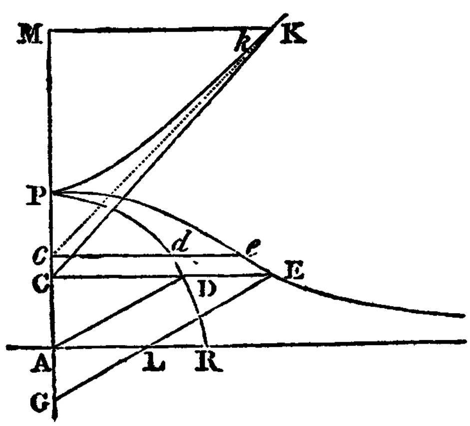
Caput X. in D, et compleatur parallelogrammum ACDI; eritque AC$=z$, CD$=v$ & area quæſita AGEC $(=s-xv=$ ACDP$-$ACDI$)=$ IDP.

13. Exempl. 3. Sit AGE Ciſſois, ad circulum ADQ, diametro AQ deſcriptum, pertinens. Agatur DCE diametro normalis, et Curvis occurrens in D et E. Et nominatis AC$=z$, CE$=y$, & AQ$=a$; propter CD, CA, CE continuè proportionales, erit CE, ſive $y,=\frac{zz}{\sqrt{az-zz}}$; ac dividendo per z, fit $y=\frac{z}{\sqrt{az^{-1}-1}}$. Eſt itaque $z^{-1}=z^{\eta}$ ſive $-1=\eta$; et inde $y=\frac{z^{-2\eta-1}}{\sqrt{az^{\eta}-1}}$; æquatio tertiæ ſpeciei quarti ordinis poſterioris



Catalogi. Collatiſque terminis fit $d=1$, $e=-1$, & $f=a$. Adeoque $z\ (=\frac{1}{z^{\eta}})=x$, $\sqrt{ax-xx}=v$, & $3s-2xv=t$. Quare eſt AC $=x$, CD$=v$, et inde ACDH$=s$, adeoque 3ACDH-4ΔADC $(=3s-2xv=t)$ $=$ areæ ciſſoidali ACEGA. Vel quod perinde eſt, 3 ſegment. ADHA $=$ areæ ADEGA, ſive 4 ſegment. ADHA$=$areæ AHDEGA.

14. Exempl. 4. Eſto PE Prima Conchoides Veterum, centro G, aſymptoto AL, et intervallo LE deſcripta; age GAP axem ejus, ac demitte EC ordinatim ap-



plicatam: dictiſque AC$=z$, CE$=y$, GA$=b$ &, AP$=c$; propter proportionales AC:CE$-$AL::GC:CE, erit CE ſive $y=\frac{b+z}{z}\sqrt{c^2-z^2}$ (*). Jam ut ejus area PEC exhinc inveniatur, partes applicatæ CE, ſeorſim conſiderandæ ſunt. Et quidem ſi illa CE ita dividatur in D, ut ſit CD$=\sqrt{c^2-z^2}$, ac DE $=\frac{b}{z}\sqrt{c^2-z^2}$ erit CD ordinatim applicata circuli centro A et intervallo AP deſcripti: adeoque pars areæ PDC innoteſcet, et reſtabit pars altera DPED invenienda.

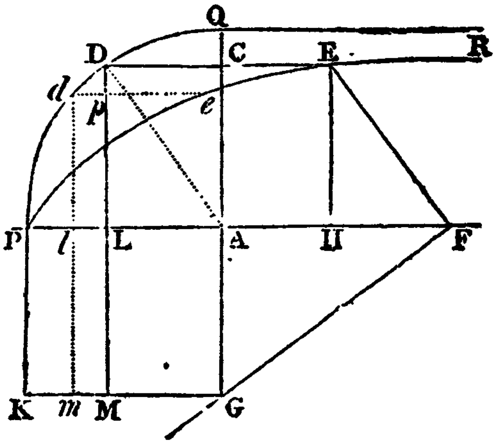
Cùm

(*) Propter EL : EG $=$ AC : CG, erit EG $=\frac{\text{EL}\times\text{CG}}{\text{AC}}$, hoc eſt EG $=c\times\frac{b+z}{z}$. Quare EG$^2=$

Cùm itaque DE (pars applicatæ quâcum deſcribitur) valeat $\frac{b}{z}\sqrt{c^2-z^2}$ ſuppone $2=\eta$, et evadet $\frac{b}{z}\sqrt{c^2-z^\eta}=$ DE, æquatio primæ ſpeciei tertii ordinis poſterioris Catalogi. Collatiſque terminis fiet $d=b$, $e=c^2$ et $f=-1$. Atque adeo $\frac{1}{z}=\sqrt{\frac{1}{z^\eta}}=x$; $\sqrt{-1+c^2x^2}=v$, et $2bc^2s-\frac{bv^3}{x}=t$. His inventis, redige ad juſtum dimenſionum numerum, multiplicando terminos nimis depreſſos, ac dividendo nimis altos per datam quamvis quantitatem. Id quod ſi fiet per c, prodibit $\frac{c^2}{z}=x$, $\sqrt{-c^2+x^2}\pm v$, et $\frac{2bs}{c}-\frac{bv^3}{cx}=t$. Et horum conſtructio eſt ejuſmodi. CURVÆ CUM CONICIS COMPARANDÆ.

Centro A, vertice principali P & parametro 2AP Hyperbolam PK deſcribe: deinde à puncto C age rectam CK quæ tangat hyperbolam in K, et erit ut AP ad 2AG ita area CKPC ad aream quæſitam DPED.

15. EXEMPL. 5. Normâ GFE ita circa polum G rotante, ut ejus punctum angulare F ſuper rectâ AF, poſitione datâ, continuò prolabatur: concipe Curvam PE à puncto quolibet E in crure EF ſito deſcribi. Jam ut inveniatur hujus area, demitte GA & EH ad rectam AF perpendiculares; et, completo parallelogrammo AHEC, dic AC$=z$, CE$=y$, AG$=b$ & EF$=c$: et propter proportionales HF:EH::AG:AF, erit AF $=\frac{bz}{\sqrt{c^2-z^2}}$: adeoque CE, ſive y, $=\frac{bz}{\sqrt{c^2-z^2}}-\sqrt{c^2-z^2}$. Cùm autem $\sqrt{c^2-z^2}$ ſit ordinatim applicata circuli, ſemidiametro c deſcripti circa centrum A, deſcribe talem circulum PDQ; eique CE producta occurrat in D, et erit



DE $=\frac{bz}{\sqrt{c^2-z^2}}$; cujus æquationis ope reſtat area PDEP, vel DERQ, determinanda. Supponatur ergo $\eta=2$ et evadet DE $=\frac{bz^{\eta-1}}{\sqrt{c^2-z^\eta}}$ æquatio primæ ſpeciei quarti ordinis prioris Catalogi: et collatis terminis fiet $b=d$, $c^2=e$ & $-1=f$; adeoque $b\sqrt{c^2-z^2}\,(=-b\text{R})=t$.

EG$^2=\overline{\frac{b+z}{z}}\Big|^2\times c^2$. Sed CG$^2=\overline{\frac{b+z}{z}}\Big|^2\times z^2$. Quare CE$^2=$ EG$^2-$CG$^2=\overline{\frac{b+z}{z}}\Big|^2\times c^2-z^2$. Et CE $=\frac{b+z}{z}\sqrt{c^2-z^2}$.

Jam

Jam cùm valor t negativus exiſtat, et inde area per t deſignata jaceat ultra lineam DE, ut ejus limes initialis inveniatur, quære illam ipſius z longitudinem, quâ t evadit nulla: et invenies eſſe c. Quare produc AC ad Q ut ſit AQ$=c$, et erige applicatam QR, et erit DQRED area illa cujus valor jam inventus eſt $=-l\sqrt{c-z^2}$. Quòd ſi quantitatem areæ PDE juxta baſin AC poſitæ, et cum eâ coextenſæ deſideres, poſſis, ignoto limite QR, ſic determinare. A valore quem t ad baſis longitudinem AC ſortita eſt, ſubduc valorem ejus ad initium baſis hoc eſt, $a-b\sqrt{c^2-z^2}$ ſubduc $-bc$, et proveniet quantitas $bc-b\sqrt{c^2-z^2}$ quam quæris. Comple ergo parallelogrammum PAGK, et ad AP demitte normalem DM, quæ cum GK occurrat in M, et erit parallelogrammum PMNK æquale areæ PDE.

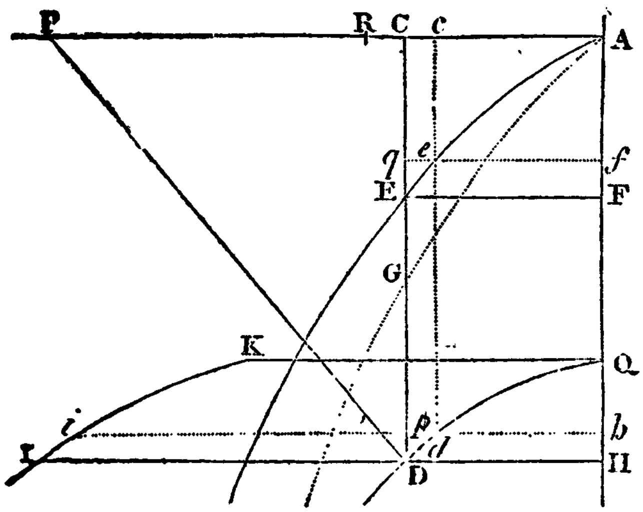
16. Siquando æquatio Curvam aliquam definiens non reperiatur in Catalogis, neque ad ſimpliciores terminos ope diviſionis, vel alio pacto, reduci poſſit, transformanda eſt in alias affinium Curvarum æquationes, pro more in Prob. VIII. oſtenſo; donec tandem obvenerit aliqua, cujus area ex catalogis innoteſcat. Et conatibus omnimodò inſtitutis, ſi nulla talis obveniat, certum eſt Curvam propoſitam neque cum figuris rectilineis, neque cum conicis ſectionibus, comparari poſſe.

17. Ad eundem modum cùm de Curvis Mechanicis agitur, illæ imprimis transformandæ ſunt in æquales Geometricas, prout in eodem Prob. VIII. oſtenſum fuit, ac deinde Geometricarum areæ ex Catalogis eliciendæ. Cujus rei accipe ſequens exemplum.

18. Exempl. 6. Proponatur figura arcuum cujuſvis conicæ ſectionis ad ſinus rectos applicatorum determinanda. Utpote ſit A centrum conicæ ſectionis, AQ et AR ſemiaxes, CD ordinatim applicata ad axem AR, et PD perpendiculum ad punctum D. Sit etiam AE dicta figura mechanica occurrens CD in E, et ex ejus naturâ præfinita erit CE$=$ arcui QD. Quæritur itaque area AEC, vel parallelogrammo ACEF completo, quæritur exceſſus AEF. In quem finem ſit a latus rectum conicæ ſectionis et b latus tranſverſum ſive 2AQ; ſit etiam AC$=z$ & CD$=y$ eritque $\sqrt{\frac{1}{4}b^2+\frac{b}{a}z^2}=y$

(c) Hamilton. Conic. Lib. 2. Prop. XXIV.

æquatio



æquatio ad conicam ſectionem, ut notum eſt. Erit etiam PC $= \frac{b}{a} z$ ( ) et inde PD$= \sqrt{\frac{1}{4}b^2 + \frac{b^2+ab}{a^2} z^2}$. *Curvæ cum Conicis Comparandæ.*

Atque adeo cùm ſit fluxio arcûs QD ad fluxionem baſis AC, ut PD ad CD, ſi fluxio baſis ſupponatur 1, erit arcûs illius QD ſive applicatæ CE fluxio = $\sqrt{\dfrac{\frac{1}{4}b^2 + \frac{b^2+ab}{a^2} z^2}{\frac{1}{4}b^2 + \frac{b}{a} z^2}}$. Hanc duc in FE, ſive z, et proveniet $z\sqrt{\dfrac{\frac{1}{4}b^2 + \frac{b^2+ab}{aa} z^2}{\frac{1}{4}b^2 + \frac{b}{a} z^2}}$ fluxio areæ AEF; adeoque ſi in applicatâ DC capias CG $= z\sqrt{\dfrac{\frac{1}{4}b^2 + \frac{b^2+ab}{a^2} z^2}{\frac{1}{4}b^2 + \frac{b}{a} z^2}}$, area AGC, quam illa CG ſuper AC incedens deſcribet, æquabitur areæ AEF; et erit AG Curva Geometrica. Quæritur itaque area AGC. Et in hunc finem ſubſtituatur z^n pro z^2 in æquatione noviſſimâ, et evadet $z^{n-1}\sqrt{\dfrac{\frac{1}{4}b^2 + \frac{b^2+ab}{a^2} z^n}{\frac{1}{4}b^2 + \frac{b}{a} z^n}}$ = CG; æquatio ſecundæ ſpeciei undecimi ordinis poſterioris Catalogi. Et collatis útrobique terminis, fit $d = 1$, $e = \frac{1}{4}b^2 = g$, $f = \frac{b^2+ab}{a^2}$, et $h = \frac{b}{a}$; adeoque $\sqrt{\frac{1}{4}b^2 + \frac{b}{a} z^2} = x$ et $\sqrt{-\frac{b^3}{4a} + \frac{a+b}{a} x^2} = v$ & $\frac{a}{b} s = t$. Hoc eſt CD$=x$, DP$=v$ et $\frac{a}{b} s = t$. Et inventorum talis eſt conſtructio.

Ad Q erige QK perpendicularem et æqualem QA, et huic parallelam, æqualem verò DP, age HI per punctum D. Et linea KI, in quam HI terminatur, erit ſectio conica, areaque comprehenſa HIKQ ad aream quæſitam AEF, ut b ad a, ſive ut PC ad AC.

CAPUT X.

19. Nota ſi mutes ſignum b, ſectio conica cujus arcui recta CG æquatur, evadet Ellipſis: et præterea ſi fiat $b=-a$ Ellipſis evadet Circulus: in quo caſu linea KI fit recta parallela AQ.

20. POSTQUAM Curvæ alicujus area ſic inventa fuerit, de conſtructionis demonſtratione conſulendum eſt, quâcum ſine computo algebraico, quantùm liceat, contextâ ornetur Theorema, ut evadat publicæ notitiæ dignum (ϛ). Eſtque demonſtrandi methodus generalis, quam ſequentibus exemplis illuſtrare conabor.

21. *Demonſtratio conſtructionis in Exemplo* 5.

In arcu PQ (vide fig. p. 493.) ſume punctum d proximum ad D, et age dc ac dm parallelas DE ac DM, et occurrentes DM et AP in p et l: et erit DEed momentum areæ PDEP, et LMml momentum areæ LMKP. Age ſemidiametrum AD, et concipe indefinitè exiguum arcum Dd eſſe inſtar rectæ, et triangula Dpd et ALD erunt ſimilia: adeoque Dp : pd : : AL : LD. Eſt autem HF : EH : : AG : AF; hoc eſt, AL : LD : : ML, DE. Et proinde Dp : pd : : ML : DE. Quare Dp × DE $=$ pd × ML. Hoc eſt, momentum DEed æquale momento LMml. Et cùm hoc de quibuſlibet contemporaneis momentis indeterminatè demonſtratur, patet ſingula momenta areæ DPEP eſſe ſingulis contemporaneis momentis areæ PLMK æqualia, adeoque totas areas ex iſtis momentis compoſitas æquari. Q. E. D.

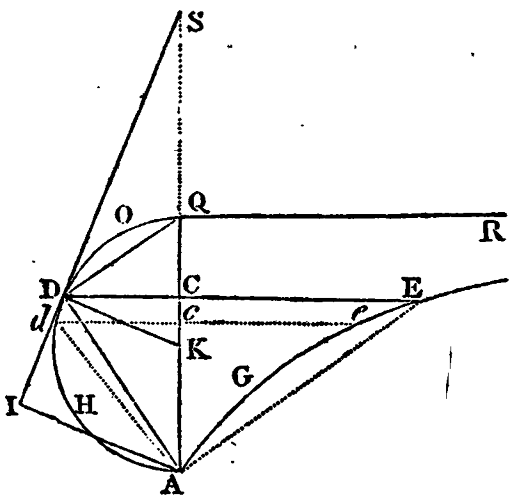
22. *Demonſtratio conſtructionis in Exemplo* 3.

Eſto DEed momentum ſuperficiei AHDE ac AdDA contem-

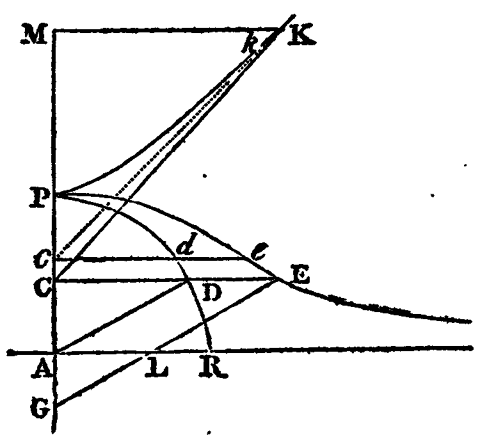
(ϛ) "Et verò non omnis effectio geometrica concinna eſt. Singula enim Problemata ſuas habent elegantias: verùm ea cæteris antefertur, quæ compoſitionem operis non ex æqualitate, ſed æqualitatem ex compoſitione arguit, et demonſtrat; ipſa verò ſe ipſam. Itaque artifex Geometra, quanquam analyticam edoctus, illud diſſimulat; & tanquam de opere efficiundo cogitans profert ſuum ſyntheticum problema, et explicat: deinde, Logiſtis auxiliaturus, de proportione vel æqualitate in eo agnitâ concipit et demonſtrat Theorema." *Vieta. Iſagog. Cap.* VII. *De Officio Rheticês.*

"Monemus tantùm Viros Clariſſimos, ut, ſepoſitis tantiſper ſpeciebus analyſeos, problemata Geometrica viâ Euclideanâ et Apollonianâ exequantur; ne pereat paulatim elegantia et conſtruendi et demonſtrandi, cui præcipuè operam dediſſe Veteres innuunt ſatis et Data Euclidis, & alii à Pappo enumerati Analyſeos Libri." *Fermatius in Epiſtolâ quâdam ſuâ ad Kenelm Digby*, quæ ſeptima eſt â quadrageſimâ in Commercio Epiſtolico Walliſii.

poraneum momentum ſegmenti ADH. Age ſemidiametrum DK, et *de* occurrat AQ in *c*, eritque $Cc : Dd :: DC : DK$ (σ). Præterea eſt $DC : QA$ (2DK) $:: AC : DE$ (τ). Adeoque $Cc : 2Dd$ ($:: DC : 2DK$) $:: AC : DE$; et $Cc \times DE = 2Dd \times AC$. Jam ad peripheriæ momentum D*d* rectâ productum, i. e. ad tangentem circuli, demitte normalem AI, et erit AI æqualis AC (υ), adeoque $2Dd \times AC = 2Dd \times AI = 4$ triangulis AD*d*. Quare 4 triang. $ADd = Cc \times DE =$ momento DE*ed*. Spatii ergo AHDE ſingula momenta ſunt quadrupla momentorum contemporaneorum ſegmenti ADH, et proinde totum illud ſpatium quadruplum totius ſegmenti. Q. E. D.



23. *Demonſtratio conſtructionis in Exemplo* 4.



Parallelam CE age indefinitè parùm diſtantem *ce*, et hyperbolæ tangentem C*k*, ac demitte KM rectam ad AP: et ex Hyperbolæ naturâ erit $AC : AP :: AP : AM$ (φ). Adeoque $AG^q : GL^q$ ($:: AC^q : LE^q$ ſive AP^q) $:: AP^q : AM^q$: ac diviſim $AG^q : AL^q$ (DE^q) $:: AP^q : AM^q - AP^q$ (MK^q) et inverſè (χ) $AG : AP :: DE : MK$. Eſt autem areola DE*ed* ad triangulum

(σ) Excerpt. ex Epiſt. not. (k) § 5.

(τ) E naturâ Ciſſoideos $QC : CD = CA : CE$. Quare $QA : DE = QC : CD$. (El. v. 12.) Sed propter angulos QDA, QCD rectos, $QC : CD = CD : CA$. Quare $QA : DE = CD : AC$. (El. v. 11.) Permutando $QA : CD = DE : AC$. Invertendo $CD : QA = AC : DE$.

(υ) Namque angulus ADI angulo AQD æqualis eſt (El. III. 32.) et angulus AQD æqualis eſt angulo ADC. (El. VI. 8.) Anguli igitur ADI, ADC inter ſe æquales ſunt. Sed et ACD, AID inter ſe æquales. Rectus enim uterque. Et triangulis ADC, AID latus AD eſt commune. Quare et reliqua eorum latera ſingulatim inter ſe æqualia erunt; eà nempe lege, ut æqualia latera æqualibus angulis obverſa ſint. (El. I. 26.) Quare latera AI, AC inter ſe æqualia. Q. E. D.

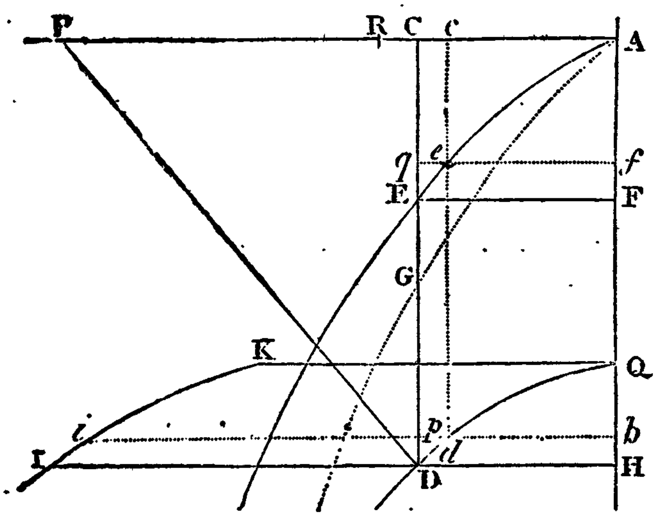
(φ) Hamilton. Conic. I. 49.

(χ) *Permutando* potiùs.

CAPUT X. CK*c*, ut altitudo DE ad ſemiſſem altitudinis KM: hoc eſt, ut AG ad $\frac{1}{2}$AP. Quare omnia ſpatii PDE momenta ad omnia contemporanea momenta ſpatii PKC ſunt ut AG ad $\frac{1}{2}$AP. Et proinde tota illa ſpatia ſunt in eâdem ratione. Q. E. D.

24. *Demonſtratio conſtructionis in Exemplo* 6.

Parallelam & proximam CD age *cd* occurrentem Curvæ AE in *e*. Age *bi* et *fe* occurrentes DC in *p* et *q*. Et erit ex hypotheſi D*d*=E*q*: et ex ſimilitudine triangulorum D*pd*, DCP, erit D*p*:D*d*(E*q*)::CP:PD(HI). Adeoque D*p*×HI=E*q*×CP; et inde D*p*×HI (moment. HI*ib*): E*q*×AC (moment. EF*fe*)::E*q*×CP:E*q*×AC::CP:AC. Quare cùm PC et AC ſint



in datâ ratione lateris tranſverſi ad latus rectum conicæ ſectionis QD(ψ), et arearum HIKQ et AEF momenta, HI*ib* et EF*fe*, in illâ ratione, erunt ipſæ areæ in eâdem ratione. Q. E. D.

25. In hujuſmodi demonſtrationibus obſervandum eſt, quòd quantitates pro æqualibus habeo, quarum ratio eſt æqualitatis. Et ratio æqualitatis cenſenda eſt, quæ minùs differt ab æqualitate quàm quælibet inæqualis ratio poteſt aſſignari. Sic in poſtremâ demonſtratione, poſui rectangulum E*q*×AC ſive FE*qf* æquale ſpatio FE*ef*; quia, propter differentiam E*qe* infinitè minorem ipſis, ſive reſpectu

(ψ) Hamilton Conic. (ω) Indiviſibilium.

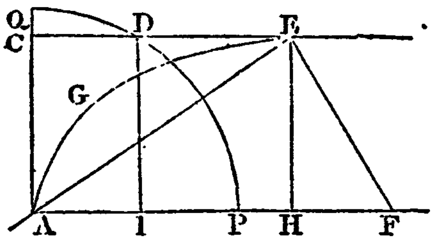
(αα) Et per § 4. Introduct. ad Quadrat. Curv.

(ϐϐ) Junctâ enim AD, cùm triangula duo rectangularia, AID, FHE, hypotenuſas AD, FE inter ſe æquales habeant, necnon latera EH, ID inter ſe æqualia, erunt et tertia eorundem latera AI, HF inter ſe æqualia. Quare FH : HE = AI : ID. Sed propter angulos AEF, AHE rectos, FH : HE = HE : AH vel CE. (El. VI. 8.) Quare AI : ID = HE vel ID : CE.

(γγ) NIMIRUM ad demonſtrationes ex fluxionum doctrinâ componendas nil conferunt lineæ illæ

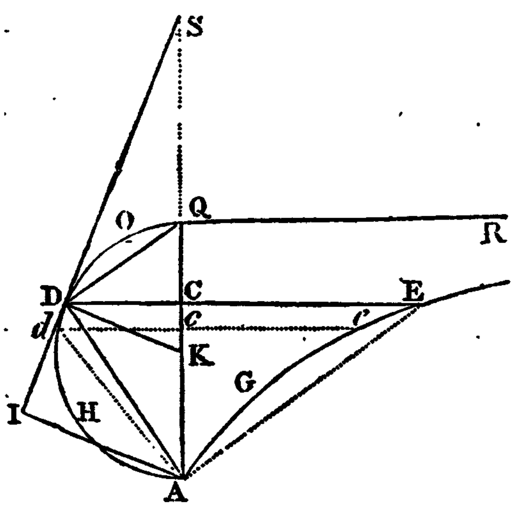
reſpectu ipſarum nullam, non habent rationem inæqualitatis. Et eâdem de cauſâ, poſui $Dp \times HI = HIib$, et ſic in aliis. De Componendis Demonſtrationibus.

26. Hâc methodo probandi Curvas, per æqualitatem vel datam rationem momentorum, æquales eſſe, vel datam rationem habere, hîc uſus ſum, quòd cum methodis in his rebus uſitatis (^{u}) affinitatem habeat. Sed magis naturalis videtur, quæ geneſi ſuperficierum ex fluendi motu innititur. Sic ſi conſtructio in



Exemplo 2, demonſtranda ſit: Ex naturâ circuli (^{zz}) eſt fluxio rectæ ID ad fluxionem rectæ IP ut AI ad ID: eſtque AI : ID :: ID : CE, ex naturâ curvæ AGE ($^{\zeta\zeta}$). Et proinde CE × flux. ID = ID × flux. IP. Sed CE × flux. ID = fluxioni areæ ACEG; et ID × flux. IP = fluxioni areæ PDI. Et propterea areæ illæ, æqualiter fluendo genitæ, æquales erunt. Q. E. D.

27. Plenioris illuſtrationis gratiâ, adjiciam demonſtrationem conſtructionis, quâ Ciſſoidis area in Exemplo 3, determinatur.



Lineæ punctis notatæ in ſchemate deleantur ($^{\gamma\gamma}$), et agatur DQ, et Ciſſoidis aſymptotos QR. Et ex naturâ circuli, eſt $DQ^{q} = AQ \times CQ$. Et inde per Prob. I. 2DQ × flux. ipſius DQ = AQ × flux. CQ. Adeoque AQ : DQ = 2 flux. DQ : flux. CQ. Eſt et ex naturâ Ciſſoidis ED : AD = AQ : DQ ($^{\delta\delta}$). Quare ED : AD :: 2 flux. DQ : flux. CQ. Et ED × flux. CQ = AD × 2 flux. DQ, ſive $4 \times \frac{1}{2}$ AD × flux. DQ. Jam cùm DQ perpendicularis ſit ad terminum ipſius AD

illæ viciniſſimæ, ſine quibus nihil unquam moliri auſa eſt *Indiviſibilium*, quæ dicitur, Geometria. Meliùs autem ſine illis rem agi, cùm ex illis demonſtrationibus, quæ in notis noſtris ad quartum Excerptorum ex Epiſtolis poſuimus, ſatis pateat, tum plurimarum exemplis, quas ad varia PRINCIPIORUM loca brevi proferemus, clariùs etiam eluceſcet.

($\delta\delta$) Nam è naturâ Ciſſoidis CE : CA = CA : CD. Junctâ igitur EA, angulus EAD rectus erit. Atque idcirco ED : DA = DA : DC. (El. VI. 8.) Sed propter angulos ADQ, ACD rectos DA : DC = AQ : QD. (El. VI. 8.) Quare ED : DA = AQ : QD.

CAPUT X. circa A gyrantis, eſt $\frac{1}{2}$ AD × flux. DQ = fluxioni generanti aream ADOQ. Eſt et ejus quadruplum, ED × flux. CQ, = fluxioni generanti ciſſoidalem aream QREDO. Et proinde area illa infinitè longa QREDO generatur quadrupla alterius ADOQ. Q. E. D.

SCHOLIUM.

28. Per præcedentes Catalogos non tantùm areæ Curvarum, ſed et aliæ cujuſcunque generis quantitates, analogâ fluendi ratione generatæ, è fluxionibus derivari poſſunt. Idque mediante hoc Theoremate. *Quòd quantitas cujuſcunque generis ſit ad unitatem congeneram, ut area curvæ ad unitatem ſuperficialem; ſi modò fluxio quantitatem illam generans ſit ad unitatem ſui generis, ut fluxio generans aream ad unitatem ſui generis; hoc eſt ut linea ſuper baſi normaliter incedens, quâ area illa deſcribitur, ad unitatem linearem.* Et proinde ſi fluxio qualiſcunque exponatur per ejuſmodi lineam incedentem, quantitas, ab illâ fluxione generata, exponetur per aream ab illâ incedente deſcriptam. Vel ſi fluxio per eoſdem terminos algebraicos cum incedente lineâ exponatur, quantitas generata exponetur per eoſdem cum areâ deſcriptâ. Æquatio itaque, quæ fluxionem cujuſcunque generis exhibet, quærenda eſt in primâ columnâ Catalogorum, et valor t, in ultimâ columnâ, indicabit quantitatem generatam.

29. Quemadmodum ſi $\sqrt{1+\frac{9z}{4a}}$ fluxionem cujuſcunque generis exhibeat, pone æqualem y; et ut ad formam æquationum in Catalogis reducatur, ſubſtitue z^n pro z: ſic enim evadet $z^{n-1}\sqrt{1+\frac{9z^n}{4a}}=y$, æquatio primæ ſpeciei tertii ordinis prioris Catalogi. Et collatis terminis, fiet $d=1$; $e=1$; $f=\frac{9}{4a}$: et inde $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}\ (=\frac{2d}{3nf}\text{R}^3)=t$. Eſt itaque $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}$ quantitas, quæ generatur fluxione $\sqrt{1+\frac{9z}{4a}}$.

30. Atque ita ſi $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ deſignet fluxionem, per debitam reductionem (extrahendo $z^{\frac{2}{3}}$ è radicali, et ſcribendo z^n pro $z^{-\frac{2}{3}}$)

5 habebitur

habebitur $\frac{1}{z^{n+1}}\sqrt{z^n+\frac{16}{9a^{\frac{3}{2}}}}=y$, æquatio ſecundæ ſpeciei tertii ordinis poſterioris Catalogi. Et collatis terminis fit $d=1$; $e=\frac{16}{9a^{\frac{3}{2}}}$; et $f=1$. Adeoque $z^{\frac{2}{3}}=\frac{1}{z^n}=xx$; $\sqrt{1+\frac{16xx}{9a^{\frac{3}{2}}}}=v$; et $3s=\frac{-2d}{n}s=t$. DE USU CATALOGORUM.

Quibus inventis, quantitas per fluxionem $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{3}{2}}}}$ generata innoteſcet, ponendo eſſe ad unitatem ſui generis ut area $3s$ ad unitatem ſuperficialem. Vel quod eodem recidit, ponendo quantitatem t non ampliùs ſuperficiem ſignificare, ſed alterius generis quantitatem, quæ eſt ad unitatem ejuſdem generis ut ſuperficies illa ad unitatem ſuperficialem. Sic poſito quòd $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ deſignet fluxionem linearem, imaginor t non ampliùs ſuperficiem ſed lineam jam ſignificare; eam nempe, quæ ad unitatem linearem eſt ut area, quam t juxta Catalogos deſignat, ad unitatem ſuperficialem; hoc eſt, eam quæ producitur applicando aream illam ad linearem unitatem. Quâ ratione, ſi linearis unitas ſtatuatur e, longitudo, per præfatam fluxionem generata, erit $\frac{3s}{e}$. Et hoc fundamento Catalogi illi, ad longitudines Curvarum, contenta Solidorum, et alias quaſcunque quantitates, æquè ac areas Curvarum, determinandas, applicari poſſunt.

CAPUT XI.

De Quæſtionibus cognatis.

SECT. I.

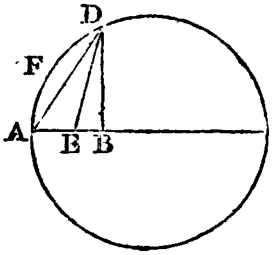
Curvarum areas per Mechanicam approximare (u)

1. METHODUS eſt, ut duarum pluriumve rectilinearum figurarum valores ita componantur inter ſe, ut valorem areæ Curvæ quàm proximè conſtituant. Sic ad circulum AFD quem æquatio $x-xx=rr$ deſignat, poſtquam inventus eſt areæ AFDB valor $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$ &c. quærendi ſunt aliquot rectangnlorum valores, CONSTRUCTIONES MECHANICÆ.

(u) Vide Excerpt. IV. ſub fin.

quales

CAPUT XI. quales ſunt ipſius BD × AB valor $x\sqrt{x-xx}$, ſive $x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{5}{2}}-\frac{1}{8}x^{\frac{7}{2}}-\frac{1}{16}x^{\frac{9}{2}}$ &c. ac ipſius AD × AB valor $x\sqrt{x}$, ſive $x^{\frac{3}{2}}$. Dein hi valores per literas quaſlibet diverſas, quæ numeros indefinitè deſignent, multiplicandi ſunt et addendi; ſummæque termini cum correſpondentibus terminis valoris areæ AFDB comparandi; ut quantùm liceat evadant æquales. Quemadmodum ſi per e & f multiplicentur, fiet ſumma $\begin{matrix}+e\\+f\end{matrix}x^{\frac{3}{2}}-\frac{e}{2}x^{\frac{5}{2}}-\frac{e}{8}x^{\frac{7}{2}}$ &c; cujus terminis cum terminis hiſce $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}$ &c. collatis, prodit $e+f=\frac{2}{3}$; et $-\frac{e}{2}=-\frac{1}{5}$; ſive $e=\frac{2}{5}$, et $f=\frac{2}{3}-e=\frac{4}{15}$. Adeoque eſt $\frac{2}{5}$BD × AB + $\frac{4}{15}$AD × AB = areæ AFDB proximè. Scilicet $\frac{2}{5}$BD × AB + $\frac{4}{15}$AD × AB valet $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{20}x^{\frac{7}{2}}-\frac{1}{40}x^{\frac{9}{2}}$ &c. quod ab areâ AFDB ſubductum relinquit ſolummodò errorem $\frac{1}{70}x^{\frac{7}{2}}+\frac{1}{90}x^{\frac{9}{2}}$ &c.



2. Sic biſectâ AB in E, rectanguli AB × DE valor erit $x\sqrt{x-\frac{3}{4}xx}$, ſive $x^{\frac{3}{2}}-\frac{3}{8}x^{\frac{5}{2}}-\frac{9}{128}x^{\frac{7}{2}}-\frac{27}{1024}x^{\frac{9}{2}}$ &c. Et hoc collatum cum rectangulo AD × AB dat $\frac{8\mathrm{DE}+2\mathrm{AD}}{15}\times$ AB = areæ AFDB, errore tantùm exiſtente $\frac{1}{560}x^{\frac{7}{2}}+\frac{1}{5760}x^{\frac{9}{2}}$ &c. qui ſemper minor eſt quàm $\frac{1}{1500}$ totius areæ, etiamſi AFDB ponatur quadrans circuli. Hoc autem Theorema ſic enunciari poteſt. *Ut* 3 *ad* 2 *ita rectangulum* AB *in* DE *plus quintâ parte differentiæ inter* AD *ac* DE *ad aream* AFDB *proximè.*

3. Atque ita conferendo duo rectangula AB × ED et AB × BD, vel omnia tria rectangula inter ſe, vel adhibendo adhuc alia rectangula, poſſunt aliæ regulæ excogitari, eæque tanto exactiores quo plura rectangula adhibentur. Et idem de areâ Hyperbolæ ac aliarum Curvarum intelligendum eſt. Imo et per unicum tantùm rectangulum area plerumque commodè exhiberi poteſt: ut in prædicto Circulo, ſi capiatur AE ad AB ut $\sqrt{10}$ ad 5, rectangulum AB × ED erit ad aream AFDB ut 3 ad 2, errore tantùm exiſtente $\frac{1}{175}x^{\frac{7}{2}}+\frac{11}{2250}x^{\frac{9}{2}}$ &c.

SECT. II.

Ex Datâ Areâ, Baſem et Incedentem Lineam determinare (ss).

1. Ubi area per finitam æquationem exhibetur, nihil occurrit

(ss) Vide Analyſ. per Æquat. Infinit. C. VII. Sect. 1.

diffi-

difficultatis. Ubi verò per infinitam exhibetur, affecta radix ex- **Basis ex Area.**
trahenda eſt, quæ baſem deſignat. Sic ad Hyperbolam, quam æquatio $\frac{ab}{a+x}=y$ deſignat; poſtquam inventum eſt $z=bx-\frac{bxx}{2a}+\frac{bx^3}{3aa}$ &c. ut ex datâ areâ, z, viciſſim innoteſcat baſis, x, extrahe radicem affectam, et proveniet $x=\frac{z}{b}+\frac{zz}{2abb}+\frac{z^3}{6aab^3}+\frac{z^4}{24a^3b^4}+\frac{z^5}{96a^4b^5}$ &c. Et præterea ſi incedens, y, deſideretur; divide ab per $a+x$, hoc eſt per $a+\frac{z}{b}+\frac{zz}{2abb}+\frac{z^3}{6aab^2}$ &c. et emerget $y=b-\frac{z}{a}+\frac{zz}{2aab}-\frac{z^3}{6a^3bb}+\frac{z^4}{24a^4b^3}$ &c.

2. Sic ad Ellipſin, quam æquatio $ax-\frac{a}{c}xx=yy$ deſignat; poſtquam inventa fuerit area $z=\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}-\frac{a^{\frac{1}{2}}x^{\frac{5}{2}}}{5c}-\frac{a^{\frac{1}{2}}x^{\frac{7}{2}}}{28cc}-\frac{a^{\frac{1}{2}}x^{\frac{9}{2}}}{72c^3}$ &c. ſcribe v^3 pro $\frac{3z}{2a^{\frac{1}{2}}}$, ac t pro $x^{\frac{1}{2}}$, et evadet $v^3=t^3-\frac{3t^5}{10c}-\frac{3t^7}{56cc}-\frac{t^9}{48c^3}$ &c. Et, extractâ radice, $t=v+\frac{v^3}{10c}+\frac{81v^5}{1400cc}+\frac{1171v^7}{25200c^3}$ &c. Cujus quadratum, $vv+\frac{v^4}{5c}+\frac{22v^6}{175cc}+\frac{823v^8}{7875c^3}$ &c. valet x. Et hoc valore pro x in æquatione, $ax-\frac{a}{c}xx=yy$, ſubſtituto, et extractâ radice, proveniet $y=a^{\frac{1}{2}}v-\frac{2a^{\frac{1}{2}}v^3}{5c}-\frac{38a^{\frac{1}{2}}v^5}{175cc}-\frac{407a^{\frac{1}{2}}v^7}{2250c^3}$ &c. Adeoque ex datâ areâ z, et inde v, ſive $\sqrt{\text{cub.}}\,\frac{3z}{2a^{\frac{1}{2}}}$, dabitur Baſis x et Incedens y. Quæ omnia ad Hyperbolam etiam accommodantur, ſi modò ſignum quantitatis c ubique mutetur, ubi exiſtit imparium dimenſionum.

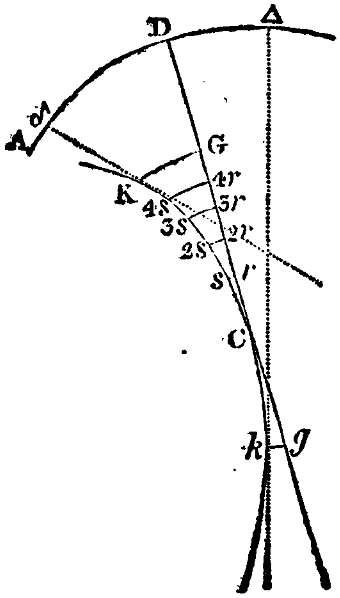
CAPUT XII.

PROBLEMA X.

Curvas pro arbitrio multas invenire, quarum Longitudines per finitas Æquationes deſignari poſſunt.

AD hujus reſolutionem via per ſequentes poſitiones ſternitur.

1. Si recta DC, in Curvam quamvis AD perpendiculariter inſiſtens, moveri concipiatur; ſingula ejus puncta G, g, r, &c. deſcribent



deſcribent alias æquidiſtantes ſibique perpendiculares curvas GK, *gk*, *rs*, &c.

2. Si recta illa hinc inde indefinitè producatur, ejus extremitates movebuntur ad contrarias plagas; et punctum quod diſtinguit inter contrarios motus, quodque ideo dici poteſt centrum Motionis, idem eſt cum centro Curvaturæ, quam Curva AD habet ad punctum D, ut ſuprà diximus. Iſtud autem punctum eſto C.

3. Si lineam AD non circularem eſſe, ſed difformiter incurvatam ſupponamus, puta magis curvam in δ, et minùs in Δ; illud centrum continuò mutabitur, propiùs accedens ad partes magis curvas, ut in K, et longiùs recedens à partibus minùs curvis, ut in *k*: eoque pacto lineam aliquam, qualis KC*k*, deſcribet.

4. Hanc à centro Curvaturæ deſcriptam lineam recta DC continuò tanget. Nam ſi rectæ illius punctum D moveat verſus δ, ejus punctum G, quod interea tranſit ad K et ſitum eſt ad eandem partem centri C, movebit verſus eandem plagam (per poſitionem ſecundam).. Deinde ſi idem D moveat verſus Δ, punctum *g*, quod interea tranſit ad *k*, et ſitum eſt ad contrariam partem centri C, movebit ad contrariam, hoc eſt ad eandem plagam ad quam G in priori caſu movebat, dum tranſit ad K. Et proinde K et *k* jacent ad eandem partem rectæ DC. Quare cùm K et *k* indeterminatè pro quibuſlibet punctis ſumantur, patet totam illam Curvam jacere ad eandem partem rectæ DC, proindeque ab illâ non ſecari, ſed tangi tantùm.

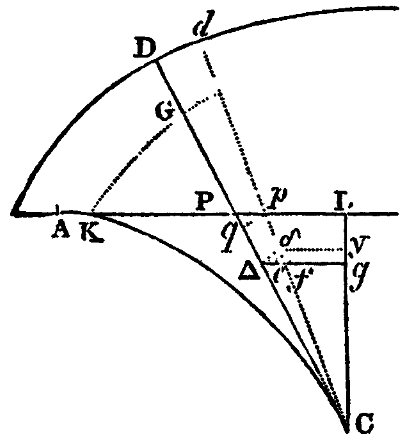
Hîc ſupponitur lineam δDΔ magis curvam eſſe à parte δ continuò, et minùs à parte Δ. Quòd ſi maxima minimave Curvatura fuerit ad ipſum D, tunc recta DC ſecabit curvam KC, ſed in angulo tamen qui ſit quovis rectilineo minor: quod perinde eſt ac ſi tangere dicatur. Imo punctum C, in hoc caſu, terminus eſt inſtar cuſpidis, ad quem partes Curvæ, obliquiſſimo concurſu deſinentes, ſe mutuò contingunt; proindeque à rectâ DC, quæ angulum illum contactûs dividit, rectiùs dicatur tangi quàm ſecari.

5. Recta CG æquatur Curvæ CK. Nam concipe rectæ illius ſingula

ſingula puncta r, $2r$, $3r$, $4r$, &c. deſcribere Curvarum arcus rs, $2r2s$, $3r3s$, &c. interea dum per motum rectæ illius accedant ad Curvam CK; et arcus illi, cùm (per poſitionem primam) ſint perpendiculares ad rectas quæ (per poſit. 4.) tangunt Curvam CK, erunt etiam perpendiculares ad Curvam illam. Quare partes iſtius CK, inter arcus illos interjectæ, quæ propter infinitam parvitatem pro rectis haberi poſſint, æquantur intervallis eorundem arcuum, hoc eſt (per poſit. 1) totidem partibus rectæ CG. Et, additis utrinque æqualibus, tota CK æquabitur toti CG. Inventio CURVARUM quæ Evolutioni ſunt capaces.

6. Idem conſtare poteſt, imaginando ſingulas partes rectæ CG, inter movendum, ſucceſſivè applicari ad ſingulas partes Curvæ CK, eaſque menſurare: perinde ut rotæ, ſuper planum per gyros promoventis, circumferentia diſtantiam metitur, quam punctum contactûs tranſigit.

7. Ex his pateat Problema reſolvi poſſe, aſſumendo pro lubitu Curvam quamvis AδDΔ, et inde determinando alteram Curvam KCk, in quâ aſſumptæ centrum curvaturæ verſatur.



Ad rectam itaque quamvis poſitione datam, AB, demiſſis perpendiculis, DB, CL, et in AB ſumpto quovis puncto A, dictiſque AB $= x$, et BD$=y$, pro Curvâ AD definiendâ aſſumatur relatio quævis inter x et y, et inde (per Prob. V.) elicietur punctum C, quo et Curva CK et ejus longitudo GC determinatur.

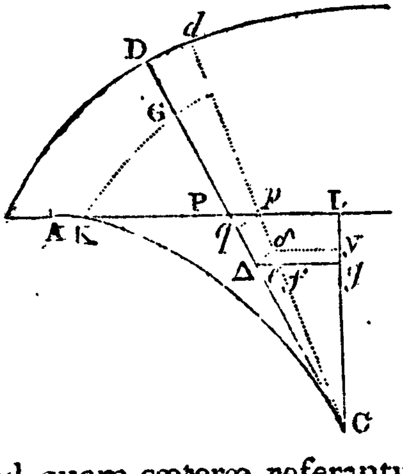
8. EXEMPL. Sit $ax=y^2$ æquatio ad Curvam AD, Parabolam nempe Apollonianam. Et per Prob. V. invenientur AL $= \frac{1}{2}a + 3x$, CL $= \frac{4y^3}{aa}$, ac DC $= \frac{a+4x}{a}\sqrt{\frac{1}{4}a^2+ax}$. Quibus habitis, Curva KC determinatur per AL ac LC, et longitudo ejus per DC: utpote cùm liberum ſit ubivis in Curvâ KC aſſumere puncta K & C, ſupponamis K eſſe centrum curvaturæ Parabolæ ad verticem; et poſitis perinde AB & BD, ſeu x & y, nullis, evadit DC$=\frac{1}{2}a$: eſtque hæc longitudo AK, vel DG, quæ, ſubducta à ſuperiori indefinito valore DC, relinquit GC, ſeu KC, $=\frac{a+4x}{a}\sqrt{\frac{1}{4}a^2+ax}-\frac{1}{2}a$.

9. Jam ſi qualis ſit hæc Curva, quantaque ejus longitudo, non

CAPUT XII ampliùs habitâ ratione ad Parabolam, ſcire deſideratur: dic $KL = z$, et $LC = v$; et erit $z = (AL - \frac{1}{2}a) = 3x$; ſeu $\frac{1}{3}z = x$, et $\frac{az}{3} (= ax) = y^2$; adeoque $4\sqrt{\frac{z^3}{27a}} \left(= \frac{4y^3}{aa} = CL\right) = v$, ſive $\frac{16z^3}{27a} = v^2$. Quod indicat Curvam KC eſſe Parabolam ſecundi generis. Et pro ejus longitudine prodit $\frac{3a+4z}{3a}\sqrt{\frac{1}{4}aa + \frac{1}{3}az} - \frac{1}{2}a$, ſcribendo $\frac{1}{3}z$ pro x in valore CG.

9. Poteſt etiam Problema reſolvi per aſſumptionem æquationis, quæ relationem inter AP et PD (poſitâ nempe P interſectione baſis et perpendiculi) definiant.

10. Nam dictis $AP = x$, et $PD = y$, concipe CPD per ſpatium quàm minimum moveri; puta ad locum cpd: inque CD & cd ſumpto $C\Delta$, et $c\delta$, ejuſdem cujuſvis datæ longitudinis, puta 1, et ad CL demiſſis Δg, δy perpendiculis, quorum Δg (quod dic z) occurrat cd in f, et completo parallelogrammo $gy\delta\varepsilon$; erit $\Delta e : \Delta f \; (:: \overline{\Delta e}^2 : \overline{\Delta\delta}^2 :: \overline{Cg}^2 : \overline{C\Delta}^2) :: \frac{\overline{Cg}^2}{C\Delta} : C\Delta$. Et $\Delta f : Pp :: C\Delta : CP$. Et ex æquo $\Delta e : Pp :: \frac{\overline{Cg}^2}{C\Delta} : CP$. Eſt autem Pp momentum baſis AP, cujus additamento evadit Ap; ac Δe contemporaneum momentum perpendiculi Δg, cujus ablatione evadit δy: adeoque Δe et Pp ſunt ut fluxiones linearum Δg (z) et AP (x); hoc eſt, ut $\dot{x}$ et $\dot{z}$. Quare $\dot{z} : \dot{x} :: \frac{\overline{Cg}^2}{C\Delta} : CP$. Et proinde cùm ſit $\overline{Cg}^2$ $(= \overline{C\Delta}^2 - \overline{\Delta g}^2) = 1 - zz$; et $C\Delta = 1$; erit $CP = \frac{\dot{x} - \dot{x}z^2}{\dot{z}}$. Et inſuper cùm è tribus $\dot{x}$, $\dot{y}$ et $\dot{z}$ quamlibet pro uniformi fluxione, ad quam cæteræ referantur, habere liceat; ſi iſta ponatur $\dot{x}$ ejuſque quantitas unitas, evadet $CP = \frac{1 - zz}{\dot{z}}$.



11. Præterea eſt $C\Delta$ (1) : Δg (z) :: CP : PL; et $C\Delta$ (1) : Cg ($\sqrt{1 - zz}$) :: CP : CL; adeoque ſit $PL = \frac{z - z^3}{\dot{z}}$, et $CL = \frac{1 - zz}{\dot{z}}\sqrt{1 - zz}$. Ac denique actâ pq parallelâ arcui infinitè parvo Dd, ſeu perpendiculari DC, erit Pq momentum ipſius DP, cujus additamento evadit dp, ſimul ac AP evadit Ap. Et idcirco Pp & Pq ſunt ut fluxiones ipſarum AP (x) et PD (y), hoc eſt ut 1 & $\dot{y}$. Atque adeo cùm propter

propter ſimilia triangula Ppq et $C\Delta g$, $C\Delta$ ac Δg, ſeu 1 & z, ſint in eâdem ratione, erit $\dot{y}=z$. Unde talis evadit Problematis reſolutio. Inventio CURVARUM quæ Evolutionum ſint capaces.

12. E propoſitâ æquatione, quæ relationem inter x et y deſignat, quære relationem fluxionum $\dot{x}$ et $\dot{y}$ (per Prob. 1.) Et poſito $\dot{x}=1$ habebitur valor $\dot{y}$, cui z æquatur. Dein ſubſtituto z pro $\dot{y}$, ope æquationis noviſſimæ, quære relationes fluxionum $\dot{x}$, $\dot{y}$ et $\dot{z}$ (per idem Prob. 1.) et iterum ſubſtituto 1 pro $\dot{x}$, obtinebitur valor $\dot{z}$. Quibus habitis, fac $\frac{1-\dot{y}\dot{y}}{\dot{z}}=CP$; $z\times CP=PL$; et $CP\times\sqrt{1-\dot{y}\dot{y}}=CL$, et erit C ad Curvam, cujus pars quævis, KC, æquatur rectæ CG, differentiæ nempe tangentium ductarum à punctis C et K perpendiculariter ad Curvam Dd.

13. EXEMPL. Sit $ax=yy$ æquatio quæ relationem inter AP et PD deſignet, et per Prob. 1. primò erit $a\dot{x}=2y\dot{y}$, ſeu $a=\frac{2y\dot{y}}{\dot{x}}=2yz$. Deinde $0=2\dot{y}z+2y\dot{z}$, ſeu $\frac{-zz}{y}=\dot{z}$. Indeque fit $CP=(\frac{1-\dot{y}\dot{y}}{\dot{z}}=)\,y-\frac{4y^3}{aa}$; $PL=(z\times CP=)\,\frac{1}{2}a-\frac{2yy}{a}$; et $CL=\frac{aa-4yy}{2aa}\sqrt{4yy-aa}$. Et à CP ac PL ablatis y et x, reſtat $CD=-\frac{4y^3}{aa}$, et $AL=\frac{1}{2}a-\frac{3yy}{a}$. Aufero autem y et x, quòd CP et PL, ubi valores habent affirmativos, cadant ad partes puncti P verſus D et A, et tunc diminui debent auferendo affirmativas quantitates PD et AP. Ubi verò negativos valores obtinent, cadunt ad contrarias partes puncti P; et tunc augeri debent: id quod etiam fit auferendo affirmativas quantitates PD et AP.

14. Jam ut Curvæ, in quâ punctum C locatur, longitudo inter duo quævis puncta, K et C, noſcatur; quæro longitudinem tangentis ad punctum K, & aufero à CD. Quemadmodum ſi K ſit punctum, ad quod tangens terminatur, ubi $C\Delta$ et Δg, ſeu 1 & z, ponuntur æquales, quodque proinde in ipſâ baſi AP ſitum eſt; ſcribe 1 pro z in æquatione $a=2yz$, & prodit $a=2y$. Quare pro y ſcribe $\frac{1}{2}a$ in valore CD, nempe in $-\frac{4y^3}{aa}$, et oritur $-\frac{1}{2}a$. Eſtque hæc longitudo tangentis ad punctum K, ſive ipſius DG; inter quam et ſuperiorem indefinitum valorem CD differentia, $\frac{4y^3}{aa}-\frac{1}{2}a$, eſt GC, cui Curvæ pars KC æquatur.

15. Ut inſuper pateat qualis ſit hæc Curva, ab AL (mutato priùs ſigno ut evadat affirmativa) aufer AK, quæ erit $\frac{1}{4}a$; et reſtabit $KL=\frac{3yy}{a}-\frac{3}{4}a$, quam dic t; et in valore lineæ CL, quam dic v, ſcribe $\frac{4at}{3}$

pro

CAPUT XII. pro $4yy-aa$, & prodibit $\frac{2t}{3a}\sqrt{\frac{4}{3}at}=v$, ſeu $\frac{16t^3}{27a}=vv$, æquatio ad Parabolam ſecundi generis, ut ſuprà.

16. Siquando relatio inter t & v minùs commodè ad æquationem redigi poſſit, ſufficit inveſtigaſſe tantùm longitudines PC & PL. Quemadmodum ſi pro relatione inter AP & PD [vid. fig. p. 506] aſſumatur æquatio $3a^2x+3a^2y-y^3=0$. Inde (per Prob. I.) primò prodit $a^2+a^2z-y^2z=0$; deinde $a^2\dot{z}-2yz^2-y^2\dot{z}=0$. Adeoque eſt $z=\frac{aa}{y-a}$, & $\dot{z}=\frac{2yz^2}{a^2-y^2}$. Unde dantur PC$=\frac{1-\dot{y}\dot{y}}{\dot{z}}$, & PL$=z\times$PC; quibus punctum C, quod ad Curvam ſitum eſt, determinatur. Et longitudo Curvæ inter duo ejuſmodi puncta è differentiâ correſpondentium duarum tangentium DC, ſive PC$-y$, innoteſcit.

EX. GR. Si ponatur $a=1$, et ad determinandum aliquod Curvæ punctum C, ſumatur $y=2$; evadet AP ſeu x $(=\frac{y^3-3a^2y}{3aa})=\frac{2}{3}$; $z=\frac{1}{3}$; $\dot{z}=-\frac{4}{9}$, PC$=-2$, & PL$=-\frac{2}{3}$. Deinde ad aliud punctum determinandum, ſi ſumatur $y=3$, evadet AP$=6$; $z=\frac{1}{8}$; $\dot{z}=\frac{-3}{256}$; PC$=-84$, & PL$=-10\frac{1}{2}$. Quibus habitis, ſi auferatur y à PC, reſtabit -4 in priori caſu, et -87 in ſecundo caſu, pro longitudinibus DC; quarum differentia, 83, eſt longitudo Curvæ inter inventa duo puncta C et c.

Hæc ita intelligenda ſunt, ubi Curva inter puncta duo C & c vel K & C continuatur, ſine terminò quem cuſpidi aſſimilavimus. Sed ubi unus vel plures ejuſmodi termini interjacent iſtis punctis (qui termini inveniuntur, per determinationem Maximæ aut Minimæ PC vel DC) longitudines ſingularum partium Curvæ, inter illos et puncta C vel K, ſeorſim inveſtigari debent & addi.

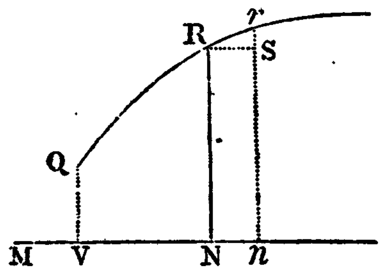
PROBLEMA XI.

17. *Curvas invenire quotaſcunque, quarum Longitudines cum propoſitæ alicujus Curvæ Longitudine, vel cum Areâ ejus ad datam Lineam applicatâ, ope finitarum Æquationum, comparari poſſunt.*

18. Peragitur, involvendo longitudinem areamve propoſitæ Curvæ in æquatione, quæ in præcedente Problemate aſſumitur ad determinandam relationem inter AP & PD. Sed ut z et $\dot{z}$ inde per Prob. I. eliciantur, fluxio longitudinis vel areæ illius priùs inveſtigari debet.

19. Fluxio

19. Fluxio longitudinis ejus determinatur ponendo æqualem radici quadraticæ ſummæ quadratorum à fluxionibus baſis & perpendiculariter incedentis. Sit RN linea perpendiculariter incedens ſuper baſi MN, et QR Curva propoſita ad quam RN terminatur. Dictiſque $MN=s$, $NR=t$ & $QR=v$; concipe lineam NR ad locum quàm proximum, nr, promoveri; et demiſſo ad nr perpendiculo RS, erunt RS, sr & Rr, contemporanea momenta linearum MN, NR, & QR, quorum additamentis evadunt Mn, nr, & Qr. Et cùm hæc ſint inter ſe ut earundem linearum fluxiones, ac propter angulum rectum RSr ſit $\sqrt{RS^2+sr^2}=Rr$, erit $\sqrt{\dot{s}^2+\dot{t}^2}=\dot{v}$. COMPARARE CURVARUM LONGITUDINES.



20. Ad determinandas autem fluxiones $\dot{s}$ & $\dot{t}$, duæ requiruntur æquationes; una, quæ definiat relationem inter MN & NR, ſeu s & t: unde relatio inter fluxiones $\dot{s}$ & $\dot{t}$ eruenda eſt: et alia, quæ definiat relationem inter MN vel NR ad datam figuram, et AP ſeu x ad quæſitam; unde relatio fluxionis $\dot{s}$, vel $\dot{t}$, ad fluxionem $\dot{x}$, ſeu 1, innoteſcit.

21. Invento $\dot{v}$, fluxiones $\dot{y}$ et $\dot{z}$ per aſſumptam tertiam æquationem, quâ longitudo PD ſive y definitur, inveſtigandæ ſunt; & capienda $PC=\frac{1-\dot{y}^2}{\dot{z}}$; $PL=\dot{y}\times PC$; ac $DC=PC-y$, ut in præcedente Problemate.

22. EXEMPL. 1. Sit $as-ss=tt$, æquatio ad datam Curvam QR, utpote circulum; $xx=as$ relatio inter lineas AP et MN; et $\frac{2}{3}v=y$ relatio inter longitudinem datæ Curvæ, QR, et rectæ PD. Per primam, fit $a\dot{s}-2s\dot{s}=2t\dot{t}$, ſeu $\frac{a-2s}{2t}\dot{s}=\dot{t}$. Et inde $\frac{a\dot{s}}{2t}(\sqrt{\dot{s}\dot{s}+\dot{t}\dot{t}})=\dot{v}$. Per ſecundam fit $2x=a\dot{s}$: adeoque eſt $\frac{x}{t}=\dot{v}$. Et per tertiam fit $\frac{2}{3}\dot{v}=\dot{y}$, hoc eſt $\frac{2x}{3t}=z$, dein hinc fit $\frac{2}{3t}-\frac{2x\dot{t}}{3tt}=\dot{z}$. Quibus inventis, capienda ſunt $PC=\frac{1-\dot{y}\dot{y}}{\dot{z}}$; $PL=\dot{y}\times PC$; ac $DC=PC-y$, ſive $PC-\frac{2}{3}QR$; ubi patet longitudinem datæ Curvæ QR inveniri non poſſe, quin ſimul innoteſcat longitudo rectæ DC, indeque longitudo Curvæ ad quam punctum C cadit. Et contrà.

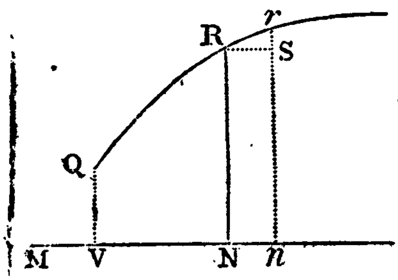
23. EXEMPL. 2. Stante $as-ss=tt$, ponatur $x=s$, & $vv-4ax=4ay$.

Perque

 Perque primam invenietur $\frac{a\dot{s}}{2t}=\dot{v}$, ut ſuprà. Per ſecundam verò $1=\dot{s}$; atque adeo $\frac{a}{2t}=\dot{v}$. Et per tertiam $2\dot{v}v-4a=4a\dot{y}$, ſeu (eliminato $\dot{v}$) $\frac{v}{4t}-1=z$; dein hinc $\frac{\dot{v}}{4t}-\frac{v\dot{t}}{4tt}=\dot{z}$.

24. Exempl. 3. Ponantur tres æquationes $aa=st$; $a+3s=x$; et $x+v=y$. Et per primam (quæ Hyperbolam denotat) evadit $0=\dot{s}t+\dot{t}s$, ſeu $-\frac{\dot{s}t}{s}=\dot{t}$: et inde $\frac{1}{s}\sqrt{ss+tt}\ (=\sqrt{\dot{s}^2+\dot{t}^2})=\dot{v}$. Per ſecundam evadit $3\dot{s}=1$; adeoque eſt $\frac{1}{3s}\sqrt{s^2+t^2}=\dot{v}$. Et per tertiam fit $1+\dot{v}=\dot{y}$, ſive $1+\frac{1}{3s}\sqrt{ss+tt}=\dot{z}$. Dein hinc fit $\dot{\phi}=\dot{z}$, poſitâ ſcilicet $\dot{\phi}$ fluxione radicalis $\frac{1}{3s}\sqrt{s^2+t^2}$; quæ ſi fingatur æqualis ϕ, ſive $\frac{1}{9}+\frac{tt}{9s^2}=\phi\phi$, proveniet inde $\frac{2t\dot{t}}{9ss}-\frac{2tt\dot{s}}{9s^3}=2\phi\dot{\phi}$. Et ſubſtituto imprimis $-\frac{\dot{s}t}{s}$ pro $\dot{t}$, deinde $\frac{1}{3}$ pro $\dot{s}$, factâque diviſione per 2ϕ, habebitur $\frac{-2t^2}{27\phi s^3}=(\dot{\phi}=)\dot{z}$. Inventis $\dot{y}$ & $\dot{z}$ cætera peraguntur ut in Exemplo Primo.

25. Quòd ſi à quovis Curvæ puncto Q, perpendiculum QV ad MN demittatur, et Curva invenienda ſit, cujus longitudo ex longitudine, quæ oritur applicando aream QRNV ad datam aliquam lineam, innoteſcat: ponatur illa data linea E; longitudo $\frac{\text{QRNM}}{\text{E}}$, quæ ex applicatione oritur, v. Et cùm fluxio areæ QRNV ſit ad fluxionem areæ parallelogrammi rectanguli ſuper VN ad altitudinem E conſtituti ut incedens linea NR, ſeu t, quâ hæc deſcribitur, ad incedentem lineam E, quâ illud eodem tempore deſcribitur; et longitudinum, quæ oriuntur applicando areas illas ad datam E, hoc eſt, linearum v et VN, ſeu s, fluxiones $\dot{v}$ et $\dot{s}$ ſint in eâdem ratione, erit $\dot{v}=\frac{\dot{s}t}{\text{E}}$. Per hanc itaque Regulam valor $\dot{v}$ inquirendus eſt, cæteraque ut in præcedentibus Exemplis peragenda.

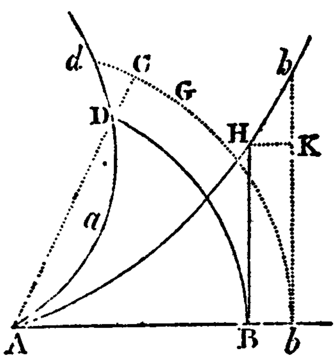


26. Exempl. 4. Sit QR Hyperbola quam æquatio $aa+\frac{ass}{c}=tt$ definit, et inde juxta Prob. I. evadet $\frac{as\dot{s}}{c}=t\dot{t}$, ſive $\frac{as\dot{s}}{ct}=\dot{t}$. Dein, ſi pro aliis duabus æquationibus aſſumantur $x=s$, & $y=v$, prior dabit $1=\dot{s}$; unde fit $\dot{v}\ (=\frac{\dot{s}t}{\text{E}})=\frac{t}{\text{E}}$; et poſterior dabit imprimis $\dot{y}=\dot{v}$, ſive

$z = \frac{t}{E}$; dein hinc $\dot{z} = \frac{\dot{t}}{E}$; et ſubſtituto $\frac{a\dot{s}}{ct}$, ſive $\frac{as}{ct}$, pro $\dot{t}$, evadet $\dot{z} = \frac{as}{Ect}$. Inventis $\dot{y}$ et $\dot{z}$, fac $\frac{1-\dot{y}\dot{y}}{\dot{z}}$ = CP, & $\dot{y}\times$ CP = PL, ut in præcedentibus; et inde punctum C, adeoque Curva, in quam omnia ejuſmodi puncta cadunt, determinabitur: cujus Curvæ longitudo ex longitudine DC, quæ valet $\acute{b}$P$-v$, innoteſcat, uti ſatis oſtendimus. COMPARARE CURVARUM LONGITUDINES.

27. Eſt et alia methodus quâ Problema reſolvitur, quærendo nempe Curvas, quarum fluxiones vel æquentur fluxioni Curvæ propoſitæ, vel ex illius et aliarum linearum fluxionibus componantur. Et hæc aliquando uſui eſſe poteſt, præſertim in convertendo Mechanicas Curvas, in æquales Geometricas, cujus rei inſigne eſt exemplum in Spiralibus.

28. Sit AB recta poſitione data; BD arcus ſuper AB tanquam baſi incedens, ac interea retinens A pro centro; ADd Spiralis, ad quam arcus ille perpetim terminatur; bd arcus quàm proximus, ſive locus in quem arcus BD, dum incedit, proximè movetur; DC perpendicularis ad arcum bd; dG differentia arcuum; AH alia Curva Spirali AD æqualis; BH recta ſuper AB normaliter incedens ac terminata ad curvam AH; bb locus quàm proximus in quem recta illa incedit; et HK perpendicularis ad bb. Et in triangulis infinitè parvis DCd ac HKb, cùm DC & HK æqualia ſint eidem tertio Bb, indeque ſibi mutuò æqualia; ac Dd & Hb ex hypotheſi ſint correſpondentes partes æqualium Curvarum, et inde etiam æqualia; nec non anguli ad C & K recti; tertia etiam latera, dC & bK, æqualia erunt. Quare cùm inſuper ſit AB : BD :: Ab : bC :: Ab−AB (Bb) : bC−BD (CG); adeoque $\frac{BD \times Bb}{AB}$ = CG; ſi hoc auferatur à dG, reſtabit dG$-\frac{BD \times Bb}{AB}$ (=dC) = bK. Dic itaque AB=z, BD=v, & BH=y, et cùm Bb, dG, & HK ſint earundem contemporanea momenta quorum additamentis evadunt Ab, bd, & bb, et proinde inter ſe ſint ut fluxiones, ideo pro momentis in æquatione noviſſimâ ſubſtituantur fluxiones, juxta et notæ pro lineis, et emerget $\dot{v} - \frac{v\dot{z}}{z} = \dot{y}$. Ubi ſi è fluxionibus $\dot{z}$ pro æ-



quabili-

CAPUT XII. quabili habeatur & ſupponatur unitas eſſe, ad quam cæteræ referantur, evadit $\dot{v} - \frac{v}{z} = \dot{y}$.

Quamobrem datâ per æquationem aliquam relatione inter AB & BD (ſive x et v) quà Spiralis definiatur, dabitur (per Prob. I.) fluxio $\dot{v}$, et inde etiam fluxio $\dot{y}$, ponendo æqualem $\dot{v} - \frac{v}{z}$. Atque hæc per Prob. II. dabit lineam y, ſive BH, cujus eſt fluxio.

29. EXEMPL. I. Si detur $\frac{zz}{a} = v$ æquatio nempe ad Spiralem Archimedeam, inde (per Prob. I.) elicictur $\frac{2z}{a} = \dot{v}$. A quo aufer $\frac{v}{z}$, ſive $\frac{z}{a}$, et reſtabit $\frac{z}{a} = \dot{y}$, et inde per Prob. II. fit $\frac{zz}{2a} = y$. Quod indicat Curvam AH, cui hæc Spiralis AD æquatur, eſſe Parabolam Apollonianam, cujus latus rectum exiſtit $2a$, ſive cujus incedens, BH, perpetuò æquatur ſemiſſi arcûs BD.

30. EXEMPL. 2. Si proponatur Spiralis, quam æquatio $z^3 = av^2$, ſive $v = \frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ definit, emerget, per Prob. I. $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{v}$. A quo ſi auferatur $\frac{v}{z}$, ſeu $\frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, reſtabit $\frac{z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$, et inde per Prob. II. producetur $\frac{z^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = y$. Hoc eſt $\frac{1}{3}$BD=BH, exiſtente AH Parabolâ ſecundi generis.

31. EXEMPL. 3. Si ad Spiralem ſit $z\sqrt{\frac{a+z}{c}} = v$. Exinde per Prob. I. elicietur $\frac{2a+3z}{2\sqrt{ac+cz}} = \dot{v}$. A quo ſi auferatur $\frac{v}{z}$, ſive $\sqrt{\frac{a+z}{c}}$, reſtabit $\frac{z}{2\sqrt{ac+zc}} = \dot{y}$. Jam cùm quantitas hâc fluxione $\dot{y}$ generata nequeat inveniri per ea quæ in Prob. II. habentur, niſi fiat reſolutio in infinitam ſeriem, juxta tenorem Scholii Prob. IX. reduco ad formam æquationum in primâ columnâ Catalogorum, ſubſtituendo z^2 pro z; et evadit $\frac{z^{2-1}}{2\sqrt{ac+cz^2}} = y$: æquatio nempe ſecundæ ſpeciei quarti ordinis prioris Catalogi. Et conferendo terminos fit $d = \frac{1}{2}$; $e = ac$; & $f = c$: adeoque $\frac{z-2a}{3c}\sqrt{ac+cz}\ (=t) = y$, quæ æquatio eſt ad Curvam Geometricam AH, cui Spiralis AD æquatur.

PROB.

PROBLEMA XII.

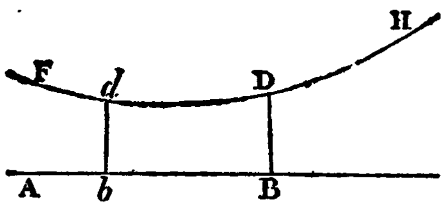
CURVARUM LONGITUDINES.

Curvarum Longitudines determinare.

33. Fluxionem Curvæ lineæ in ſuperiore Problemate oſtendimus æqualem eſſe radici quadraticæ ſummæ quadratorum à fluxionibus baſis et perpendiculariter incedentis. Et proinde ſi baſis fluxionem pro uniformi ac determinatâ menſurâ, nimirum unitate, ad quam cæteræ fluxiones referantur, habeamus; et inſuper per æquationem, quæ Curvam definit, quæramus fluxionem Incedentis, habebitur fluxio Curvæ lineæ, à quâ longitudo ejus per Prob. II. elicienda eſt.

34. EXEMPL. I. Proponatur Curva FDH, quam æquatio $\frac{z^3}{aa} + \frac{aa}{12z} = y$ definit; poſito ſcilicet z=baſi AB, ac y= incedenti DB. Et ex æquatione illâ per Prob. I. elicietur $\frac{3zz}{aa} - \frac{aa}{12z^2} = \dot{y}$, exiſtente nimirum 1 fluxione ipſius z. Dein additis fluxionum quadratis, fit ſumma $\frac{9z^4}{a^4} + \frac{1}{2} + \frac{a^4}{144z^4} = \dot{t}\dot{t}$; et extractâ radice, $\frac{3zz}{aa} + \frac{aa}{12zz} = \dot{t}$, indeque per Prob. II. $\frac{z^3}{aa} - \frac{aa}{12z} = t$.

35. Itaque ſi cujuſvis portionis Curvæ hujus, puta dD, longitudo deſideretur; à punctis d ac D demitte ad AB perpendicula db ac DB, et in valore t ſubſtitue quantitates Ab et AB ſeorſim pro z, ac differentia productorum erit longitudo quæſita dD.



Quemadmodum ſi ſit A$b=\frac{1}{2}a$, et AB$=a$, ſcripto $\frac{1}{2}a$ pro z, evadet $t=-\frac{a}{24}$. Dein ſcripto a pro z, evadet $t=\frac{11a}{12}$: à quo ſi prior valor auferatur, reſtabit $\frac{23a}{24}$ pro longitudine dD. Vel ſi Ab tantùm definiatur eſſe $\frac{1}{2}a$, et AB ſpectetur indefinitè, reſtabit $\frac{z^3}{aa} - \frac{aa}{12z} + \frac{a}{24} = d$D.

36. Quòd ſi cupias noſcere portionem Curvæ quam t deſignat, finge valorem t æquari nihilo; et evadet $z^4 = \frac{a^4}{12}$, ſive $z = \frac{a}{\sqrt[4]{12}}$; adeoque

CAPUT XII. ſi ſumatur $Ab = \frac{a}{\sqrt[4]{12}}$, et erigatur bd, longitudo arcûs dD erit t, ſive $\frac{z^3}{aa} - \frac{aa}{12z}$. Et hæc de aliis Curvis generaliter intelligenda ſunt.

37. Ad eundem modum, quo hujus longitudinem determinavimus, ſi pro aliâ Curvâ definiendâ proponatur æquatio $\frac{z^4}{a^3} + \frac{a^3}{32z^2} = y$, proveniet $\frac{z^4}{a^3} - \frac{a^3}{32z^2} = t$. Vel ſi proponatur $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} - \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = y$, proveniet $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = t$. Vel generaliter ſi ſit $cz^{\theta} + \frac{z^{2-\theta}}{4\theta\theta c - 8\theta c} = y$, ubi θ pro quolibet numero, ſive integro, ſive fracto, deſignando adhibetur, erit $cz^{\theta} - \frac{z^{2-\theta}}{4\theta\theta c - 8\theta c} = t$.

38. EXEMPL. 2. Proponatur Curva, quam æquatio $\frac{2a^2 + 2z^2}{3a^2}\sqrt{a^2 + z^2} = y$ definit: et per Prob. I. obtinebitur $\dot{y} = \frac{4a^4z + 8a^2z^3 + 4z^5}{3a^4y}$; ſive exterminato y, $\dot{y} = \frac{2z}{aa}\sqrt{a^2 + z^2}$: cujus quadrato adde I, et ſumma erit $1 + \frac{4zz}{aa} + \frac{4z^4}{a^4}$; ejuſque radix $1 + \frac{2z^2}{aa} = \dot{t}$, unde per Prob. II. obtinetur $z + \frac{2z^3}{3a^2} = t$.

39. EXEMPL. 3. Proponatur Parabola ſecundi generis, ad quam æquatio eſt $z^3 = ay^2$, ſeu $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} = y$: et inde (per Prob. I.) elicietur $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$; adeoque eſt $\sqrt{1 + \frac{9z}{4a}}\ (\sqrt{1 + \dot{y}\dot{y}}) = \dot{t}$. Jam cùm longitudo per fluxionem $\dot{t}$ generata nequeat inveniri per Prob. II. abſque reductione in infinitam ſeriem ſimplicium terminorum, conſulo Catalogos ad Prob. VIII. et juxta ea quæ in Scholio ejus habentur, prodit $t = \frac{8a + 18z}{27} \times \sqrt{1 + \frac{9z}{4a}}$.

40. Et ſic Parabolarum $z^5 = ay^4$, $z^7 = ay^6$, $z^9 = ay^8$, &c. longitudines inveniri poſſunt.

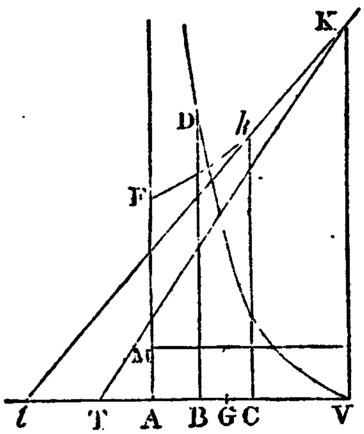
41. EXEMPL. 4. Proponatur Parabola, ad quam æquatio eſt $z^4 = ay^3$, ſive $\frac{z^{\frac{4}{3}}}{a^{\frac{1}{3}}} = y$; et inde per Prob. I. orietur $\frac{4z^{\frac{1}{3}}}{3a^{\frac{1}{3}}} = \dot{y}$. Adeoque $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}\ (= \sqrt{\dot{y}\dot{y} + 1}) = \dot{t}$. Quo invento, conſulo Catalogos juxta Scholium prædictum, et factâ collatione cum ſecundo Theoremate tertii ordinis poſterioris catalogi, prodit $z^{\frac{1}{3}} = x$; $\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$,

et $3ſ=t$. Ubi x deſignat baſem, y ordinatim applicatam, & $ſ$ aream Hyperbolæ, atque t longitudinem quæ oritur applicando aream $3ſ$ ad unitatem linearem. CURVARUM LONGITUDINES.

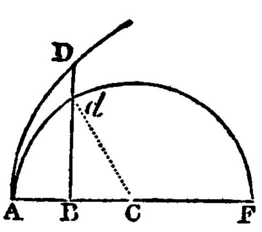
42. Eâdem methodo Parabolarum $z^6=ay^5$, $z^8=ay^7$, $z^{10}=ay^9$, &c. longitudines etiam per aream Hyperbolæ determinantur.

43. EXEMPL. 5. Proponatur Ciſſois Veterum; et exiſtente ad eam æquatione $\frac{aa-2az+zz}{\sqrt{az-zz}}=y$, inde per Prob. I. elicietur $\frac{-a-2z}{2zz}\sqrt{az-zz}=\dot{y}$: et conſequenter $\frac{a}{2z}\sqrt{\frac{a+3z}{z}}\ (=\sqrt{\dot{y}\dot{y}+1})=\dot{t}$; quæ, ſcribendo z^{η} pro $\frac{1}{z}$ ſeu z^{-1}, evadit $\frac{a}{2z}\sqrt{az^{\eta}+3}=\dot{t}$, æquatio primæ ſpeciei tertii ordinis poſterioris Catalogi. Et collatis terminis fit $\frac{a}{z}=d$, $3=e$, et $a=f$; adeoque $z\ (=\frac{1}{z^{\eta}})=x^2$, $\sqrt{a+3x^2}=v$, & $6ſ-\frac{2v^3}{x}\ (=\frac{4de}{\eta f}\text{ in }\frac{v^3}{2ex}-ſ)=t$. Et adhibitâ a pro unitate, per cujus multiplicationem vel diviſionem hæ quantitates ad juſtum dimenſionum numerum reducantur, evadit $az=xx$, $\sqrt{aa+3xx}=v$, & $\frac{6ſ}{a}-\frac{2v^3}{ax}=t$. Quorum hæc eſt conſtructio.

44. Exiſtente VD Ciſſoide, AV diametro circuli ad quam aptatur, AF aſymptoto ejus, ac DB perpendiculari ad AV; cum ſemiaxe AF=AV, & ſemiparametro AG=$\frac{1}{3}$AV, deſcribatur Hyperbola FkK: et inter AB & AV ſumptâ AC mediâ proportionali, erigantur ad C & V perpendicula Ck & VK, et agantur kt & KT rectæ tangentes Hyperbolam in k et K, et occurrentes AV in t ac T; et ad AV conſtituatur rectangulum VM æquale ſpatio TKkt; et Ciſſoidis VD longitudo erit ſextupla altitudinis AM.



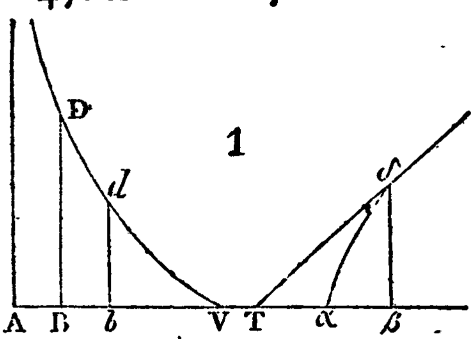
45. EXEMPL. 6. Exiſtente Ad Ellipſi, quam æquatio $\sqrt{az-2zz}=y$, definit: proponatur Curva Mechanica AD, talis ut ſi Bd, ſeu y, producatur donec huic Curvæ ad D occurrat, ſit BD æqualis arcui elliptico Ad. Jam quo hujus longitudo determinetur, æquatio $\sqrt{az-2zz}=y$, dabit $\frac{a-4z}{2\sqrt{az-2zz}}=\dot{y}$. Cujus quadrato ſi 1 addatur, prodit $\frac{a^2-4az+8z^2}{4az-8z^2}$ quadratum fluxionis arcûs

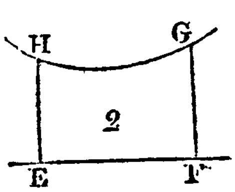


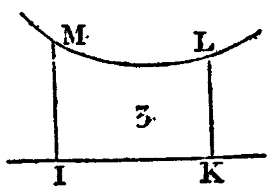
CAPUT XII. Ad: et huic ſi iterum addatur 1, provenit $\frac{aa}{4az-8zz}$; cujus radix $\frac{a}{2\sqrt{az-2zz}}$ eſt fluxio Curvæ lineæ AD. Ubi ſi è radicali extrahatur z, & pro z^{-1} ſcribatur z^n, habebitur $\frac{a}{2z\sqrt{az^n-2}}$ fluxio primæ ſpeciei quarti ordinis poſterioris Catalogi. Collatiſque terminis exibunt $d=\frac{1}{2}a$, $e=-2$, & $f=a$: adeoque $z\,(=\frac{1}{z^{-1}})=x$, $\sqrt{ax-2x^2}=v$, & $\frac{8s}{a}-\frac{4xv}{a}+v\;(=\frac{8de}{nff}$ in $s+\frac{1}{2}xv-\frac{fv}{4e})=t$.

46. Quorum conſtructio eſt, ut ad Ellipſis centrum D actâ rectâ dc, conſtituatur ſuper AC parallelogrammum æquale ſectori ACd, et duplum altitudinis ejus ponatur eſſe longitudo Curvæ AD.

47. EXEMPL. 7. Exiſtente $\alpha\beta=\varphi$, & $\alpha\delta$ Hyperbolâ ad quam æquatio ſit $\sqrt{-a+b\varphi\varphi}=\beta\delta$, actâque δT tangente ejus: proponatur Curva, VdD, cujus baſis AB ſit $\frac{1}{\varphi\varphi}$, & normaliter incedens, BD, longitudo, quæ oritur applicando aream $\alpha\delta$Tα ad unitatem linearem. Jam ut hujus aD longitudo determinetur, quæro fluxionem areæ $\alpha\delta$Tα, cùm AB uniformiter fluit; & invenio eſſe $\frac{a}{4bz}\sqrt{b-az}$, poſitâ AB$=z$, & fluxione ejus unitate. Nam eſt αT$=\frac{a}{b\varphi}=\frac{a}{b}\sqrt{z}$; ejuſque fluxio $\frac{a}{2b\sqrt{z}}$: cujus dimidium ductum in altitudinem $\beta\delta$, ſeu $\sqrt{-a+\frac{b}{z}}$, eſt fluxio areæ αTδ deſcriptæ per tangentem δT. Quare fluxio illa eſt $\frac{a}{4bz}\sqrt{b-az}$, atque hæc applicata ad unitatem ſit fluxio incedentis BD. Hujus quadrato $\frac{aab-a^3z}{16b^2z^2}$ adde 1, quadratum fluxionis ipſius AB, et prodit $\frac{a^2b-a^3z+16b^2z^2}{16b^2z^2}$; cujus radix $\frac{1}{4bz}\sqrt{a^2b-a^3z+16b^2z^2}$ eſt fluxio curvæ VD. Eſt autem hæc fluxio primæ ſpeciei ſeptimi ordinis poſterioris Catalogi. Collatiſque terminis exeunt $\frac{1}{4b}=d$, $aab=e$, $-a^3=f$, $16b^2=g$; adeoque $z=x$, & $\sqrt{a^2b-a^3x+16b^2x^2}=v$ (æquatio ad unam conicam ſectionem, puta GH, (fig. 2) cujus area EFGH ſit s, exiſtente







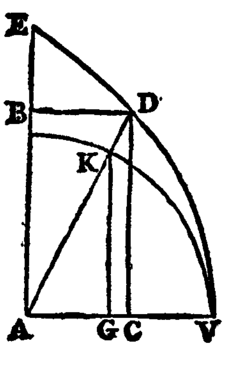
exiſtente EF$=x$, & FG$=v$). Item $\frac{1}{z}=\xi$, et $\sqrt{16b^2-a^3\xi+ab\xi^2}=\Upsilon$ (æquatio ad aliam conicam ſectionem, puta ML (fig. 3) cujus area IKLM ſit σ, exiſtente IK$=\xi$, & KL$=\Upsilon$) Denique $\frac{2aabb\xi\Upsilon-a\,b\Upsilon-a^4v-4a^2b^2\sigma-32abbs}{64b^4-a^4}=t$. CURVARUM LONGITUDINES.

48. Quare ut Curvæ VD portionis cujuſcunque, Dd, longitudo noſcatur; demitte db normalem ad AB, fingeque AB$=z$, & exinde per jam inventa quære t: dein finge BA$=z$, & exinde etiam quære t: & horum duorum t differentia erit longitudo Dd.

49. EXEMPL. 8. Proponatur Hyperbola, ad quam æquatio eſt $\sqrt{aa+bzz}=y$; et inde (per Prob. I.) elicietur $\dot{y}=\frac{bz}{y}$, ſeu $\frac{bz}{\sqrt{aa+bzz}}$. Cujus quadrato adde 1, & ſummæ radix erit $\sqrt{\frac{aa+bzz+bbzz}{aa+bzz}}=\dot{t}$. Hanc fluxionem, cùm non reperiatur in Tabulis, reduco in infinitam ſeriem: & primò, per diviſionem, evadit $\dot{t}=\sqrt{1+\frac{b^2z^2}{a^2}-\frac{b^3z^4}{a^4}+\frac{b^4z^6}{a^6}-\frac{b^5z^8}{a^8}}$ &c. dein per extractionem radicis $\dot{t}=1+\frac{b^2z^2}{2a^2}-\frac{4b^3+b^4}{8a^4}z^4+\frac{8b^4+4b^5+b^6}{16a^5}z^6$ &c. Et hinc per Prob. II. obtinetur t, ſeu longitudo Hyperbolæ $=z+\frac{b^2}{6a^2}z^3-\frac{4b^3+b^4}{40a^4}z^5+\frac{8b^4+4b\ +b^6}{112a^6}z^7$ &c.

50. Quòd ſi Ellipſis, $\sqrt{aa-bzz}=y$, proponatur; debet ſignum ipſius b ubique mutari: & habebitur $z+\frac{b^2}{6a^2}z^3+\frac{4b^3-b^4}{40a^4}z^4+\frac{8b^4-4b^5+b^6}{112a^6}z^5$ &c. pro longitudine ejus. Et poſitâ inſuper unitate pro b, emerget $z+\frac{z^3}{6a^2}+\frac{3z^5}{40a^4}+\frac{5z^7}{112a^6}$ &c. pro longitudine Circuli: cujus ſeriei numerales coefficientes in infinitum inveniuntur, multiplicando continuò terminos hujus progreſſionis, $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, $\frac{9\times9}{10\times11}$, &c.

51. EXEMPL. 9. Proponatur denique Quadratrix, VDE, cujus vertex eſt V, exiſtente A centro, et AV ſemidiametro Circuli interioris ad quam aptatur, atque angulo VAE recto. Actâ jam rectâ quâlibet AKD ſecante Circulum iſtum in K et Quadraticem in D, demiſſiſque ad AV, AE normalibus KG, DB; dic AV, a; KG, z; VK, x; & BD, y; eritque ut in ſuperiore exemplo, $x=z+\frac{z^3}{6a^2}+\frac{3z^5}{40a^4}+\frac{5z^7}{112a^6}$ &c. Extrahe radicem z, et emerget $z=x-\frac{x^3}{6a^2}+\frac{x^5}{120a^4}-\frac{x^7}{5040a^6}$ &c. Cujus qua-



dratum

Caput XII. dratum aufer de $\overline{AK}^2$, et reſidui radix $a - \frac{x^2}{2a} + \frac{x^4}{24a^3} - \frac{x^6}{720a^5} + \&c.$ erit AG. Jam cùm ex naturâ Quadratricis ſit $AB = VK = x$, ſitque etiam $GK : AG :: AB : BD\ (y)$, divide $AB \times AG$ per GK, et orietur $y = a - \frac{x^2}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5}$ &c. Et inde per Prob. I. $\dot{y} = -\frac{2x}{3a} - \frac{4x^3}{45a^3} - \frac{4x^5}{315a^5}$ &c. Cujus quadrato adde 1, et ſummæ radix erit $1 + \frac{2x^2}{9a^2} + \frac{14x^4}{405a^4} + \frac{604x^6}{127575a^6}$ &c. $= \dot{t}$; unde per Prob. II. obtinetur t ſeu Quadratricis arcus $VD = x + \frac{2x^3}{27a^2} + \frac{14x^5}{2025a^4} + \frac{604x^5}{893025a^6}$ &c.

FINIS GEOMETRIÆ ANALYTICÆ.

METHODUS

METHODUS

DIFFERENTIALIS.

METHODUS DIFFERENTIALIS.

PROP. I.

Si figuræ curvilineæ Abſciſſa componatur ex quantitate quâvis datâ A, *& quantitate indeterminatâ x, & Ordinata conſtet ex datis quotcunque quantitatibus b, c, d, e, &c. in totidem terminos hujus progreſſionis geometricæ x,* x^2, x^3, x^4, *&c. reſpectivè ductis, & ad Abſciſſæ puncta totidem data erigantur ordinatim applicatæ: dico quòd Ordinatarum differentiæ primæ dividi poſſint per earum intervalla; & differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum binarum intervalla; & harum differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum ternarum intervalla; & ſic deinceps in infinitum.*

ETENIM ſi pro Abſciſſæ parte indeterminatâ, *x*, ponantur quantitates quævis datæ *p*, *q*, *r*, *s*, *t*, &c. ſucceſſivè, & ad Abſciſſarum ſic datarum terminos erigantur Ordinatæ α, β, γ, ε, δ, &c. Hæ Abſciſſæ & Ordinatæ & ordinatarum differentiæ diviſæ per Abſciſſarum differentias (quæ utique ſunt Ordinatarum intervalla) & quotorum differentiæ diviſæ per Ordinatarum alternarum differentias, & ſic deinceps, exhibentur per Tabulam ſequentem.

Prop. II.

Abſciſſæ	Ordinatæ
$A+p$	$A+bp+cp^2+dp^3+ep^4=\alpha$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\epsilon$
Diviſor. Diff. Ord.	**Quoti per diviſionem prodeuntes.**
$p-q)\ \alpha-\beta$	$b+c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r)\ \beta-\gamma$	$b+c\times\overline{q+r}+d\times\overline{qq+qr+rr}+e\times\overline{q^3+q^2r+qr^2+r^3}=\eta$
$r-s)\ \gamma-\delta$	$b+c\times\overline{r+s}+d\times\overline{rr+rs+ss}+e\times\overline{r^3+r^2s+rs^2+s^3}=\theta$
$s-t)\ \delta-\epsilon$	$b+c\times\overline{s+t}+d\times\overline{ss+st+tt}+e\times\overline{s^3+s^2t+st^2+t^3}=\varkappa$
$p-r)\ \zeta-\eta$	$c+d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s)\ \eta-\theta$	$c+d\times\overline{q+r+s}+e\times\overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t)\ \theta-\varkappa$	$c+d\times\overline{r+s+t}+e\times\overline{rr+rs+ss+rt+st+tt}=\nu$
$p-s)\ \lambda-\mu$	$d+e\times\overline{p+q+r+s}=\xi$.
$q-t)\ \mu-\nu$	$d+e\times\overline{q+r+s+t}=\pi$
$p-t)\ \xi-\pi$	$e=\sigma$

PROP. II.

Iiſdem poſitis, & quòd numerus terminorum b, c, d, e, &c. ſit finitus, dico quòd Quotorum ultimus æqualis erit ultimo terminorum b, c, d, e, &c. et quòd per quotos reliquos dabuntur termini reliqui b, c, d, e, &c. et his datis dabitur Linea Curva generis Parabolici, quæ per Ordinatarum omnium terminos tranſibit.

Etenim in Tabulâ ſuperiore quotus ultimus, σ, æqualis erat termino ultimo e. Et hic terminus ductus in ſummam datam $\overline{p+q+r+s}$, & ablatus de quoto ξ relinquit terminum penultimum d. Et quantitates jam datæ $d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}$, ſi auferantur de quoto λ, relinquunt terminorum antepenultimum c. Et quantitates jam datæ $c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+ppq+pqq+q^3}$, ſi auferantur de quoto ζ, relinquunt terminum b. Et ſimili computo, ſi plures eſſent termini, colligerentur omnes per quotorum ordines totidem. Deinde quantitates datæ $bp+cpp+dp^3+ep^4$, ſi ſubducantur de Ordinatâ primâ α, relinquunt Abſciſſæ terminum primum A. Et quantitas $A+bx+cx^2+dx^3+ex^4+$ &c. eſt Ordinata Curvæ generis Parabolici, quæ per Ordinatarum omnium datarum terminos tranſibit, exiſtente Abſciſſâ $A+x$.

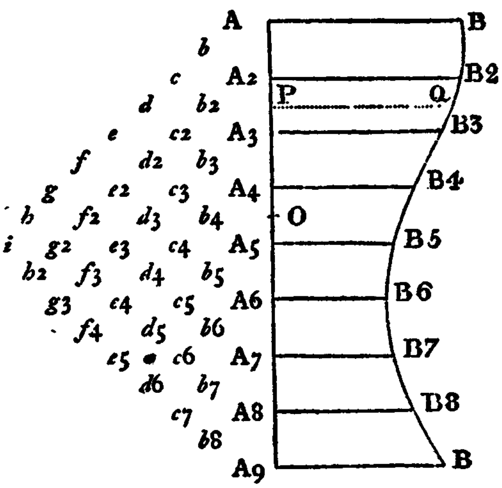
Ex his Propoſitionibus quæ ſequuntur facilè colligi poſſunt.

PROP.

PROP. III.

Si recta aliqua AA9 *in æquales quotcunque partes* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici, quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Erectarum AB, A2B2, A3B3, &c. quære differentias Primas, *b*, *b2*, *b3*, &c. Secundas *c*, *c2*, *c3*, &c. Tertias *d*, *d2*, *d3*, &c. et ſic deinceps uſque dum veneris ad ultimam differentiam, quæ hîc ſit *i*.



Tunc incipiendo ab ultimâ differentiâ excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum; & Arithmetica media inter duas medias reliquarum, ordine pergendo uſque ad ſeriem primorum terminorum AB, A2B2, A3B3, &c. ſint hæc *k*, *l*, *m*, *n*, *o*, *p*, *q*, *r*, *s*, &c. quorum ultimus ſignificet ultimam differentiam; penultimus medium arithmeticum inter duas penultimas differentias; antepenultimus mediam trium antepenultimarum differentiarum, & ſic deinceps uſque ad primum, quod erit vel medius terminorum A, A2, A3, &c. vel arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A2, A3, &c. eſt impar; poſterius ubi par.

CAS. I.

In Caſu priori, ſit A5B5 iſte medius terminus, hoc eſt, A5B5 $=k$; $\frac{b4+b5}{2}=l$; $c4=m$; $\frac{d3+d4}{2}=n$; $e3=o$; $\frac{f2+f3}{2}=p$; $g2=q$; $\frac{h+h2}{2}=r$; $i=s$. Et erectâ ordinatim applicatâ PQ, dic $A5P=x$; & duc terminos hujus progreſſionis $1 \times \frac{x}{1} \times \frac{x}{2} \times \frac{x^2-1}{3x} \times \frac{x}{4} \times \frac{x^2-4}{5x} \times \frac{x}{6} \times$

 x^2-9

PROP. III. $\frac{x^2-9}{7x} \times \frac{x}{8} \times \frac{x^2-16}{9x} \times \frac{x}{10} \times \frac{x^2-25}{11x} \times \frac{x}{12} \times \frac{x^2-36}{13x}$ &c. in ſe continuò; & orientur termini $1 . x . \frac{x^2}{2} . \frac{x^3-x}{6} . \frac{x^4-x^2}{24} . \frac{x^5-5x^3+4x}{120} . \frac{x^6-5x^4+4x^2}{720} \frac{x^7-14x^5+49x^3-36x}{5040}$ &c. per quos ſi termini ſeriei k, l, m, n, o, p, &c. reſpectivè multiplicentur, aggregatum factorum $k+xl+\frac{x^2}{2}m+\frac{x^3-x}{6}n+\frac{x^4-x^2}{24}o+\frac{x^5-5x^3+4x}{120}p+$ &c. erit longitudo ordinatim applicatæ PQ.

CAS. II.

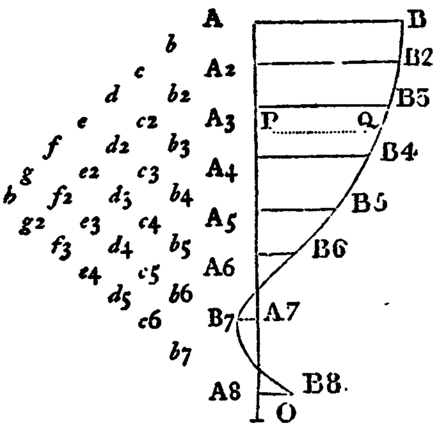
In Caſu poſteriori, ſint A4B4, A5B5 duo medii termini; hoc eſt, ſit $\frac{A4B4+A5B5}{2}=k$; $b4=l$; $\frac{c3+c4}{2}=m$; $d3=n$; $e2+e3=o$; $f2=p$; $\frac{g+g2}{2}=q$; & $h=r$. Et erectâ ordinatim applicatâ PQ, biſeca A4A5 in O, & dicto OP$=x$, duc terminos hujus progreſſionis $1 \times \frac{x}{1} \times \frac{xx-\frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx-\frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx-\frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx-\frac{49}{4}}{8x}$, &c. in ſe continuò; et orientur termini $1 . x . \frac{4xx-1}{8} . \frac{4x^3-x}{24} . \frac{16x^4-40x^2+9}{384}$. &c. per quos ſi termini ſeriei k, l, m, n, o, p, q, &c. reſpectivè multiplicentur, aggregatum factorum $k+xl+\frac{4x^2-1}{8}m+\frac{4x^3-x}{24}n+\frac{16x^4-40x^2+9}{384}o+$ &c. erit longitudo ordinatim applicatæ PQ.

Sed hîc notandum eſt, quòd intervalla AA2, A2A3, A3A4, &c. hîc ſupponantur eſſe unitates, & quòd differentiæ colligi debent auferendo inferiores quantitates de ſuperioribus, A2B2 de AB, A3B3 de A2B2, $b2$ de b, &c. et faciendo ut ſint AB$-$A2B2$=b$, A2B2$-$A3B3$=b2$, $b-b2=c$, &c. adeoque quando differentiæ illæ hoc modo prodeunt negativæ, ſigna earum mutanda ſunt.

PROP. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. invenire Curvam Geometricam generis Parabolici, quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abſciſſam quamvis, AA7, demitte Ordinatas perpendiculariter BA, B2A2, &c. Et



Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$, $\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$, $\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$, $\frac{-A7B7 - A8B8}{A7A8} = b7$. Prop. IV.
Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c. Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c. Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.

Sic pergendum eſt ad ultimam differentiam.

Differentiis ſic collectis & diviſis per intervalla ordinatim applicatarum; in alternis earum columnis, ſive ſeriebus vel ordinibus, excerpe medias, incipiendo ab ultimâ; & in reliquis columnis excerpe media arithmetica inter duas medias, pergendo uſque ad ſeriem primorum terminorum, AB, A2B2, &c. Sunto hæc k, l, m, n, o, p, q, r, &c. quorum ultimus terminus ſignificet ultimam differentiam; penultimus medium arithmeticum inter duas penultimas; antepenultimus mediam trium antepenultimarum, &c. Et primus k erit media ordinatim applicata, ſi numerus datorum punctorum eſt impar; vel medium arithmeticum inter duas medias, ſi numerus eorum eſt par.

CAS. I.

In Caſu priori, ſit A4B4 iſta media ordinatim applicata; hoc eſt, ſit $A4B4 = k$; $\frac{b3 + b4}{2} = l$; $c3 = m$; $\frac{d2 + d3}{2} = n$; $e2 = o$; $\frac{f + f2}{2} = p$; $g = q$. Et erectâ ordinatim applicatâ PQ, & in baſi AA5 ſumpto quovis puncto, O, dic $OP = x$, & duc in ſe gradatim terminos hujus progreſſionis, $1 \times \overline{x - OA4} \times \overline{x - \frac{OA3 + OA5}{2}} \times \frac{\overline{x - OA3} \times \overline{x - OA5}}{x - \frac{1}{2}\overline{OA3 + OA5}} \times \overline{x - \frac{OA2 + OA6}{2}} \times$ &c. et ortam progreſſionem aſſerva; vel quod perinde eſt, duc terminos hujus progreſſionis $1 \times \overline{x - OA4} \times \overline{x - OA3} \times \overline{x - OA5} \times \overline{x - OA2} \times \overline{x - OA6} \times \overline{x - OA} \times \overline{x - OA7} \times$ &c. in ſe gradatim; & terminos exinde ortos duc reſpectivè in terminos hujus progreſſionis, $1.\ x - \frac{+OA3 + OA5}{2}.\ x - \frac{+OA2 + OA6}{2}.\ x - \frac{+OA + OA7}{2}$, &c. et orientur

termini

Prop. IV. termini intermedii, totâ progreſſione exiſtente $1.\ \overline{x-\mathrm{OA4}}.\ x^2 - \frac{+\mathrm{OA3}+2\mathrm{OA4}+\mathrm{OA5}}{2}x + \frac{\mathrm{OA3}+\mathrm{OA5}}{2}\times\mathrm{OA4}$, &c.

Vel dic $\mathrm{OA}=\alpha$; $\mathrm{OA2}=\beta$; $\mathrm{OA3}=\gamma$; $\mathrm{OA4}=\delta$; $\mathrm{OA5}=\varepsilon$; $\mathrm{OA6}=\zeta$; $\mathrm{OA7}=\eta$: $\frac{\mathrm{OA3}+\mathrm{OA5}}{2}=\theta$; $\frac{\mathrm{OA2}+\mathrm{OA6}}{2}=\chi$; $\frac{\mathrm{OA}+\mathrm{OA7}}{2}=\lambda$. Et ex progreſſione $1\times\overline{x-\delta}\times\overline{x-\gamma}\times\overline{x-\varepsilon}\times\overline{x-\beta}\times\overline{x-\zeta}\times\overline{x-\alpha}\times\overline{x-\eta}$ &c. collige terminos. Quibus multiplicatis per $1.\ x-\theta,\ x-\chi,\ x-\lambda$, &c. collige alios terminos intermedios; totâ ſerie prodeunte $1,\ x-\delta,\ x^2-\overline{\delta+\theta}x+\delta\theta,\ x^3-\overline{\delta+2\theta}x^2+\overline{\gamma\varepsilon+2\delta\theta}x-\gamma\delta\varepsilon$, &c. per cujus terminos multiplica ſeries $k,\ l,\ m,\ n,\ o$, &c. Et aggregatum productorum $k+\overline{x-\delta}\times l+\overline{x^2-\overline{\delta+\theta}x+\delta\theta}\times m+$ &c. erit longitudo ordinatim applicatæ PQ.

CAS. II.

In Caſu poſteriori, ſint A4B4, A5B5 duæ mediæ ordinatim applicatæ; hoc eſt, $\frac{A4B4+A5B5}{2}=k$; $b4=l$; $\frac{c3+c4}{2}=m$; $d3=n$; $\frac{e2+e3}{2}=o$; $f2=p$; &c. Et alternorum $k,\ m,\ o,\ q$, &c coefficientes orientur ex multiplicatione terminorum hujus progreſſionis in ſe; $1\times\overline{x-\mathrm{OA4}}\times\overline{x-\mathrm{OA5}}\times\overline{x-\mathrm{OA3}}\times\overline{x-\mathrm{OA6}}\times\overline{x-\mathrm{OA2}}\times\overline{x-\mathrm{OA7}}\times\overline{x-\mathrm{OA}}\times\overline{x-\mathrm{OA8}}$ &c. Et reliquorum coefficientes ex multiplicatione horum per terminos hujus progreſſionis; $x-\frac{+\mathrm{OA4}+\mathrm{OA5}}{2}$, $x-\frac{+\mathrm{OA3}+\mathrm{OA6}}{2}$, $x-\frac{+\mathrm{OA2}+\mathrm{OA7}}{2}$, $x-\frac{+\mathrm{OA}+\mathrm{OA8}}{2}$, &c. Hoc eſt, erit $k+\overline{x-\frac{+\mathrm{OA4}+\mathrm{OA5}}{2}}\times l+\overline{x^2-\overline{\mathrm{OA4}+\mathrm{OA5}}x+\mathrm{OA4}\times\mathrm{OA5}}\times m$ &c. ordinatim applicata PQ. Vel PQ=

$$k+\begin{array}{l}x\\-\tfrac{1}{2}\mathrm{OA4}\\-\tfrac{1}{2}\mathrm{OA5}\end{array}\times l+\begin{array}{l}x\\-\mathrm{OA4}\end{array}\times+\begin{array}{l}x\\-\mathrm{OA5}\end{array}\times m+\begin{array}{l}x\\-\mathrm{OA4}\end{array}\times+\begin{array}{l}x\\-\mathrm{OA5}\end{array}\times+\begin{array}{l}x\\-\tfrac{1}{2}\mathrm{OA3}\\-\tfrac{1}{2}\mathrm{OA6}\end{array}\times n\ \&c.$$

Sive dic $x-\frac{+\mathrm{OA4}+\mathrm{OA5}}{2}=\pi$, $\overline{x-\mathrm{OA4}}\times\overline{x-\mathrm{OA5}}=\varrho$,

$\varrho\times\overline{x-\frac{+\mathrm{OA3}+\mathrm{OA6}}{2}}=\sigma$, $\varrho\times\overline{x-\mathrm{OA3}}\times\overline{x-\mathrm{OA6}}=\tau$,

$\tau\times\overline{x-\frac{+\mathrm{OA2}+\mathrm{OA7}}{2}}=\upsilon$, $\tau\times\overline{x-\mathrm{OA2}}\times\overline{x-\mathrm{OA7}}=\varphi$,

$\varphi\times\overline{x-\frac{+\mathrm{OA}+\mathrm{OA8}}{2}}=\chi$, $\varphi\times\overline{x-\mathrm{OA}}\times\overline{x-\mathrm{OA8}}=\psi$,

Et erit $k+\pi l+\varrho m+\sigma n+\tau o+\upsilon p+\varphi q+\chi r+\psi s=$ PQ.

PROP. V.

Prop. V & VI.

Datis aliquot terminis ſeriei cujuſcunque ad data intervalla diſpoſitis, invenire terminum quemvis intermedium quamproximè.

Ad rectam poſitione datam erigantur termini dati in dato angulo, interpoſitis datis intervallis, & per eorum puncta extima, per propoſitiones præcedentes, ducatur Linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per ſeriem totam.

PROP. VI.

Figuram quamcunque curvilineam quadrare quamproximè, cujus Ordinatæ aliquot inveniri poſſunt.

Per terminos Ordinatarum ducatur Linea Curva generis Parabolici, ope propoſitionum præcedentium. Hæc enim figuram terminabit, quæ ſemper quadrari poteſt, et cujus area æquabitur areæ figuræ propoſitæ quamproximè.

SCHOLIUM.

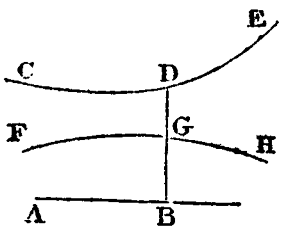
Utiles ſunt hæ Propoſitiones ad tabulas conſtruendas per interpolationem ſerierum, ut & ad ſolutiones Problematum quæ à quadraturis Curvarum dependent; præſertim ſi Ordinatarum intervalla & parva ſint & æqualia inter ſe, & Regulæ computentur, & in uſum reſerventur pro dato quocunque numero Ordinatarum. Ut ſi quatuor ſint Ordinatæ ad æqualia intervalla ſitæ, ſit A ſumma primæ & quartæ, B ſumma ſecundæ & tertiæ, & R intervallum inter primam & quartam; & Ordinata nova in medio omnium erit $\frac{9B - A}{16}$, & Area tota inter primam & quartam erit $\frac{A + 3B}{8}$ R.

Et nota, quod ubi Ordinatæ ſtant ad æquales ab invicem diſtantias, ſumendo ſummas Ordinatarum quæ ab Ordinatâ mediâ hinc inde æqualiter diſtant, & duplum Ordinatæ mediæ, componitur Curva nova, cujus area per pauciores Ordinatas determinatur, & æqualis eſt areæ Curvæ prioris, quam invenire oportuit. Quinetiam ſi pro Ordinatis novis ſumantur ſumma Ordinatæ primæ & ſecundæ,

SCHOLIUM. ſecundæ, et ſumma tertiæ & quartæ, & ſumma quintæ & ſextæ, & ſic deinceps; vel ſi ſumantur ſumma trium primarum Ordinatarum, & ſumma trium proximarum, & ſumma trium quæ ſunt deinceps; vel ſi ſumantur ſummæ quaternarum Ordinatarum, vel ſummæ quinarum; Area Curvæ novæ æqualis erit areæ Curvæ primò propoſitæ. Et ſic habitis Curvæ quadrandæ Ordinatis quotcunque, quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per data verò puncta quotcunque non ſolùm Curvæ lineæ generis Parabolici, ſed etiam Curvæ lineæ innumeræ diverſorum generum duci poſſunt.

Sunto CDE, FGH Curvæ duæ Abſciſſam habentes communem AB, et Ordinatas in eâdem rectâ jacentes BD, BG; & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque, per quæ Curva CDE tranſire debet, & per æquationem illam dabuntur puncta totidem nova, per quæ Curva FGH tranſibit. Per propoſitiones ſuperiores deſcribatur Curva FGH generis Parabolici, quæ per puncta illa omnia nova tranſeat; & per æquationem eandem dabitur Curva CDE, quæ per puncta omnia primò data tranſibit.



ENUMERATIO

LINEARUM TERTII ORDINIS.

ARGUMENTI CAPITUM HUJUS LIBRI.

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ENUMERATIO LINEARUM TERTII ORDINIS.

CAPUT PRIMUM.

Linearum Ordines.

LINEÆ Geometricæ ſecundum numerum dimenſionum æquationis, quâ relatio inter Ordinatas & Abſciſſas definitur, vel (quod perinde eſt) ſecundum numerum punctorum in quibus à lineâ rectâ ſecari poſſunt, optimè diſtinguuntur in Ordines. Quâ ratione linea Primi Ordinis erit recta ſola; eæ Secundi ſive quadratici Ordinis erunt Sectiones Conicæ & Circulus; & eæ Tertii ſive cubici Ordinis Parabola Cubica, Parabola *Neiliana*, Ciſſois Veterum, & reliquæ quas hîc enumerare ſuſcepimus. Curva autem primi generis (ſiquidem recta inter curvas non eſt numeranda) eadem eſt cum lineâ ſecundi ordinis; & Curva ſecundi generis eadem cum lineâ ordinis tertii. Et linea ordinis infiniteſimi, ea eſt quam recta in punctis infinitis ſecare poteſt; qualis eſt Spiralis, Cycloïs, Quadratrix, & linea omnis quæ per radii, vel rotæ, revolutiones infinitas generatur.

CAPUT SECUNDUM.

Proprietates Sectionum Conicarum competunt Curvis ſuperiorum generum.

SECTIONUM conicarum proprietates præcipuæ à Geometris paſſim traduntur. Et conſimiles ſunt proprietates Curvarum ſecundi generis, & reliquarum, ut ex ſequenti proprietatum præcipuarum enumeratione conſtabit.

CAPUT II. 1. *De Curvarum ſecundi generis Ordinatis, Diametris, Verticibus, Centris, Axibus.*

Si rectæ plures parallelæ, & ad Conicam Sectionem utrinque terminatæ, ducantur, recta duas earum biſecans biſecabit alias omnes, ideoque dicitur *Diameter* figuræ; & rectæ biſectæ dicuntur *Ordinatim Applicatæ* ad diametrum; & concurſus omnium diametrorum eſt *Centrum* figuræ; & interſectio Curvæ & Diametri *Vertex* nominatur; & diameter illa *Axis* eſt, cui ordinatim applicatæ inſiſtunt ad angulos rectos. Et ad eundem modum in Curvis ſecundi generis, ſi rectæ duæ quævis parallelæ ducantur occurrentes Curvæ in tribus punctis: recta, quæ ita ſecat has parallelas, ut ſumma duarum partium, ex uno ſecantis latere ad Curvam terminatarum, æquetur parti tertiæ ex altero latere ad Curvam terminatæ, eodem modo ſecabit omnes alias his parallelas, Curvæque in tribus punctis occurrentes rectas; hoc eſt, ita ut ſumma partium duarum, ex uno ipſius latere, ſemper æquetur parti tertiæ ex altero latere. Has itaque tres partes, quæ hinc inde æquantur, *Ordinatim Applicatas*; & rectam ſecantem, cui ordinatim applicantur, *Diametrum*; & interſectionem diametri & Curvæ, *Verticem*; & concurſum duarum diametrorum, *Centrum* nominare licet. Diameter autem ad ordinatas rectangula, ſi modò aliqua ſit, etiam *Axis* dici poteſt; & ubi omnes diametri in eodem puncto occurrunt, iſtud erit *Centrum Generale.*

2. *De Aſymptotis & earum proprietatibus.*

Hyperbola primi generis duas *Aſymptotos*, ea ſecundi tres, ea tertii quatuor & non plures habere poteſt, & ſic in reliquis. Et quemadmodum partes lineæ cujuſvis rectæ, inter Hyperbolam Conicam & duas ejus Aſymptotos, ſunt hinc inde æquales: ſic in Hyperbolis ſecundi generis, ſi ducatur recta quævis ſecans tam Curvam quàm tres ejus Aſymptotos in tribus punctis, ſumma duarum partium iſtius rectæ, quæ à duobus quibuſvis Aſymptotis in eandem plagam ad duo puncta Curvæ extenduntur, æqualis erit parti tertiæ, quæ à tertiâ Aſymptoto in plagam contrariam ad tertium Curvæ punctum extenditur.

3. *De Lateribus Rectis & Tranſverſis.*

De Curvarum Analogia.

Et quemadmodum in Conicis Sectionibus non Parabolicis, quadratum ordinatim applicatæ, hoc eſt, rectangulum Ordinatarum, quæ ad contrarias partes diametri ducuntur, eſt ad rectangulum partium diametri, quæ ad vertices Ellipſeos vel Hyperbolæ terminantur, ut data quædam linea, quæ dicitur *Latus Rectum*, ad partem diametri, quæ inter Vertices jacet & dicitur *Latus Tranſverſum*: ſic in Curvis non Parabolicis ſecundi generis, parallelepipedum ſub tribus ordinatim applicatis eſt ad parallelepipedum ſub partibus diametri, ad ordinatas & tres vertices figuræ abſciſſis, in ratione quâdam datâ: in quâ ratione ſi ſumantur tres rectæ ad tres partes diametri, inter vertices figuræ ſitas, ſingulæ ad ſingulas, tunc illæ tres rectæ dici poſſunt *Latera Recta* figuræ; & illæ partes diametri inter vertices, *Latera Tranſverſa*. Et ſicut in Parabolâ Conicâ, quæ ad unam & eandem diametrum unicum tantùm habet verticem, rectangulum ſub ordinatis æquatur rectangulo ſub parte diametri, quæ ad ordinatas & verticem abſcinditur, & rectâ quâdam datâ, quæ *Latus Rectum* dicitur: ſic in Curvis ſecundi generis, quæ non niſi duos habent vertices ad eandem diametrum, parallelopipedum ſub ordinatis tribus æquatur parallelopipedo ſub duabus partibus diametri, ad ordinatas & vertices illos duos abſciſſis, & rectâ quàdam datâ, quæ proinde *Latus Rectum* dici poteſt.

4. *De Ratione contentorum ſub parallelarum ſegmentis.*

Denique ſicut in Conicis Sectionibus, ubi duæ parallelæ ad Curvam utrinque terminatæ ſecantur à duabus parallelis ad Curvam utrinque terminatis, prima à tertiâ, & ſecunda à quartâ, rectangulum partium primæ eſt ad rectangulum partium tertiæ ut rectangulum partium ſecundæ ad rectangulum partium quartæ: ſic ubi quatuor tales rectæ occurrunt Curvæ ſecundi generis, ſingulæ in tribus punctis, parallelopipedum partium primæ rectæ erit ad parallelopipedum partium tertiæ, ut parallelopipedum partium ſecundæ ad parallelopipedum partium quartæ.

5. De

Caput III. 5. *De Cruribus Hyperbolicis & Parabolicis & eorum plagis.*

Curvarum ſecundi & ſuperiorum generum, æquè atque primi, crura omnia, in infinitum progredientia, vel *Hyperbolici* ſunt generis vel *Parabolici*. Crus *Hyperbolicum* voco, quod ad Aſymptoton aliquam in infinitum appropinquat; *Parabolicum*, quod Aſymptoto deſtituitur. Hæc crura ex tangentibus optimè dignoſcuntur. Nam ſi punctum contactûs in infinitum abeat, tangens cruris Hyperbolici cum Aſymptoto coincidet, & tangens cruris Parabolici in infinitum recedet, evaneſcet, & nullibi reperietur. Invenitur igitur Aſymptotos cruris cujuſvis, quærendo tangentem cruris illius ad punctum infinitè diſtans. Plaga autem cruris infiniti invenitur, quærendo poſitionem rectæ cujuſvis, quæ tangenti parallela eſt, ubi punctum contactûs in infinitum abit. Nam hæc recta in eandem plagam cum crure infinito dirigitur.

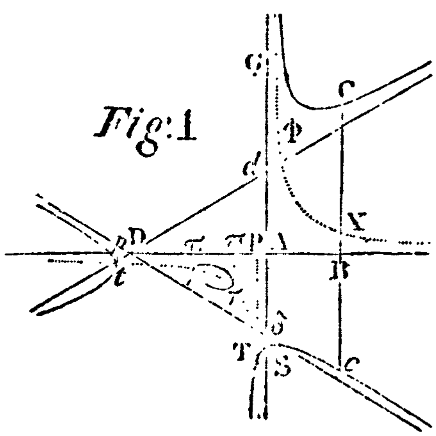
CAPUT TERTIUM.

Reductio curvarum omnium generis ſecundi ad æquationum caſus quatuor.

1. LINEÆ omnes Ordinis primi, tertii, quinti, ſeptimi & imparis cujuſque, duo habent ad minimum crura in infinitum verſus plagas oppoſitas progredientia. Et lineæ omnes Tertii ordinis duo habent ejuſmodi crura in plagas oppoſitas progredientia, in quas nulla alia earum crura infinita (præterquam in Parabolâ *Carteſianâ*) tendunt.

CAS. I.

Fig. 1



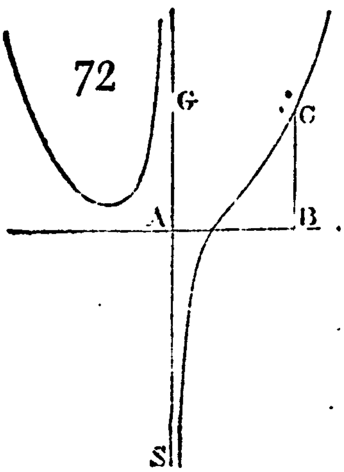
Si crura illa ſint Hyperbolici generis, ſit GAS eorum Aſymptotos; & huic parallela agatur recta quævis, CBc, ad Curvam utrinque (ſi fieri poteſt) terminata, eademque biſecetur in puncto X; & locus puncti illius X erit Hyperbola Conica, puta XΦ, cujus una Aſymptotos eſt AG. Sit ejus altera Aſymptotos AB; & æquatio, quâ relatio inter Ordinatam BC & Abſciſſam AB definitur,

finitur, ſi AB dicatur x, & BC y, ſemper induet hanc formam, $xyy+ey=ax^3+bxx+cx+d$. Ubi termini, e, a, b, c, d, deſignant quantitates datas, ſignis ſuis + & − affectas; quarum quælibet deeſſe poſſunt, modò ex earum defectu figura in ſectionem conicam non vertatur. Poteſt autem Hyperbola illa Conica cum aſymptotis ſuis coincidere, id eſt punctum X in rectâ AB locari: & tunc terminus + ey deeſt.

Genera Linearum Tertii Ordinis.

CAS. II.

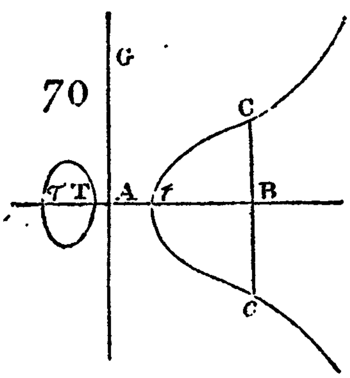
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At ſi recta illa CB*c* non poteſt utrinque ad Curvam terminari, ſed Curvæ in unico tantùm puncto occurrit: age quamvis poſitione datam rectam, AB, Aſymptoto AS occurrentem in A; ut & aliam quamvis, BC, Aſymptoto illi parallelam, Curvæque occurrentem in puncto C: & æquatio, quâ relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam, $xy=ax^3+bx^2+cx+d$.

CAS. III.

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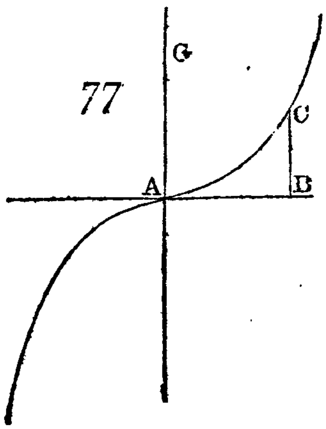
Quòd ſi crura illa oppoſita Parabolici ſint generis, recta CB*c* ad Curvam utrinque, ſi fieri poteſt, terminata in plagam crurum ducatur, & biſecetur in B: & locus puncti B erit linea recta. Sit iſta AB, terminata ad datum quodvis punctum A; & æquatio, quâ relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam, $yy=ax^3+bx^2+cx+d$.

CAS. IV.

At verò ſi recta illa CB*c* in unico tantùm puncto occurrat Curvæ,

Caput III.

77



væ, ideoque ad Curvam utrinque terminari non poſſit: ſit punctum illud C, & incidat recta illa, ad punctum B, in rectam quamvis aliam poſitione datam, & ad datum quodvis punctum A terminatam, AB: & æquatio, quâ relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam, $y=ax^3+bx^2+cx+d$.

Nomina formarum.

2. Enumerando Curvas horum caſuum, Hyperbolam vocabimus *Inſcriptam*, quæ tota jacet in Aſymptoton angulo, ad inſtar Hyperbolæ Conicæ; *Circumſcriptam*, quæ Aſymptotos ſecat, & partes abſciſſas in ſinu ſuo amplectitur; *Ambigenam*, quæ uno crure infinito inſcribitur, & altero circumſcribitur; *Convergentem*, cujus crura concavitate ſuâ ſe invicem reſpiciunt, & in plagam eandem diriguntur; *Divergentem*, cujus crura convexitate ſuâ ſe invicem reſpiciunt, et in plagas contrarias diriguntur; *Cruribus Contrariis præditam*, cujus crura in partes contrarias convexa ſunt, & in plagas contrarias infinita; *Conchoidalem*, quæ vertice concavo & cruribus divergentibus ad Aſymptoton applicatur; *Anguineam*, quæ flexibus contrariis Aſymptoton ſecat, & utrinque in crura contraria producitur; *Cruciformem*, quæ conjugatam decuſſat; *Nodatam*, quæ ſeipſam decuſſat, in orbem redeundo; *Cuſpidatam*, cujus partes duæ in angulo contactûs concurrunt, & ibi terminantur; *Punctatam*, quæ conjugatam habet Ovalem infinitè parvam, id eſt punctum; & *Puram*, quæ per impoſſibilitatem duarum radicum, Ovali, Nodo, Cuſpide & Puncto Conjugato privatur. Eodem ſenſu Parabolam quoque *Convergentem*, *Divergentem*, *Cruribus Contrariis præditam*, *Cruciformem*, *Nodatam*, *Cuſpidatam*, *Punctatam* & *Puram* nominabimus.

3. In caſu primo ſi terminus ax^3 affirmativus eſt, figura erit Hyperbola triplex cum ſex cruribus Hyperbolicis, quæ juxta tres Aſymptotos, quarum nullæ ſunt parallelæ, in infinitum progrediuntur, Linæ juxta unamquamque in plagas contrarias. Et hæ Aſymptoti ſi terminus bx^2 non deeſt, ſe mutuò ſecabunt in tribus

punctis, triangulum $\text{D}d\delta$ inter ſe continentes; ſin terminus bx^2 deeſt, convergent omnes ad idem punctum. In priori caſu, cape $\text{AD}=\frac{-b}{2a}$, & $\text{AD}=\text{A}\delta=\frac{b}{2\sqrt{a}}$, ac junge $\text{D}d$, $\text{D}\delta$; & erunt $\text{A}d$, $\text{D}d$, $\text{D}\delta$ tres Aſymptoti. In poſteriori, duc ordinatam quamvis BC, ordinatæ principali AG parallelam, & in eâ utrinque productâ cape hinc inde BF & Bf ſibi mutuò æquales, & in eâ ratione ad AB quam habet $\sqrt{a}$ ad 1; jungeque AF, Af: & erunt AG, AF, Af tres Aſymptoti. Hanc Hyperbolam vocamus *Redundantem*, quia numero crurum hyperbolicorum ſectiones conicas ſuperat. Genera Linearum Tertii Ordinis.

4. In Hyperbolâ omni Redundante, ſi neque terminus ey deſit, neque ſit $bb-4ac$ æquale $\pm ae\sqrt{a}$, Curva nullam habebit diametrum; ſin eorum alterutrum accidit, Curva habebit unicam diametrum, & tres ſi utrumque. Diameter autem ſemper tranſit per interſectionem duarum Aſymptoton, & biſecat rectas omnes, quæ ad Aſymptotos illas utrinque terminantur, & parallelæ ſunt Aſymptoto tertiæ. Eſtque abſciſſa AB diameter figuræ, quoties terminus ey deeſt. *Diametrum* vero abſolutè dictam, hîc & in ſequentibus, in vulgari ſignificatu uſurpo; nempe pro Abſciſſâ, quæ paſſim habet Ordinatas binas æquales ad idem punctum hinc inde inſiſtentes.

CAPUT QUARTUM.

Enumeratio Curvarum.

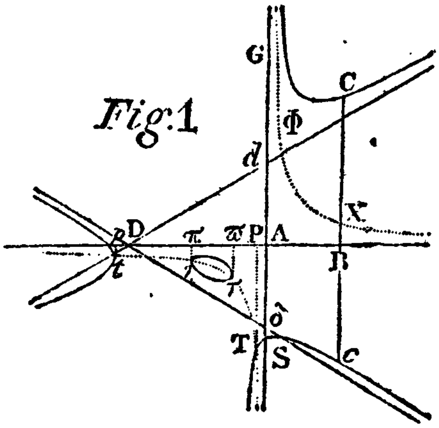
I. *De Hyperbolis novem Redundantibus, quæ diametro deſtituuntur, & tres habent Aſymptotos triangulum capientes.*

1. SI Hyperbola Redundans nullam habet diametrum, quærantur æquationis hujus $ax^4+bx^3+cx^2+dx+\frac{1}{4}ee=y$ radices quatuor, ſeu valores ipſius x. Eæ ſunto AP, Aϖ, Aτ, Ap. Erigantur Ordinatæ PT, $\varpi\tau$, π7, pt, & hæ tangent Curvam in punctis totidem T, τ, 7, t, & tangendo dabunt limites Curvæ, per quos Species ejus innoteſcet.

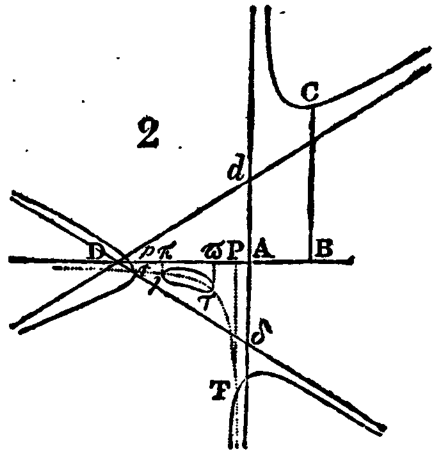
2. Nam ſi radices omnes AP, Aϖ, Ap, (fig. 1, 2) ſunt reales, ejuſdem ſigni, et inæquales, Curva conſtat ex tribus Hyperbolis, (Inſcriptâ, Circumſcriptâ & Ambigenâ) cum *Ovali*. Hyperbolarum una jacet verſus D; altera verſus d; tertia verſus δ; & Ovalis ſemper jacet intra triangulum $\text{D}d\delta$, atque etiam inter medios limites 7 & τ, in quibus utique tangitur ab Ordinatis π7 & $\varpi\tau$. Et hæc eſt *ſpecies prima*.

Caput IV.

Fig:1

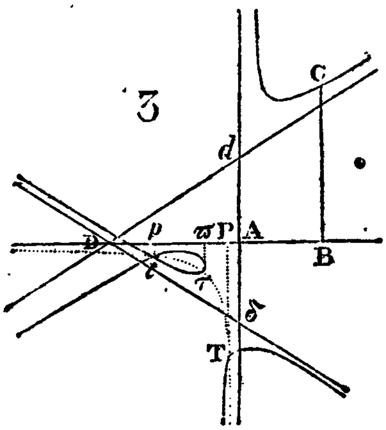


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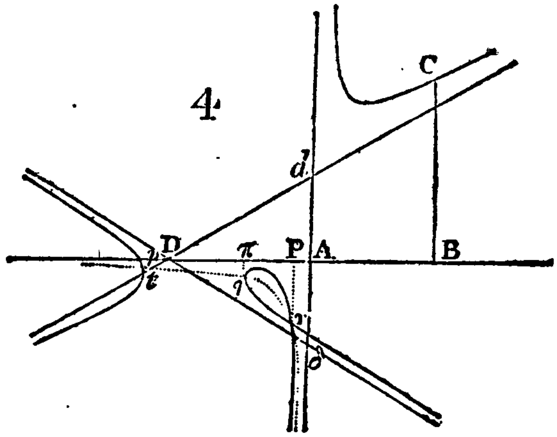


3. Si è radicibus duæ maximæ Aπ, A*p*, (fig. 3) vel duæ minimæ AP, Aϖ, (fig. 4) æquantur inter ſe, & ejuſdem ſunt ſigni cum alteris duobus, Ovalis & Hyperbola Circumſcripta ſibi

3



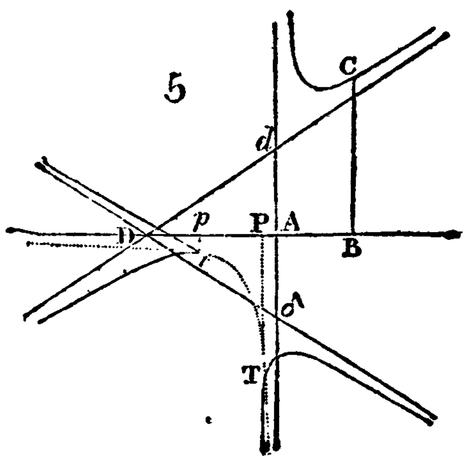
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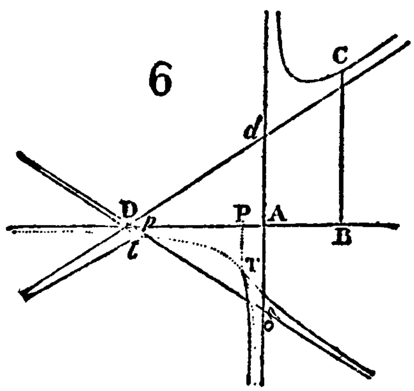
invicem junguntur, coeuntibus earum punctis contactûs, 7 & *t*, vel T & τ; & crura Hyperbolæ, ſeſe decuſſando, in Ovalem continuantur, figuram *Nodatam* efficientia. Quæ *ſpecies* eſt *ſecunda.*

4. Si è radicibus tres maximæ A*p*, Aπ, Aϖ, (fig. 5) vel tres

5



6

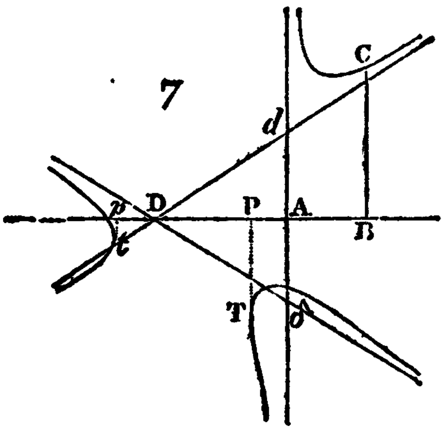


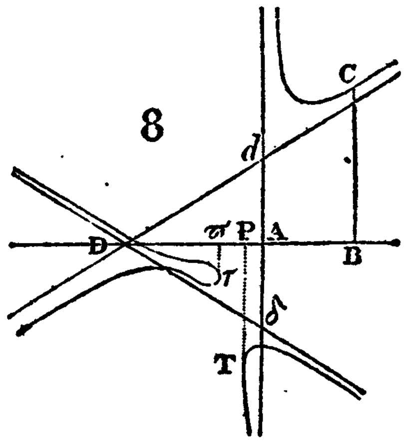
minimæ

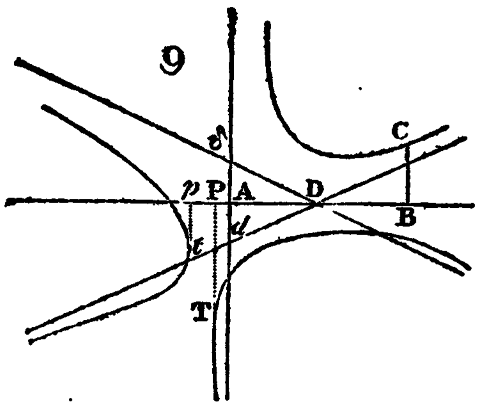
minimæ Aπ, Aϖ, AP (fig. 6) æquentur inter ſe, Nodus in *Cuſpidem* acutiſſimum convertetur. Nam crura duo Hyperbolæ Circumſcriptæ ibi in angulo contactûs concurrent, & non ultrà producentur. Et hæc eſt *ſpecies tertia.*

SPECIES LINEARUM TERTII ORDINIS.

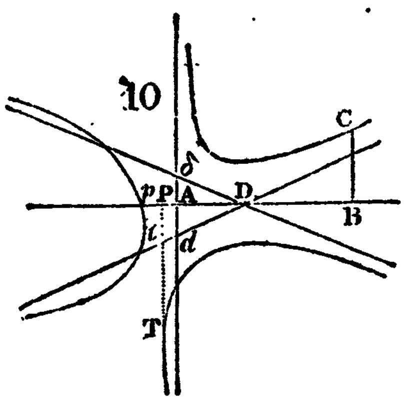
5. Si è radicibus duæ mediæ Aϖ & Aπ (fig. 7) æquentur inter





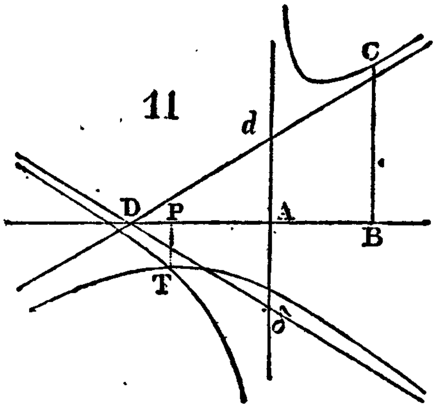


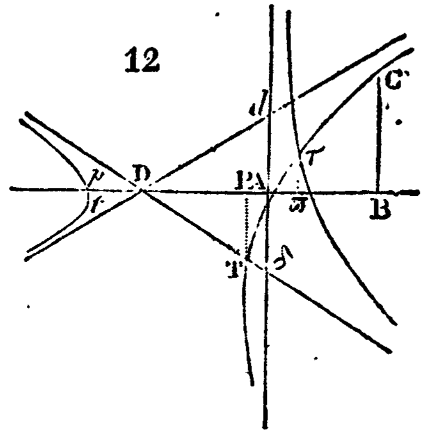
ſe, puncta contactûs τ & 7 coincidunt, & propterea Ovalis interjecta in punctum evanuit; & conſtat figura ex tribus Hyperbolis, Inſcriptâ, Circumſcriptâ, & Ambigenâ, cum *puncto* conjugato. Quæ eſt *ſpecies quarta.*

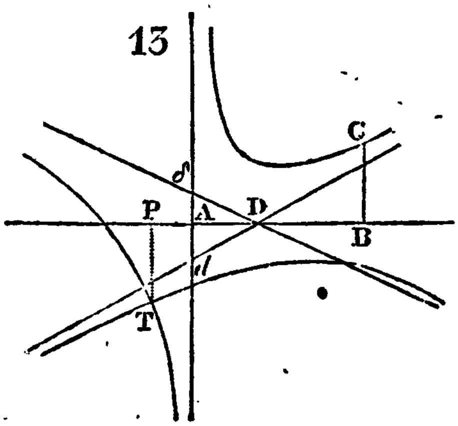


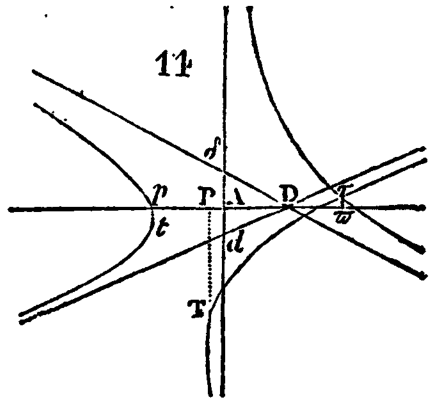
6. Si duæ ex radicibus ſunt impoſſibiles, & reliquæ duæ inæquales, & ejuſdem ſigni (nam ſigna contraria habere nequeunt), *Puræ* habebuntur Hyperbolæ tres, ſine Ovali vel Nodo vel Cuſpide vel puncto conjugato: & hæ Hyperbolæ, vel ad latera trianguli ab Aſymptotis comprehenſi, vel ad angulos ejus jacebunt; & perinde *ſpeciem* vel *quintam* (fig. 7, 8) vel *ſextam* (fig. 9, 10) conſtituent.

7. Si è radicibus duæ ſunt æquales, & alteræ duæ vel impoſſibiles ſunt (fig. 11. 13) vel reales (fig. 12. 14) cum ſignis quæ à ſignis æqualium radicum diverſa ſunt, figura *Cruciformis* habebitur:



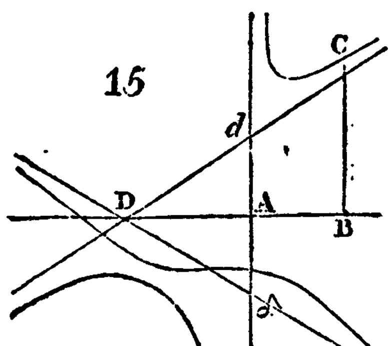


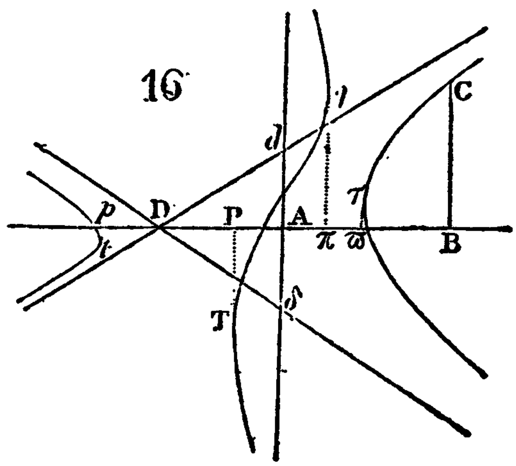




bebitur: nempe duæ ex Hyperbolis ſe invicem decuſſabunt, idque vel ad verticem trianguli ab Aſymptotis comprehenſi (fig. 13, 14) vel ad ejus baſem (fig. 11, 12). Quæ duæ *ſpecies* ſunt *ſeptima* & *octava*.

8. Si denique radices omnes ſunt impoſſibiles (fig. 15) vel ſi





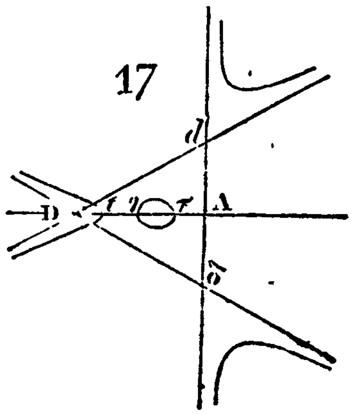
omnes

omnes ſunt reales & inæquales (fig. 16) & earum duæ ſunt aſ- *Species Linearum Tertii Ordinis.*
firmativæ, & alteræ duæ negativæ; tunc duæ habebuntur Hyperbolæ, ad angulos oppoſitos duarum Aſymptoton, cum Hyperbolâ *Anguineâ* circa Aſymptoton tertiam. Quæ *ſpecies* eſt *nona.*

9. Et hi ſunt omnes radicum caſus poſſibiles. Nam ſi duæ radices ſunt æquales inter ſe, & aliæ duæ ſunt etiam inter ſe æquales, figura evadet Sectio Conica cum lineâ rectâ.

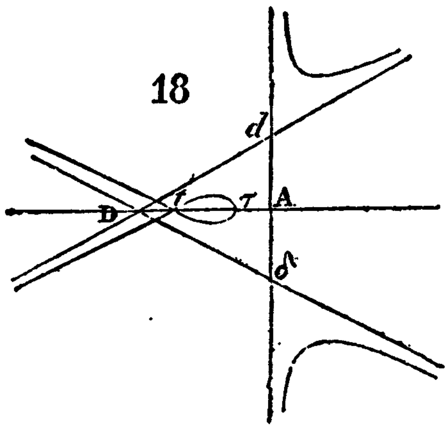
II. *De Hyperbolis duodecim Redundantibus unicam tantùm diametrum habentibus.*

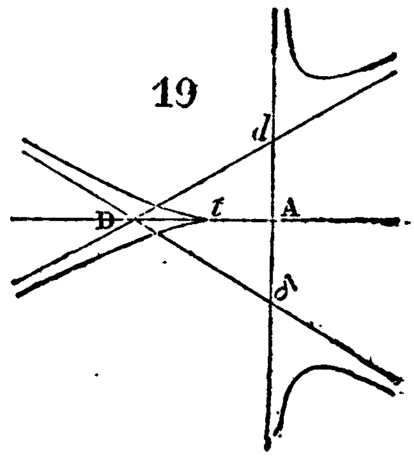
1. Si Hyperbola Redundans habet unicam tantùm diametrum, ſit ejus diameter Abſciſſa AB, & æquationis hujus $ax^3+bx^2+cx+d=0$ quære tres radices, ſeu valores x.



2. Si radices illæ ſunt omnes reales & ejuſdem ſigni, figura conſtabit ex *Ovali* intra triangulum Ddδ (fig. 17) jacente, & tribus Hyperbolis ad angulos ejus; nempe Circumſcriptâ ad angulum D, & Inſcriptis duabus ad angulos d & δ. Et hæc eſt *ſpecies decima.*

3. Si radices duæ majores ſunt æquales, & tertia ejuſdem ſigni, crura Hyperbolæ jacentis verſus D (fig. 18) ſeſe decuſſabunt in formâ *Nodi,* propter contactum Ovalis. Quæ *ſpecies* eſt *undecima.*

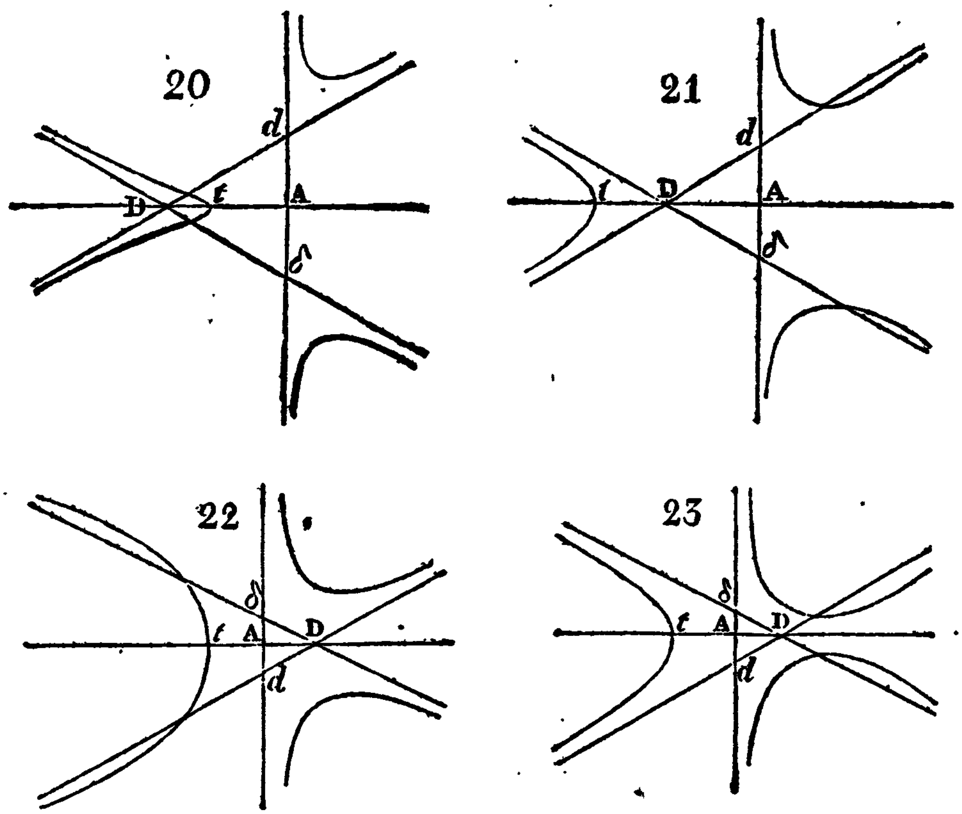




4. Si tres radices ſunt æquales, Hyperbola iſta fit *Cuſpidata* ſine Ovali (fig. 19). Quæ *ſpecies* eſt *duodecima.*

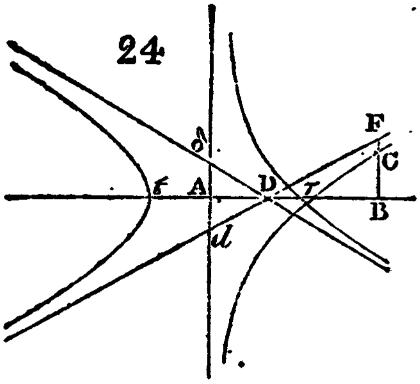
5. Si radices duæ minores ſunt æquales, & tertia ejuſdem ſigni, Ovalis in *punctum* evanuit (fig. 20). Quæ *ſpecies* eſt *decima tertia*.

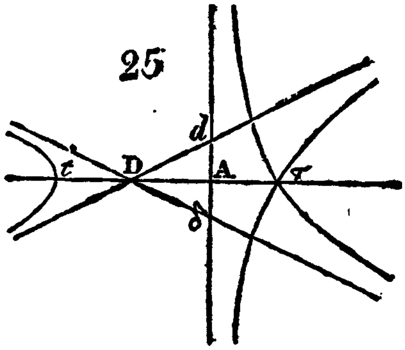
6. In ſpeciebus quatuor noviſſimis, Hyperbola, quæ jacet verſus D, Aſymptotos in ſinu ſuo amplectitur; reliquæ duæ in ſinu Aſymptoton jacent.



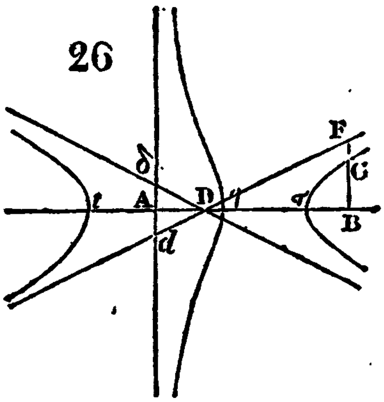
7. Si duæ ex radicibus ſunt impoſſibiles, habebuntur tres Hyperbolæ *Puræ* ſine Ovali, Decuſſatione, vel Cuſpide. Et hujus caſûs *ſpecies* ſunt quatuor: nempe *decima quarta* ſi Hyperbola Circumſcripta jacet verſus D (fig. 20), & *decima quinta* ſi Hyperbola Inſcripta jacet verſus D (fig. 21), *decima ſexta* ſi Hyperbola Circumſcripta jacet ſub baſi *dδ* trianguli D*dδ* (fig. 22), & *decima ſeptima* (fig. 23) ſi Hyperbola Inſcripta jacet ſub eâdem baſi.

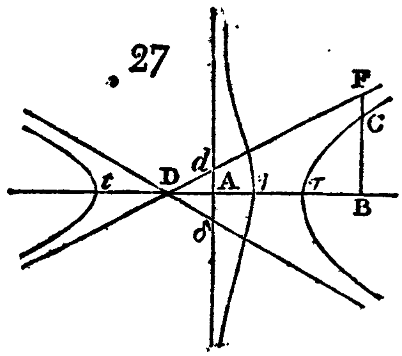
Species Linearum Tertii Ordinis.





8. Si duæ radices ſunt æquales, & tertia ſigni diverſi, figura erit *Cruciformis*. Nempe duæ ex tribus Hyperbolis ſe invicem decuſſabunt, idque vel ad verticem trianguli ab Aſymptotis comprehenſi (fig. 24) vel ad ejus baſem (fig. 25). Quæ duæ *ſpecies* ſunt *decima octava*, & *decima nona*.

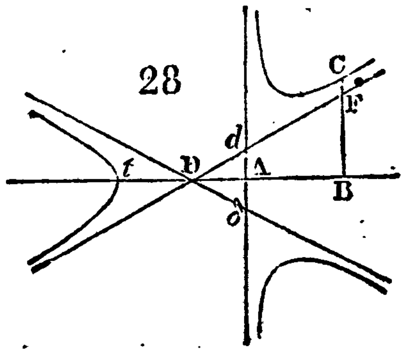


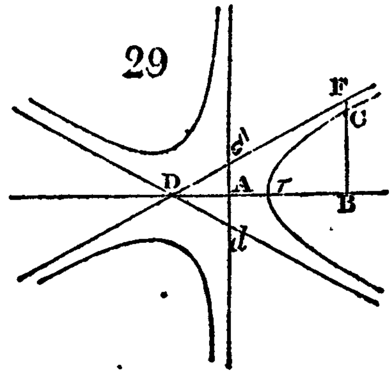


9. Si duæ radices ſunt inæquales, & ejuſdem ſigni, & tertia eſt ſigni diverſi, duæ habebuntur Hyperbolæ in oppoſitis angulis duarum Aſymptoton, cum *Conchoidali* intermediâ. *Conchoidalis* autem vel jacebit ad eaſdem partes Aſymptoti ſuæ cum triangulo ab Aſymptotis conſtituto (fig. 26) vel ad partes contrarias (fig. 27). Et hi duo caſus conſtituunt *ſpeciem vigeſimam* & *vigeſimam primam*.

III. *De*

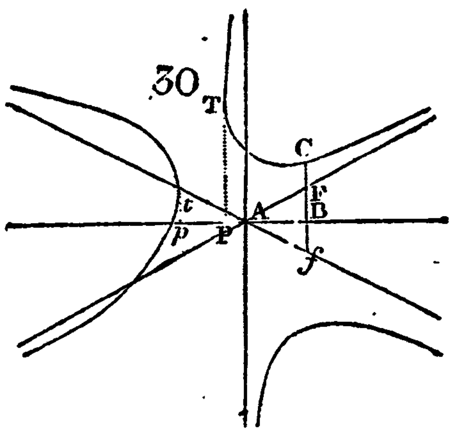
III. *De Hyperbolis duabus Redundantibus cum tribus diametris.*

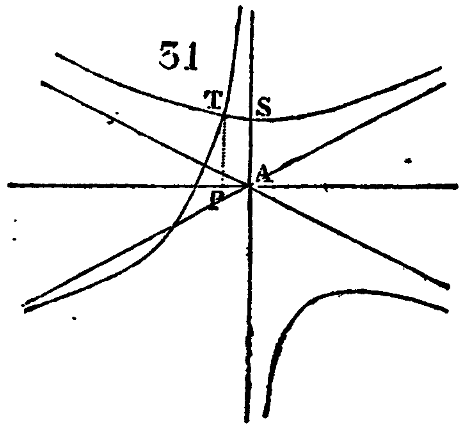


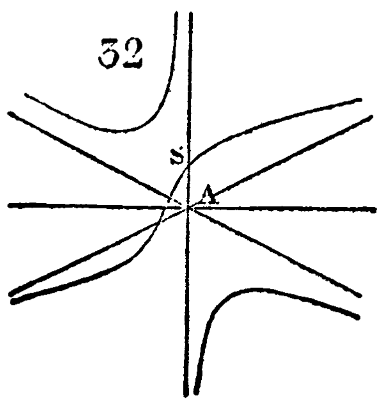


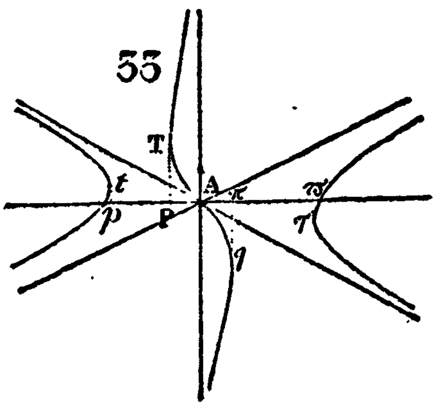
1. Hyperbola Redundans quæ habet tres diametros, conſtat ex tribus Hyperbolis in ſinubus Aſymptoton jacentibus; idque vel ad angulos trianguli ab Aſymptotis comprehenſi (fig. 28) vel ad ejus latera (fig. 29). Caſus prior dat *ſpeciem vigeſimam ſecundam*, & poſterior *ſpeciem vigeſimam tertiam*.

IV. *De Hyperbolis novem Redundantibus cum Aſymptotis tribus ad commune punctum convergentibus.*





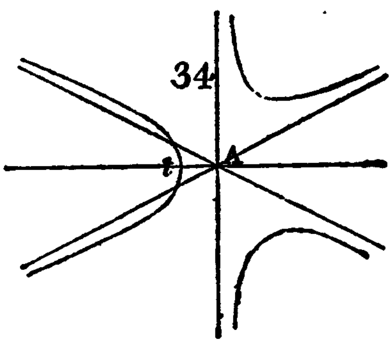


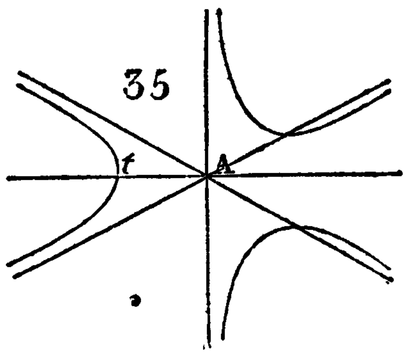


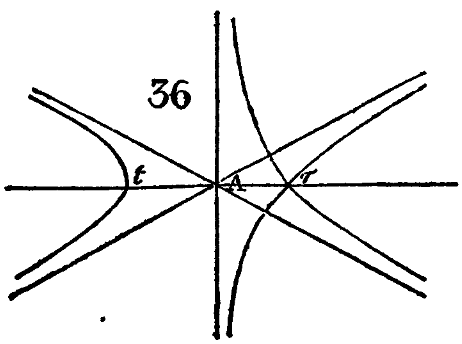
1. Si

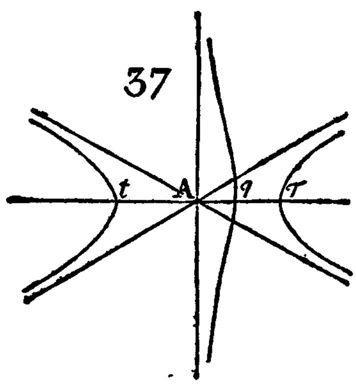
1. Si tres Aſymptoti in puncto communi ſe mutuò decuſſant, vertuntur ſpecies quinta & ſexta in *vigeſimam quartam* (fig. 30), ſeptima & octava in *vigeſimam quintam* (fig. 31), & nona in *vigeſimam ſextam* (fig. 32), ubi Anguinea non tranſit per concurſum Aſymptoton; & in *vigeſimam ſeptimam* ubi tranſit per concurſum illum (fig. 33), quo caſu termini *b* ac *d* deſunt, & concurſus Aſymptoton eſt Centrum figuræ ab omnibus ejus partibus oppoſitis æqualiter diſtans. Et hæ quatuor ſpecies diametrum non habent. SPECIES LINEARUM TERTII ORDINIS.

2. Vertuntur etiam ſpecies decima quarta ac decima ſexta in







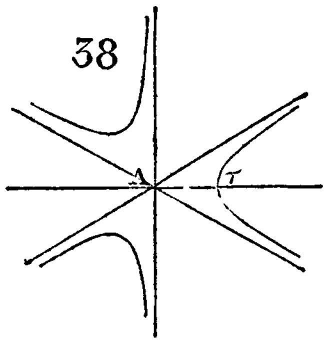


vigeſimam octavam (fig. 34); decima quinta ac decima ſeptima in *vigeſimam nonam* (fig. 35); decima octava & decima nona in *triceſimam* (fig. 36), & vigeſima cum vigeſimâ primâ in *triceſimam primam* (fig. 37). Et hæ ſpecies unicam habent diametrum.

3. Ac denique ſpecies vigeſima ſecunda & vigeſima tertia ver-

 tuntur

38



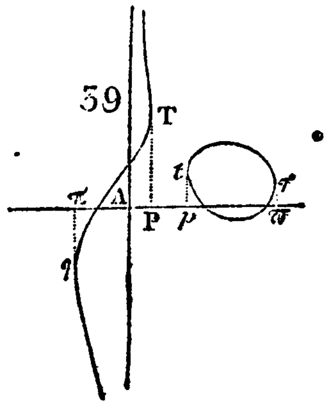
tuntur in *ſpeciem triceſimam ſecundam*, cujus tres ſunt diametri per concurſum Aſymptoton tranſeuntes (fig. 38).

4. Quæ omnes converſiones facillimè intelliguntur, faciendo ut triangulum ab Aſymptotis comprehenſum diminuatur donec in punctum evaneſcat.

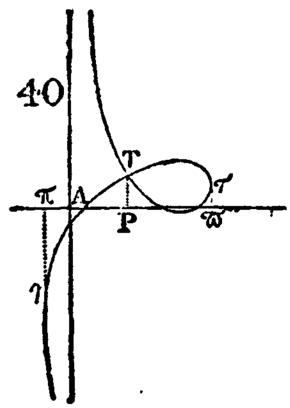
V. *De Hyperbolis ſex Defectivis diametrum non habentibus.*

1. Si in primo æquationum caſu terminus ax^3 negativus eſt, figura erit Hyperbola Defectiva unicam habens Aſymptoton & duo tantùm crura hyperbolica, juxta Aſymptoton illam, in plagas contrarias infinitè progredientia. Et Aſymptotos illa eſt ordinata

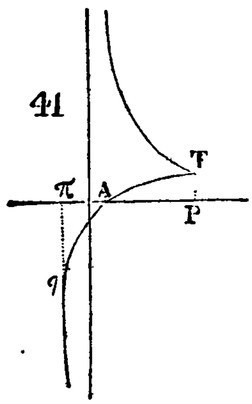
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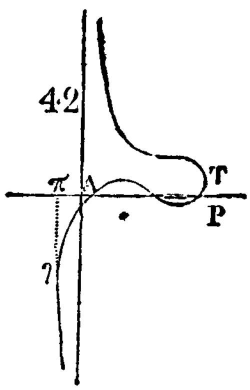
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41



42



prima & principalis AG. Si terminus *ey* non deeſt, figura nullam habebit diametrum; ſi deeſt, habebit unicam. In priori caſu ſpecies ſic enumerantur.

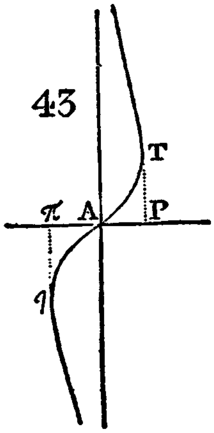
2. Si æquationis hujus $ax^4 = bx^3 + cx^2 + dx + \frac{1}{4}ee$ radices omnes $A\pi$, AP, Ap, $A\varpi$, (fig. 39) ſunt reales & inæquales, figura erit Hyperbola Anguinea, Aſymptoton flexu contrario amplexa, cum *Ovali* conjugatâ. Quæ *ſpecies* eſt *triceſima tertia*.

3. Si radices duæ mediæ AP & Ap (fig. 40) æquentur inter ſe, Ovalis & Anguinea junguntur, ſeſe decuſſantes in formâ *Nodi*. Quæ eſt *ſpecies triceſima quarta*.

4. Si

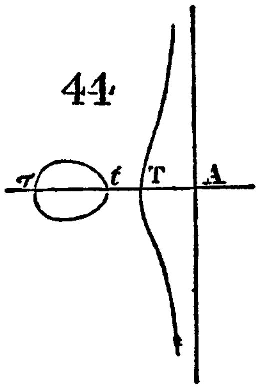
4. Si tres radices ſunt æquales, Nodus vertetur in *Cuſpidem* acutiſſimum in vertice Anguineæ (fig. 41). Et hæc eſt *ſpecies triceſima quinta*. Species Linearum Tertii Ordinis.

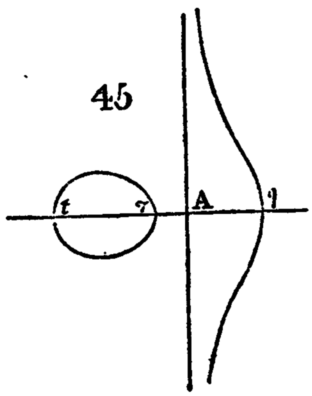
5. Si è tribus radicibus ejuſdem ſigni duæ maximæ A*p* & Aϖ (fig. 43) ſibi mutuò æquantur, Ovalis in *punctum* evanuit. Quæ *ſpecies* eſt *triceſima ſexta*.

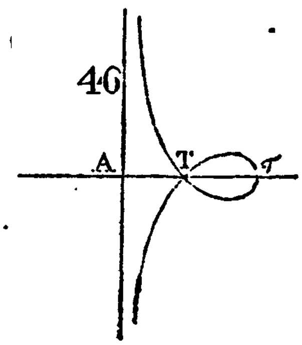


6. Si radices duæ quævis imaginariæ ſunt, ſola manebit Anguinea *Pura* ſine Ovali, Decuſſatione, Cuſpide vel puncto conjugato. Si Anguinea illa non tranſit per punctum A (fig. 42), *ſpecies* eſt *triceſima ſeptima*; ſin tranſit per punctum illud A (id quod contingit ubi termini *b* ac *d* deſunt), punctum illud A erit Centrum figuræ, rectas omnes per ipſum ductas & ad Curvam utrinque terminatas biſecans (fig. 43). Et hæc eſt *ſpecies triceſima octava*.

VI. *De Hyperbolis ſeptem Defectivis diametrum habentibus.*





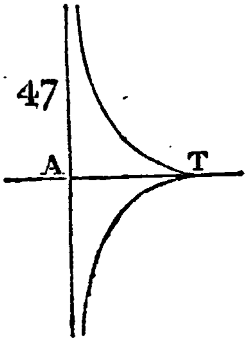


1. In altero caſu ubi terminus *ey* deeſt, & propterea figura diametrum habet, ſi æquationis hujus $ax^3=bx^2+cx+d$ radices omnes AT, A*t*, Aτ, (fig. 44) ſunt reales, inæquales & ejuſdem ſigni, figura erit Hyperbola Conchoidalis cum *Ovali* ad convexitatem. Quæ eſt *ſpecies triceſima nona*.

2. Si duæ radices ſunt inæquales & ejuſdem ſigni, & tertia eſt ſigni contrarii, *Ovalis* jacebit ad Concavitatem conchoidalis (fig. 45). Eſtque *ſpecies quadrageſima*.

3. Si radices duæ minores AT, A*t*, (fig. 46) ſunt æquales, & tertia

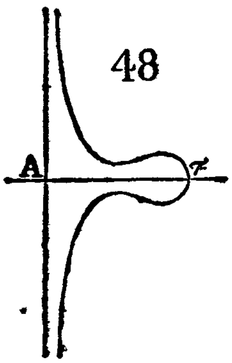
Caput IV.

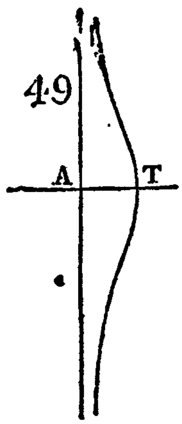


tertia Aτ eſt ejuſdem ſigni, Ovalis & Conchoidalis jungentur, ſeſe decuſſando in modum *Nodi*. Quæ *ſpecies* eſt *quadrageſima prima*.

4. Si tres radices ſunt æquales, Nodus mutabitur in *Cuſpidem*, & figura erit *Ciſſois Veterum* (fig. 47). Et hæc eſt *ſpecies quadrageſima ſecunda*.

5. Si radices duæ majores ſunt æquales, & tertia eſt ejuſdem ſigni, Conchoidalis habebit *punctum* conjugatum ad convexitatem ſuam (fig. 49). Eſtque *ſpecies quadrageſima tertia*.

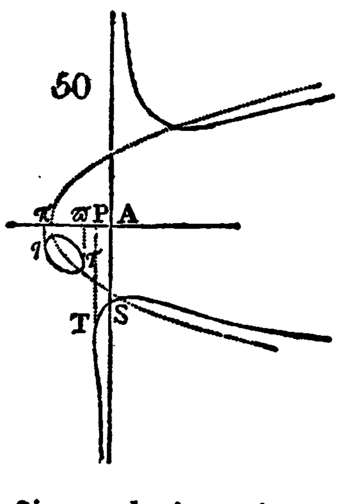


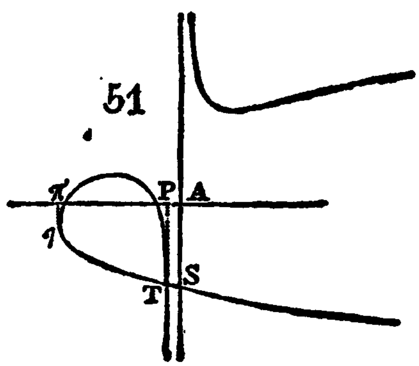


6. Si radices duæ ſunt æquales, & tertia eſt ſigni contrarii, Conchoidalis habebit *punctum* conjugatum ad concavitatem ſuam (fig. 49). Eſtque *ſpecies quadrageſima quarta*.

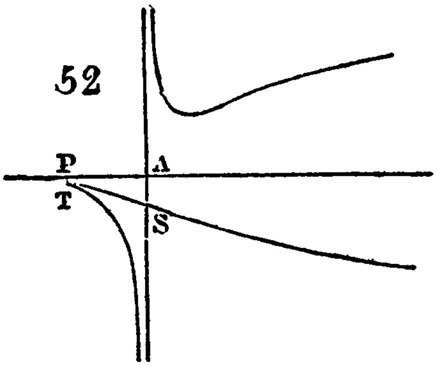
7. Si radices duæ ſunt impoſſibiles, habebitur Conchoidalis *Pura* ſine Ovali, Nodo, Cuſpide vel puncto conjugato (fig. 48, 49). Quæ *ſpecies* eſt *quadrageſima quinta*.

VII. *De Hyperbolis ſeptem Parabolicis diametrum non habentibus.*





1. Siquando in primo æquationum caſu terminus ax^3 deeſt, & terminus



terminus bx^2 non deeſt, figura erit Hyperbola Parabolica, duo habens crura Hyperbolica ad unam Aſymptoton SAG, & duo Parabolica in plagam unam & eandem convergentia. Si terminus ey non deeſt, figura nullam habebit diametrum; ſin deeſt, habebit unicam. In priori caſu ſpecies ſunt hæ.

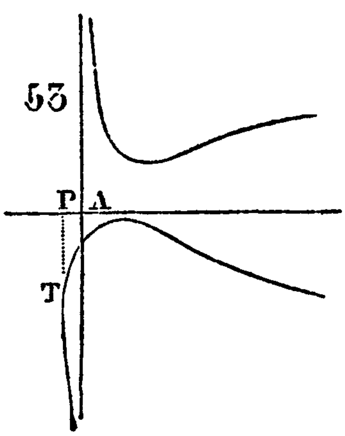
Species Linearum Tertii Ordinis.

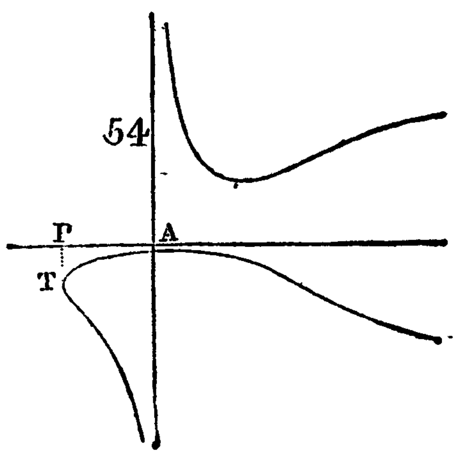
2. Si tres radices AP, Aϖ, Aπ (fig. 50) æquationis hujus $bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ ſunt inæquales & ejuſdem ſigni, figura conſtabit ex *Ovali* & aliis duabus Curvis, quæ partim Hyperbolicæ ſunt & partim Parabolicæ. Nempe crura Parabolica continuo ductu junguntur cruribus Hyperbolicis ſibi proximis. Et hæc eſt *ſpecies quadrageſima ſexta*.

3. Si radices duæ minores ſunt æquales, & tertia eſt ejuſdem ſigni, Ovalis & una Curvarum illarum hyperbolo-parabolicarum junguntur, & ſe decuſſant in formam *Nodi* (fig. 51). Quæ *ſpecies* eſt *quadrageſima ſeptima*.

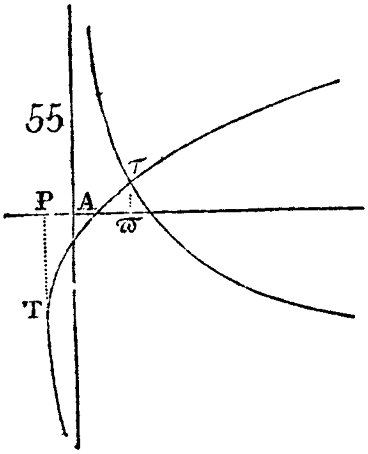
4. Si tres radices ſunt æquales, Nodus ille in Cuſpidem vertitur (fig. 52). Eſtque *ſpecies quadrageſima octava*.

5. Si radices duæ majores ſunt æquales, & tertia eſt ejuſdem ſigni, Ovalis in *punctum* conjugatum evanuit (fig. 53). Quæ *ſpecies* eſt *quadrageſima nona*.



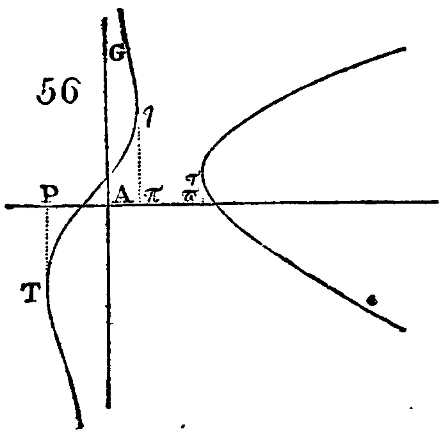


CAPUT IV.



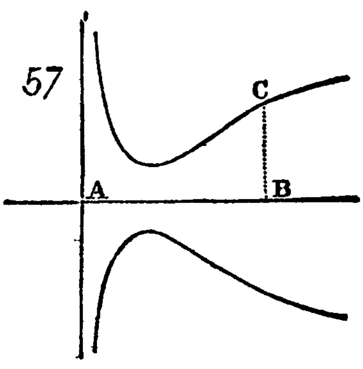
6. Si duæ radices ſunt impoſſibiles, manebunt *Puræ* illæ duæ Curvæ hyperbolo-parabolicæ ſine Ovali, Decuſſatione, Cuſpide vel puncto conjugato; & *ſpeciem quinquageſimam* conſtituent (fig. 53, 54.)

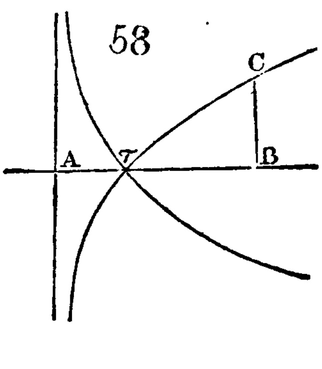
7. Si radices duæ ſunt æquales, & tertia eſt ſigni contrarii, Curvæ illæ hyperbolo-parabolicæ junguntur, ſeſe decuſſando in morem crucis (fig. 55). Eſtque *ſpecies quinquageſima prima*.

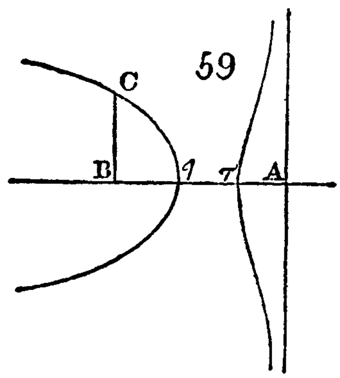


8. Si radices duæ ſunt inæquales & ejuſdem ſigni, & tertia eſt ſigni contrarii, figura evadet Hyperbola Anguinea circa Aſymptoton AG (fig. 56), cum Parabolâ conjugatâ. Et hæc eſt *ſpecies quinquageſima ſecunda*.

VIII. *De Hyperbolis quatuor Parabolicis diametrum habentibus.*







1. In altero caſu ubi terminus *ey* deeſt, & figura diametrum habet, ſi duæ radices æquationis hujus $bx^2+cx+d=0$ ſunt impoſſibiles, duæ habentur figuræ hyperbolo-parabolicæ à diametro AB (fig. 57) hinc inde æqualiter diſtantes. Quæ *ſpecies* eſt *quinquageſima tertia*.

2. Si

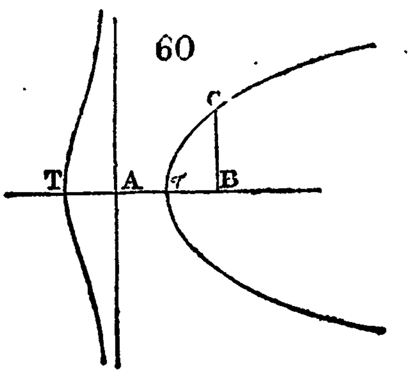
2. Si æquationis illius radices duæ ſunt impoſſibiles, figuræ hyperbolo-parabolicæ junguntur, ſeſe decuſſantes in morem crucis; & *ſpeciem quinquageſimam quartam* conſtituunt (fig. 58). SPECIES LINEARUM TERTII ORDINIS.

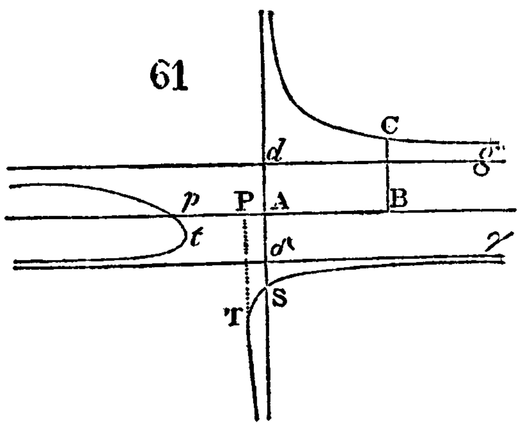
3. Si radices illæ ſunt inæquales & ejuſdem ſigni, habetur Hyperbola Conchoidalis cum Parabolâ ex eodem latere Aſymptoti (fig. 59). Eſtque *ſpecies quinquageſima quinta.*

4. Si radices illæ ſunt ſigni contrarii, habetur Conchoidalis cum Parabolâ ad alteras partes Aſymptoti (fig. 60). Quæ *ſpecies* eſt *quinquageſima ſexta.*

IX. *De quatuor Hyperboliſmis Hyperbolæ.*

1. Siquando in primo æquationum caſu terminus uterque ax^3 & bx^2 deeſt, figura erit Hyperboliſmus ſectionis alicujus conicæ. Hyperboliſmum figuræ voco, cujus Ordinata prodit applicando contentum ſub Ordinatâ figuræ illius & rectâ datâ ad Abſciſſam communem. Hâc ratione linea recta vertitur in Hyperbolam Conicam, & Sectio omnis Conica vertitur in aliquam figurarum, quas hîc Hyperboliſmos ſectionum conicarum voco. Nam æquatio ad figuras de quibus agimus, nempe $xy^2+ey=cx+d$, dat



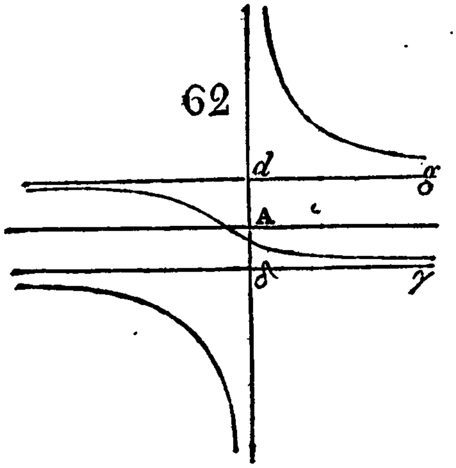


$y=\frac{e\pm\sqrt{ee+4dx+4cxx}}{2x}$, quæ generatur applicando contentum ſub Ordinatâ ſectionis conicæ $\frac{e\pm\sqrt{ee+4dx+4cxx}}{2m}$ & rectâ datâ m, ad Curvarum Abſciſſam communem x. Unde liquet, quòd figura genita Hyperboliſmus erit Hyperbolæ, Ellipſeos vel Parabolæ, perinde ut terminus ex affirmativus eſt, vel negativus, vel nullus.

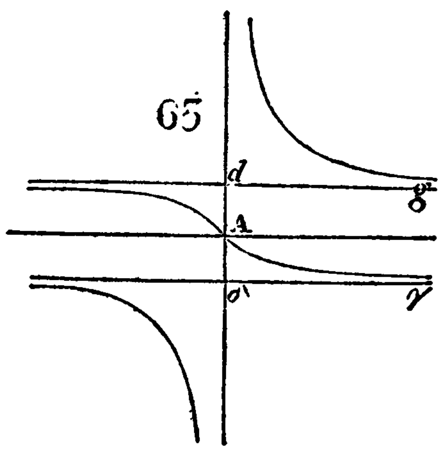
2. Hyper-

2. Hyperboliſmus Hyperbolæ tres habet Aſymptotos; quarum una eſt Ordinata prima & principalis Ad, alteræ duæ ſunt parallelæ Abſciſſæ AB, ab eâdem hinc inde æqualiter diſtantes. In Ordinatâ principali Ad, cape Ad, Aδ hinc inde æquales quantitati $\surd c$; & per puncta d ac δ age dg, $\delta\gamma$ Aſymptotos abſciſſæ AB parallelas.

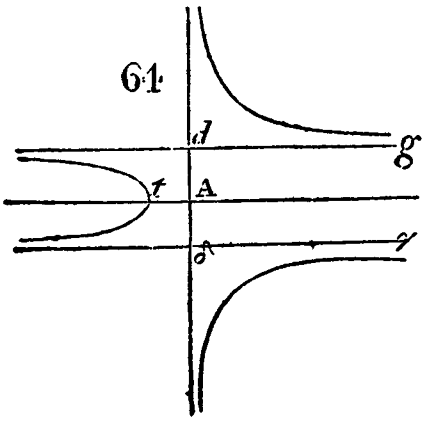
3. Ubi terminus ey non deeſt, figura nullam habet diametrum. In hoc caſu, ſi æquationis hujus $cx^2+dx+\frac{1}{4}ee=0$ radices duæ AP, Ap (fig. 61) ſunt reales & inæquales (nam æquales eſſe nequeunt niſi figura ſit Conica Sectio) figura conſtabit ex tribus Hyperbolis ſibi oppoſitis, quarum una jacet inter Aſymptotos parallelas, & alteræ duæ jacent extrà. Et hæc eſt *ſpecies quinquageſima ſeptima.*



62



63



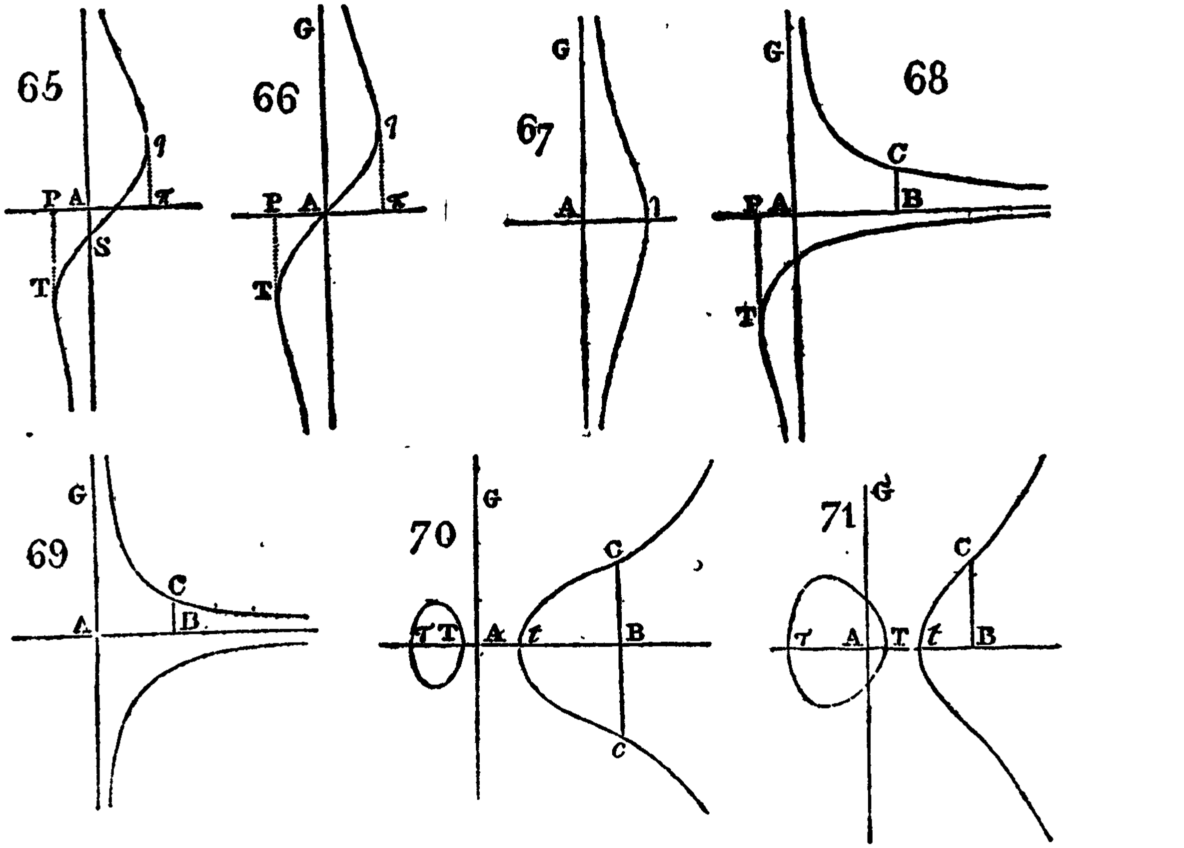
61

4. Si radices illæ duæ ſunt impoſſibiles, habentur Hyperbolæ duæ oppoſitæ, extra Aſymptotos parallelas, & Anguinea Hyperbolica intra eaſdem. Hæc figura duarum eſt ſpecierum. Nam centrum non habet ubi terminus d non deeſt (fig. 62); ſed ſi terminus ille deeſt, punctum A eſt ejus centrum (fig. 63). Prior *ſpecies* eſt *quinquageſima octava*, poſterior *quinquageſima nona.*

5. Quòd ſi terminus ey deeſt, figura conſtabit ex tribus Hyperbolis oppoſitis; quarum una jacet inter Aſymptotos parallelas, & alteræ

alteræ duæ jacent extrà, ut in ſpecie quinquageſimâ quartâ; & præterea diametrum habet, quæ eſt abſciſſa AB (fig. 64). Et hæc eſt *ſpecies ſexageſima*. SPECIES LINEARUM TERTII ORDINIS.

X. *De tribus Hyperboliſmis Ellipſeos.*



1. Hyperboliſmus Ellipſeos per hanc æquationem definitur; $xy^2+ey=ex+d$: & unicam habet Aſymptoton, quæ eſt Ordinata principalis A*d* (fig. 65). Si terminus *ey* non deeſt, figura eſt Hyperbola Anguinea ſine diametro; atque etiam ſine centro, ſi terminus *d* non deeſt. Quæ *ſpecies* eſt *ſexageſima prima*.

2. At ſi terminus *d* deeſt, figura habet centrum ſine diametro, & centrum ejus eſt punctum A (fig. 66). *Species* verò eſt *ſexageſima ſecunda*.

3. Et ſi terminus *ey* deeſt, & terminus *d* non deeſt, figura eſt Conchoidalis ad Aſymptoton AG (fig. 67) habetque diametrum ſine centro, & diameter ejus eſt abſciſſa AB. Quæ *ſpecies* eſt *ſexageſima tertia*.

XI. *De duobus Hyperboliſmis Parabolæ.*

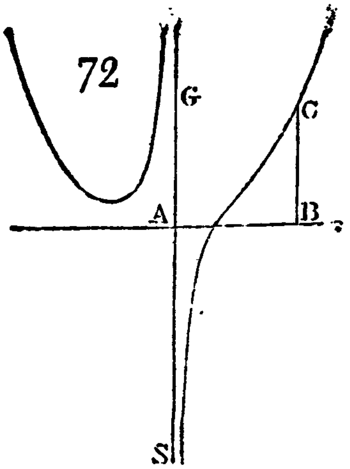
Hyperboliſmus Parabolæ per hanc æquationem definitur xy^2+

CAPUT IV. $ey=d$; & duas habet Aſymptotos, Abſciſſam AB & Ordinatam primam & principalem AG. Hyperbolæ vero in hâc figurâ ſunt duæ, non in aſymptoton angulis oppoſitis, ſed in angulis qui ſunt deinceps, jacentes; idque ad utrumque latus abſciſſæ AB, & vel ſine diametro, ſi terminus ey habetur (fig. 68), vel cum diametro ſi terminus ille deeſt (fig. 69). Quæ duæ *ſpecies* ſunt *ſexageſima quarta* & *ſexageſima quinta.*

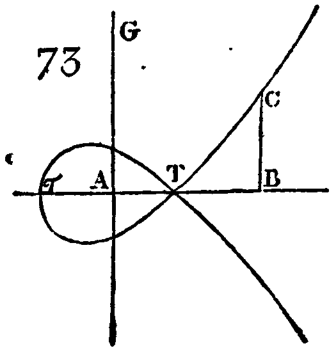
XII. *De Tridente.*

In Secundo æquationum caſu habebatur æquatio $xy=ax^3+bx^2+cx+d$. Et figura in hoc caſu habet quatuor crura infinita; quorum duo ſunt Hyperbolica, circa Aſymptoton AG (fig. 72) in contrarias partes tendentia, & duo Parabolica, convergentia, & cum

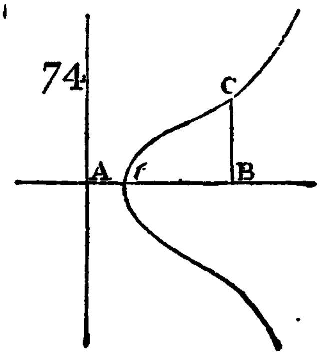
72



73



74



prioribus ſpeciem Tridentis ferè efformantia. Eſtque hæc figura Parabola illa, per quam *Carteſius* æquationes ſex dimenſionum conſtruxit. Hæc eſt igitur *ſpecies ſexageſima ſexta.*

XIII. *De Parabolis quinque Divergentibus.*

1. In Tertio caſu æquatio erat $y^2=ax^3+bx^2+cx+d$, & Parabolam deſignat, cujus crura divergunt ab invicem, & in contrarias partes infinitè progrediuntur. Abſciſſa AB eſt ejus diameter, & Species ejus ſunt quinque ſequentes.

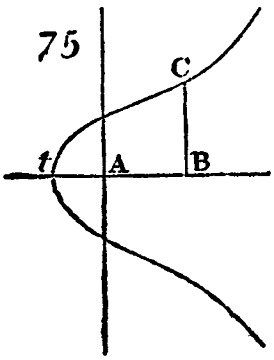
2. Si æquationis $ax^3+bx^2+cx+d=0$, radices omnes AT, AT, At ſunt reales & inæquales, figura eſt Parabola Divergens Campaniformis, cum *Ovali* ad verticem (fig. 70, 71). Et *ſpecies* eſt *ſexageſima ſeptima.*

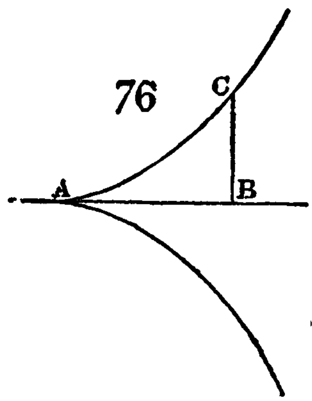
3. Si

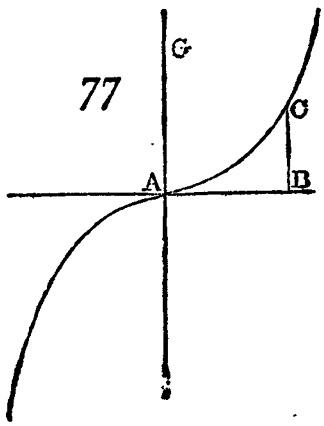
3. Si radices duæ ſunt æquales, Parabola prodit vel *Nodata* contingendo Ovalem (fig. 73), vel *Punctata* ob Ovalem infinitè parvam (fig. 74). Quæ duæ *ſpecies* ſunt *ſexageſima octava* & *ſexageſima nona*. Species Linearum Tertii Ordinis.

4. Si tres radices ſunt æquales Parabola erit *Cuſpidata* in vertice (fig. 76). Et hæc eſt Parabola *Neiliana*, quæ vulgo *Semicubica* dicitur. Et eſt *ſpecies ſeptuageſima*.

5. Si radices duæ ſunt impoſſibiles, habetur Parabola *Pura* Campaniformis (fig. 74, 75) *ſpeciem ſeptuageſimam primam* conſtituens.







XIV. *De Parabolâ Cubicâ.*

In Quarto caſu æquatio erat $y=ax^3+bx^2+cx+d$: & hæc æquatio Parabolam deſignat, quæ crura habet contraria, & *Cubica* dici ſolet (fig. 77). Et ſic *Species* omnino ſunt *ſeptuaginta duæ*.

CAPUT QUINTUM.

Geneſis Curvarum per Umbras.

1. SI in planum infinitum, à puncto lucido illuminatum, umbræ figurarum projiciantur, umbræ Sectionum Conicarum ſemper erunt Sectiones Conicæ; eæ Curvarum ſecundi generis ſemper erunt Curvæ ſecundi generis; eæ Curvarum tertii generis ſemper erunt Curvæ tertii generis, & ſic deinceps in infinitum.

2. Et quemadmodum Circulus, umbram projiciendo, generat Sectiones omnes Conicas; ſic Parabolæ quinque divergentes umbris ſuis generant & exhibent alias omnes ſecundi generis Curvas:

& ſic Curvæ quædam ſimpliciores aliorum generum inveniri poſſunt, quæ alias omnes eorundem generum Curvas, umbris ſuis à puncto lucido in planum projectis, formabunt.

CAPUT SEXTUM.

De Curvarum Punctis duplicibus.

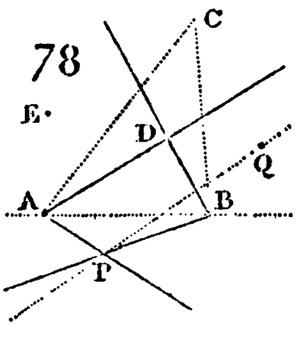
DIXIMUS Curvas ſecundi generis à lineâ rectâ in punctis tribus ſecari poſſe. Horum duo nonnunquam coincidunt. Ut cùm recta per Ovalem infinitè parvam tranſit; vel per concurſum duarum partium Curvæ, ſe mutuò ſecantium vel in Cuſpidem coeuntium, ducitur. Et ſiquando rectæ omnes in plagam cruris alicujus infiniti tendentes Curvam in unico tantum puncto ſecant, (ut fit in Ordinatis Parabolæ *Carteſianæ* & Parabolæ Cubicæ, nec non in rectis Abſciſſæ Hyperboliſmorum Hyperbolæ & Parabolæ parallelis) concipiendum eſt, quòd rectæ illæ per alia duo Curvæ puncta ad infinitam diſtantiam ſita (ut ita dicam) tranſeunt. Hujuſmodi interſectiones duas coincidentes, ſive ad finitam ſint diſtantiam ſive ad infinitam, vocabimus *Punctum Duplex*. Curvæ autem, quæ habent punctum duplex, deſcribi poſſunt per ſequentia Theoremata.

CAPUT VII.

De Curvarum deſcriptione Organicâ.

THEOR. I.

78



SI anguli duo magnitudine dati, PAD, PBD, circa polos poſitione datos, A, B, rotentur; & eorum crura, AP, BP, concurſu ſuo, P, percurrant lineam rectam; crura duo reliqua, AD, BD, concurſu ſuo, D, deſcribent Sectionem Conicam per polos A, B tranſeuntem: præterquam ubi linea illa recta tranſit per polorum alterutrum, A vel B, vel anguli BAD, ABD ſimul evaneſcunt; quibus in caſibus punctum D deſcribet lineam rectam.

THEOR. II.

Si crura prima, AP, BP, concurſu ſuo, P, percurrant Sectionem Conicam

Conicam per polum alterutrum A tranſeuntem, crura duo reliqua, AD, BD concurſu ſuo, D, deſcribent Curvam ſecundi generis per polum alterum B tranſeuntem, & punctum duplex habentem in polo primo A, per quem ſectio conica tranſit: præterquam ubi anguli BAD, ABD ſimul evaneſcunt; quo caſu punctum D deſcribet aliam Sectionem Conicam per polum A tranſeuntem. DESCRIPTIONES ORGANICÆ.

THEOR. III.

At ſi Sectio Conica, quam punctum P percurrit, tranſeat per neutrum polorum A, B; punctum D deſcribet Curvam ſecundi vel tertii generis, punctum duplex habentem. Et punctum illud duplex in concurſu crurum deſcribentium, AD, BD, invenietur, ubi anguli BAP, ABP ſimul evaneſcunt. Curva autem deſcripta ſecundi erit generis, ſi anguli BAD, ABD ſimul evaneſcunt; aliâs erit tertii generis, & alia duo habebit puncta duplicia in polis A & B.

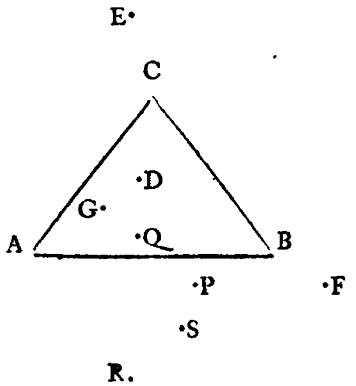
2. *Sectionum Conicarum deſcriptio per data quinque puncta.*

Jam Sectio Conica determinatur ex datis ejus punctis quinque, & per eadem ſic deſcribi poteſt. Dentur ejus puncta quinque, A, B, C, D, E. Jungantur eorum tria quævis A, B, C, & trianguli ABC rotentur anguli duo quivis, CAB, CBA, circa vertices ſuos A & B; & ubi crurum AC, BC interſectio C ſucceſſivè applicatur ad puncta duo reliqua D, E, incidat interſectio crurum reliquorum, AB & BA, in puncta P & Q. Agatur, & infinitè producatur recta PQ, & anguli mobiles ita rotentur, ut interſectio crurum, AB, BA, percurrat rectam PQ, & crurum reliquorum interſectio, C, deſcribet propoſitam Sectionem Conicam per Theorema primum.

3. *Curvarum ſecundi generis Punctum Duplex habentium deſcriptio per data ſeptem puncta.*

Curvæ omnes ſecundi generis Punctum Duplex habentes determinantur ex datis earum punctis ſeptem, quorum unum eſt Punctum illud Duplex, & per eadem puncta ſic deſcribi poſſunt. Dentur Curvæ deſcribendæ puncta quælibet ſeptem A, B, C, D, E, F, G, quorum A eſt punctum duplex. Jungantur punctum A, & alia duo quævis è punctis, puta B & C; & trianguli ABC ro-

tetur



tetur tum angulus CAB circa verticem ſuum A, tum angulorum reliquorum alteruter, ABC, circa verticem ſuum B. Et ubi crurum AC, BC concurſus, C, ſucceſſivè applicatur ad puncta quatuor reliqua D, E, F, G incidat concurſus crurum reliquorum, AB & BA, in puncta quatuor P, Q, R, S. Per puncta illa quatuor & quintum A deſcribatur ſectio conica, & anguli præfati CAB, CBA ita rotentur ut crurum, AB, BA, concurſus percurrat ſectionem illam conicam, & concurſus reliquorum crurum, AC, BC, deſcribet Curvam propoſitam per Theorema ſecundum.

Si vice puncti C datur poſitione recta BC, quæ Curvam deſcribendam tangit in B, lineæ AD, AP coincident; & vice anguli DAP habebitur linea recta circa polum A rotanda.

Si Punctum Duplex A infinitè diſtat, debebit recta ad plagam puncti illius perpetuò dirigi, & motu parallelo ferri, interea dum angulus ABC circa polum B rotatur.

Deſcribi etiam poſſunt hæ Curvæ paulo aliter per Theorema tertium, ſed deſcriptionem ſimpliciorem poſuiſſe ſufficit.

Eâdem methodo Curvas tertii, quarti & ſuperiorum generum deſcribere licet; non omnes quidem, ſed quotquot ratione aliquâ commodâ per motum localem deſcribi poſſunt. Nam Curvam aliquam ſecundi vel ſuperioris generis, punctum duplex non habentem, commodè deſcribere, Problema eſt inter difficiliora numerandum.

CAPUT OCTAVUM.

Conſtructio æquationum per deſcriptionem Curvarum.

CURVARUM uſus in Geometriâ eſt, ut per earum interſectiones Problemata ſolvantur. Proponatur æquatio conſtruenda dimenſionum novem, $x^9 * + bx^7 + cx^6 + dx^5 + ex^4 + fx^3 + gx^2 + hx +$
$+m$

$k = 0.$

Ubi

Ubi b, c, d, &c. ſignificant quantitates quaſvis datas, ſignis ſuis $+$ & $-$ affectas. Aſſumatur æquatio ad Parabolam Cubicam $x^3=y$; & æquatio prior, ſcribendo y pro x^3, evadet $y^3+bxy^2+cy^2+dx^2y+exy+my+fx^3+gx^2+hx+k=0$, æquatio ad Curvam aliam ſecundi generis. Ubi m vel f deeſſe poteſt, vel pro lubitu aſſumi. Et per harum Curvarum deſcriptiones & interſectiones dabuntur radices æquationis conſtruendæ. Parabolam Cubicam ſemel deſcribere ſufficit. CONSTRUCTIO ÆQUATIONUM.

Si æquatio conſtruenda per defectum duorum terminorum ultimorum, hx & k, reducatur ad ſeptem dimenſiones; Curva altera, delendo m, habebit punctum duplex in principio Abſciſſæ, & inde facilè deſcribi poteſt, ut ſuprà.

Si æquatio conſtruenda per defectum terminorum trium ultimorum, gx^2+hx+k, reducatur ad ſex dimenſiones, Curva altera, delendo f, evadet Sectio Conica.

Et ſi per defectum ſex ultimorum terminorum æquatio conſtruenda reducatur ad tres dimenſiones, incidetur in conſtructionem *Walliſianam* per Parabolam Cubicam & lineam rectam.

Conſtrui etiam poſſunt æquationes per Hyperboliſmum Parabolæ cum diametro. Ut ſi conſtruenda ſit hæc æquatio dimenſionum novem termino penultimo carens, $a+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+kx^8+lx^9=0$; aſſumatur æquatio ad Hyperboliſmum illum $x^2y=1$: & ſcribendo y pro $\frac{1}{xx}$, æquatio conſtruenda vertetur in hanc, $ay^3+cy^2+dxy^2+ey+fxy+mx^2y+g+hx+kx^2+lx^3=0$: quæ Curvam ſecundi generis deſignat, cujus deſcriptione Problema ſolvetur. Et quantitatum m ac g alterutra hìc deeſſe poteſt, vel pro lubitu aſſumi.

$+m$

Per Parabolam Cubicam & Curvas tertii generis conſtruuntur etiam æquationes omnes dimenſionum non pluſquam duodecim, & per eandem Parabolam & Curvas quarti generis conſtruuntur omnes dimenſionum non pluſquam quindecim; et ſic deinceps in infinitum. Et Curvæ illæ tertii, quarti & ſuperiorum generum deſcribi ſemper poſſunt, inveniendo eorum puncta per Geometriam planam. Ut ſi conſtruenda ſit æquatio, $x^{12} * +ax^{10}+bx^9+cx^8+dx^7+ex^6+fx^5+gx^4+hx^3+ix^2+kx+l=0$, & deſcripta habeat Parabola

Caput VIII. rabola Cubica; ſit æquatio ad Parabolam illam Cubicam $x^3=y$, & ſcribendo y pro x^3, æquatio conſtruenda vertetur in hanc,

$$\begin{array}{llllll} y^4 & +axy^3 & +cx^2y^2 & +fx^2y & +ix^2 & =0, \\ & +b & +dx & +gx & +kx & \\ & & +e & +b & +l & \end{array}$$

quæ eſt æquatio ad Curvam tertii generis, cujus deſcriptione Problema ſolvetur. Deſcribi autem poteſt hæc Curva, inveniendo ejus puncta per Geometriam planam, propterea quòd indeterminata quantitas, x, non niſi ad duas dimenſiones aſcendit.

LOGISTICA INFINITORUM.

AUCTORE S. HORSLEY, LL. D. ET R. S. S.

LOGISTICA INFINITORUM.

IN exſequendo per ſeries infinitas Problematum analyſin, haud rarò uſuvenit ut in ſeries ipſas inſtituendæ ſint operationes arithmeticæ; non modò faciles illæ Additionis et Subductionis, verum etiam moleſtiſſimæ Multiplicationis et Diviſionis, utriuſque quidem cum ſimplicis tum climacticæ: quæ ſanè computationes experti nôrunt quanti ſunt laboris: hujus minuendi gratiâ formulas ſequentes trado.

PROBLEMA I. MULTIPLICATIO.

Seriem infinitam $\mathrm{a}x^{\mu} + \mathrm{b}x^{\mu+\alpha} + \mathrm{c}x^{\mu+2\alpha} + \mathrm{d}x^{\mu+3\alpha} + \mathrm{e}x^{\mu+4\alpha}$ &c.
Series infinita $ax^{\nu} + bx^{\nu+\alpha} + cx^{\nu+2\alpha} + dx^{\nu+3\alpha} + ex^{\nu+4\alpha}$ &c. Multiplicet.

Fiet $\mathrm{a}ax^{\mu+\nu} \begin{matrix} +\mathrm{b}a \\ +\mathrm{a}b \end{matrix} x^{\mu+\nu+\alpha} \begin{matrix} +\mathrm{c}a \\ +\mathrm{b}b \\ +\mathrm{a}c \end{matrix} x^{\mu+\nu+2\alpha} \begin{matrix} +\mathrm{d}a \\ +\mathrm{c}b \\ +\mathrm{b}c \\ +\mathrm{a}d \end{matrix} x^{\mu+\nu+3\alpha} \begin{matrix} +\mathrm{e}a \\ +\mathrm{d}b \\ +\mathrm{c}c \\ +\mathrm{b}d \\ +\mathrm{a}e \end{matrix} x^{\mu+\nu+4\alpha}$ &c.

Poſitis igitur $\mu+\nu = \pi$, ſeries multiplicatione facta hanc formam induet

$$\mathrm{A}x^{\pi} + \mathrm{B}x^{\pi+\alpha} + \mathrm{C}x^{\pi+2\alpha} + \mathrm{D}x^{\pi+3\alpha} + \mathrm{E}x^{\pi+4\alpha} \text{ \&c.}$$

Cujus coefficientium conſtitutio hujuſcemodi erit.

$\mathrm{A} = \mathrm{a}a.$ $\mathrm{B} = \mathrm{a}b + \mathrm{b}a.$ $\mathrm{C} = \mathrm{a}c + \mathrm{b}b + \mathrm{c}a.$ $\mathrm{D} = \mathrm{a}d + \mathrm{b}c + \mathrm{c}b + \mathrm{d}a.$ $\mathrm{E} = \mathrm{a}e + \mathrm{b}d + \mathrm{c}c + \mathrm{d}b + \mathrm{e}a.$

PROBLEMA II. DIVISIONIS 1.

Seriem infinitam ſerie infinitâ dividere

Sit ſeries dividenda $\mathrm{A}x^{\pi} + \mathrm{B}x^{\pi+\alpha} + \mathrm{C}x^{\pi+2\alpha} + \mathrm{D}x^{\pi+3\alpha} + \mathrm{E}x^{\pi+4\alpha}$ &c.
Series dividens ſit $ax^{\nu} + bx^{\nu+\alpha} + cx^{\nu+2\alpha} + dx^{\nu+3\alpha} + ex^{\nu+4\alpha}$ &c.

Tum

Tum poſitis $\pi - \nu = \mu$, ſeries diviſione facta hujuſce formæ erit

$$\mathrm{a}x^{\mu} + \mathrm{b}x^{\mu+\alpha} + \mathrm{c}x^{\mu+2\alpha} + \mathrm{d}x^{\mu+3\alpha} + \mathrm{e}x^{\mu+4\alpha} \text{ \&c.}$$

Cujus Coefficientes hiſce æquationibus definiuntur: $\mathrm{a} = \frac{\mathrm{A}}{a}$. $\mathrm{b} = \frac{\mathrm{B} - \mathrm{a}b}{a}$. $\mathrm{c} = \frac{\mathrm{C} - \mathrm{a}c - \mathrm{b}b}{a}$. $\mathrm{d} = \frac{\mathrm{D} - \mathrm{a}d - \mathrm{b}c - \mathrm{c}b}{a}$. $\mathrm{e} = \frac{\mathrm{E} - \mathrm{a}e - \mathrm{b}d - \mathrm{c}c - \mathrm{d}b}{a}$. $\mathrm{f} =$ &c. Patet ex Problemate primo.

PROBLEMA III. DIVISIONIS 2.

2. Unitatem dividere à ſerie infinitâ $ax^{\nu} + bx^{\nu+\alpha} + cx^{\nu+2\alpha} + dx^{\nu+3\alpha} + ex^{\nu+4\alpha}$.

Factum puta, et ſit $\frac{1}{ax^{\nu} + bx^{\nu+\alpha} + cx^{\nu+2\alpha} + dx^{\nu+3\alpha} + ex^{\nu+4\alpha} +} = \mathrm{a}x^{\mu} + \mathrm{b}x^{\mu+\alpha} + \mathrm{c}x^{\mu+2\alpha} + \mathrm{d}x^{\mu+3\alpha} + \mathrm{e}x^{\mu+4\alpha}$ &c. Series diviſione facta cum ſerie dividente multiplicata ſeriem faciat $\mathrm{A}x^{\pi} + \mathrm{B}x^{\pi+\alpha} + \mathrm{C}x^{\pi+2\alpha} + \mathrm{D}x^{\pi+3\alpha} + \mathrm{E}x^{\pi+4\alpha}$ &c. Hujus ſumma unitati æqualis erit. Ergo

$$\begin{matrix}\mathrm{A}x^{\pi} \\ -1\end{matrix} + \mathrm{B}x^{\pi+\alpha} + \mathrm{C}x^{\pi+2\alpha} + \mathrm{D}x^{\pi+3\alpha} = 0.$$

Hoc autem ut ſemper fiat, juxta quamlibet literæ x æſtimationem, neceſſe eſt ut π ſit 0, $\mathrm{A} = 1$, & ut reliqui coefficientes B, C, D, E evadant ſinguli nihilo æquales.

Sed per Problema primum $\pi = \mu + \nu$. $\mathrm{A} = \mathrm{a}a$. $\mathrm{B} = \mathrm{a}b + \mathrm{b}a$. $\mathrm{C} = \mathrm{a}c + \mathrm{b}b + \mathrm{c}a$. &c. Ergo $\mu = -\nu$.

Series igitur diviſione facta hanc formam induet

$$\mathrm{a}x^{-\nu} + \mathrm{b}x^{\alpha-\nu} + \mathrm{c}x^{2\alpha-\nu} + \mathrm{d}x^{3\alpha-\nu} + \mathrm{e}x^{4\alpha-\nu} \text{ \&c.}$$

& Coefficientes hiſce æquationibus definientur. $\mathrm{a} = \frac{1}{a}$. $\mathrm{b} = -\frac{\mathrm{a}b}{a}$. $\mathrm{c} = -\frac{\mathrm{a}c + \mathrm{b}b}{a}$. $\mathrm{d} = -\frac{\mathrm{a}d + \mathrm{b}c + \mathrm{c}b}{a}$. $\mathrm{e} = -\frac{\mathrm{a}e + \mathrm{b}d + \mathrm{c}c + \mathrm{d}b}{a}$. f &c.

PROBLEMA IV. Multiplicatio et Diviſio Climactica.

Seriei infinitæ poteſtates formulâ generali exhibere.

Exhibenda ſit poteſtas, indice λ, ſeriei infinitæ $a + bx^{\mu} + cx^{2\mu} + dx^{3\mu} + ex^{4\mu} + fx$ [illegible] $hx^{7\mu} + kx^{8\mu} + lx^{9\mu}$ &c.

Summa nominum omnium ſeriei propoſitæ poſt primum dicatur α. Omnium poſt ſecundum, β. Omnium poſt tertium, γ; poſt quartum, δ; poſt quintum, ε; poſt ſextum, ζ; &c. Ita erit. $a + bx^{\mu} + cx^{2\mu} + dx^{3\mu}$ &c. $= a + \alpha$.

$$\alpha = bx^{\mu} + \beta$$
$$\alpha^2 = b^2x^{2\mu} + 2b\beta x^{\mu} + \beta^2$$
$$\alpha^3 = b^3x^{3\mu} + 3b^2x^{2\mu}\beta + 3bx^{\mu}\beta^2 + \beta^3$$
$$\alpha^4 = b^4x^{4\mu} + 4b^3x^{3\mu}\beta + 6b^2x^{2\mu}\beta^2 + 4bx^{\mu}\beta^3 + \beta^4$$
$$\alpha^5 = b^5x^{5\mu} + 5b^4x^{4\mu}\beta + 10b^3x^{3\mu}\beta^2 + 10b^2x^{2\mu}\beta^3 + 5bx^{\mu}\beta^4 + \beta^5$$
$$\alpha^6 = b^6x^{6\mu} + 6b^5x^{5\mu}\beta + 15b^4x^{4\mu}\beta^2 + 20b^3x^{3\mu}\beta^3 + 15b^2x^{2\mu}\beta^4 + 6bx^{\mu}\beta^5 + \beta^6$$
$$\alpha^7 = b^7x^{7\mu} + 7b^6x^{6\mu}\beta + 21b^5x^{5\mu}\beta^2 + 35b^4x^{4\mu}\beta^3 + 35b^3x^{3\mu}\beta^4 \text{ \&c.}$$
$$\alpha^8 = b^8x^{8\mu} + 8b^7x^{7\mu}\beta + 28b^6x^{6\mu}\beta^2 \text{ \&c.}$$
$$\alpha^9 = b^9x^{9\mu} + 9b^8x^{8\mu}\beta \text{ \&c.}$$

&c.

$\beta =$

$\beta = c\, x^{2\mu} + \gamma$
$\beta^2 = c^2 x^{4\mu} + 2c\, x^{2\mu}\gamma + \gamma^2$
$\beta^3 = c^3 x^{6\mu} + 3c^2 x^{4\mu}\gamma + 3c\, x^{2\mu}\gamma^2 + \gamma^3$
$\beta^4 = c^4 x^{8\mu} + 4c^3 x^{6\mu}\gamma + 6c^2 x^{4\mu}\gamma^2 + 4c x^{2\mu}\gamma^3 + \gamma^4$
$\beta^5 = c^5 x^{10\mu}$ &c.

$\gamma = d\, x^{3\mu} + \delta$
$\gamma^2 = d^2 x^{6\mu} + 2d\, x^{3\mu}\delta + \delta^2$
$\gamma^3 = d^3 x^{9\mu} + 3d^2 x^{6\mu}\delta$ &c.

$\delta = e\, x^{4\mu} + \epsilon$
$\delta^2 = e^2 x^{8\mu} + 2e x^{4\mu}\epsilon + \epsilon^2$

$\epsilon = f x^{5\mu} + \zeta$
$\epsilon^2 = f^2 x^{10\mu} +$

$\zeta = g x^{6\mu} + \eta$
$\eta = h x^{7\mu} + \theta$
$= k x^{8\mu} + \iota$
$\iota = l x^{9\mu} + \kappa$

Et denotante λ quantitatem quamvis, ſeriei propoſitæ poteſtas illa cujus index λ, ſive $\overline{a + bx^{\mu} + cx^{2\mu} + dx^{3\mu} \text{ \&c.}}|^{\lambda} = \overline{a + \alpha}|^{\lambda}$.

Sed ex formulâ generali Newtonianâ poteſtatum binomii, $\overline{a+\alpha}|^{\lambda} = a^{\lambda} + \lambda a^{\lambda-1}\alpha + \lambda \times \frac{\lambda - 1}{2}$ $a^{\lambda-2}\alpha^2 + \lambda \times \frac{\lambda-1}{2}. \times \frac{\lambda-2}{3} a^{\lambda-3}\alpha^3$ &c.

Hoc eſt, ſi pro $\lambda a^{\lambda-1}$ ſcribatur A

pro $\lambda \times \frac{\lambda-1}{2} a^{\lambda-2}$, A

pro $\lambda \times \frac{\lambda-1}{2} \times \frac{\lambda-2}{3} a^{\lambda-3}$, A

pro $\lambda. \frac{\lambda-1}{2}. \frac{\lambda-2}{3}. \frac{\lambda-3}{4} a^{\lambda-4}$, A

atque idem ſemper fiat, $\overline{a+\alpha}|^{\lambda} = a^{\lambda} + A.\alpha + A.\alpha^2 + A.\alpha^3 + A.\alpha^4 + A.\alpha^5 + A.\alpha^6 + A.\alpha^7 + A.\alpha^8 + A.\alpha^9$. &c.

Ex ſingulis membris hujus progreſſionis, ope æquationum $\alpha = bx^{\mu} + \beta$, $\alpha^2 = b^2 x^{2\mu} +$ &c. $\alpha^3 =$ &c. exterminetur primò α. Tum ex membris ſingulis ſeriei transformatæ, ope æquationum $\beta = cx^{2\mu} + \gamma$, $\beta^2 = c^2 x^{4\mu} +$ &c. $\beta^3 =$ &c. exterminetur β. Et ex membris ſingulis ſeriei denuò transformatæ, ope æquationum $\gamma = dx^{3\mu} + \delta$, $\gamma^2 = d^2 x^{6\mu} +$ &c. $\gamma^3 =$ &c. exterminetur γ. Si hoc continuò fiat, colligentur tandem $A.\alpha = A. \times bx^{\mu} + cx^{2\mu} + dx^{3\mu} + ex^{4\mu}$ &c.

$$A\alpha^2 = A \times \left\{ \begin{array}{l} b^2. x^{2\mu} + 2b.cx^{3\mu} + 2bd\, x^{4\mu} + 2bex^{5\mu} + 2bfx^{6\mu} + 2bgx^{7\mu} + 2bhx^{8\mu} + 2bkx^{9\mu} \\ \qquad + c^2 x^{4\mu} + 2cdx^{5\mu} + 2cex^{6\mu} + 2cfx^{7\mu} + 2cgx^{8\mu} + 2chx^{9\mu} \\ \qquad\qquad + d^2 x^{6\mu} + 2dex^{7\mu} + 2dfx^{8\mu} + 2dgx^{9\mu} \\ \qquad\qquad\qquad + e^2 x^{8\mu} + 2efx^{9\mu} \text{ \&c.} \end{array} \right.$$

$A\alpha^3 =$

$$\underset{3}{\mathbb{A}}\alpha^{3}=\underset{3}{\mathbb{A}}\times\left\{\begin{array}{r} b^{3}.x^{3\mu}+3b^{2}c.x^{4\mu}+3b^{2}.dx^{5\mu}+3b^{2}\ \ ex^{6\mu}+3b^{2}.\ fx^{7\mu}+3b^{2}\ \ gx^{8\mu}+3b^{2}\ \ hx^{9\mu}\ \&c. \\ +3b.c^{2}x^{5\mu}+3b.2cdx^{6\mu}+3b.\ 2cex^{7\mu}+3b.\ 2cfx^{8\mu}+3b.2cgx^{9\mu} \\ +3b.\ d^{2}x^{7\mu}+3b.\ 2dex^{8\mu}+3b.2dfx^{9\mu} \\ +3b.\ e^{2}x^{9\mu} \\ +\ c^{3}.\ x^{6\mu}+3c^{2}\ \ dx^{7\mu}+3c^{2}.\ ex^{8\mu}+\ \ 3c^{2}fx^{9\mu} \\ +3c.\ d^{2}x^{8\mu}+3c.2dex^{9\mu} \\ +\ \ \ d^{3}x^{9\mu}\ \&c. \end{array}\right.$$

&c.

$$\underset{4}{\mathbb{A}}\alpha^{4}=\underset{4}{\mathbb{A}}\times\left\{\begin{array}{r} b^{4}.x^{4\mu}+4b^{3}.cx^{5\mu}+4b^{3}.d\,x^{6\mu}+4b^{3}.\ \ ex^{7\mu}+4b^{3}\ \ fx^{8\mu}+4b^{3}\ \ gx^{9\mu} \\ +6b^{2}.c^{2}x^{6\mu}+6b^{2}.2cdx^{7\mu}+6b^{2}.2cex^{8\mu}+6b^{2}.\,2cfx^{9\mu} \\ +6b^{2}.\,d^{2}x^{8\mu}+6b^{2}.\,2dex^{9\mu} \\ +4b.\ c^{3}.x^{7\mu}+4b.3c^{2}dx^{8\mu}+4b.\,3c^{2}ex^{9\mu} \\ +4b.3cd^{2}x^{9\mu} \\ +\ c^{4}.\ \ \ x^{8\mu}+4c^{3}d.\ \ x^{9\mu} \end{array}\right.$$

$$\underset{5}{\mathbb{A}}\alpha^{5}=\underset{5}{\mathbb{A}}\times\left\{\begin{array}{r} b^{5}.x^{5\mu}+5b^{4}cx^{6\mu}+\ \ 5b^{4}dx^{7\mu}+\ 5b^{4}\ \ ex^{8\mu}+\ 5b^{4}.\ \ fx^{9\mu}\ \&c. \\ +10b^{3}c^{2}x^{7\mu}+10b^{3}.2cdx^{8\mu}+10b^{3}.\ 2cex^{9\mu}\ \&c. \\ +10b^{3}\ \ d^{2}x^{9\mu}\ \&c. \\ +10b^{2}.\ c^{3}x^{8\mu}+10b^{2}.3c^{2}dx^{9\mu}\ \&c. \\ +\ 5b\ \ \ \ c^{4}x^{9\mu}\ \&c. \end{array}\right.$$

$$\underset{6}{\mathbb{A}}\alpha^{6}=\underset{6}{\mathbb{A}}\times\left\{\begin{array}{r} b^{6}x^{6\mu}+6b^{5}.cx^{7\mu}+\ \ 6b^{5}dx^{8\mu}+\ 6b^{5}\ \ ex^{9\mu}\ \&c. \\ +15b^{4}c^{2}x^{8\mu}+15b^{4}2cdx^{9\mu}\ \&c. \\ +20b^{3}\ \ c^{3}x^{9\mu}\ \&c. \end{array}\right.$$

$$\underset{7}{\mathbb{A}}\alpha^{7}=\underset{7}{\mathbb{A}}\times\left\{\begin{array}{r} b^{7}x^{7\mu}+7b^{6}.c.x^{8\mu}+\ \ \ 7b^{6}dx^{9\mu}\ \ \&c. \\ +21.b^{5}.c^{2}x^{9\mu} \end{array}\right.$$

$$\underset{8}{\mathbb{A}}\alpha^{8}=\quad\underset{8}{\mathbb{A}}\times b^{8}x^{8\mu}+8b^{7}c.x^{9\mu}\ \ \&c.$$

$$\underset{9}{\mathbb{A}}\alpha^{9}=\quad\underset{9}{\mathbb{A}}\times b^{9}x^{9\mu}\ \ \&c.$$

Jam hiſce omnibus ſecundum indices poteſtatum literæ x diſpoſitis, conſit hæc formula generalis :

FORMULA.

MULTIPLICATIONIS ET DIVISIONIS CLIMACTICÆ.

$$\alpha^{\lambda}+\underset{1}{\mathbb{A}}\,bx^{\mu}+\begin{array}{l}\underset{1}{\mathbb{A}}.c \\ +\underset{2}{\mathbb{A}}.b^{2}\end{array}x^{2\mu}\ \begin{array}{l}+\underset{1}{\mathbb{A}}.\ d \\ +\underset{2}{\mathbb{A}}.2bc \\ +\underset{3}{\mathbb{A}}.\ b^{3}\end{array}x^{3\mu}\ \begin{array}{l}+\underset{1}{\mathbb{A}}\ e \\ +\underset{2}{\mathbb{A}}\ \overline{2bd+c^{2}} \\ +\underset{3}{\mathbb{A}}.3b^{2}c \\ +\underset{4}{\mathbb{A}}.\ b^{4}\end{array}x^{4\mu}\ \begin{array}{l}+\underset{1}{\mathbb{A}}\,f \\ +\underset{2}{\mathbb{A}}\ \overline{2be+2cd} \\ +\underset{3}{\mathbb{A}}.\overline{3b^{2}.d+3b.c^{2}} \\ +\underset{4}{\mathbb{A}}\ 4b^{3}c \\ +\underset{5}{\mathbb{A}}\ b^{5}\end{array}x^{5\mu}$$

$+\underset{1}{\mathbb{A}}.g$

$+\overset{1}{\mathfrak{A}}.g$
$+\overset{2}{\mathfrak{A}}.\overline{2bf+2ec+d^2}$
$+\overset{3}{\mathfrak{A}}.\overline{3b^2e+3b.2cd+c^3}$ $x^{6\mu}$
$+\overset{4}{\mathfrak{A}}.\overline{4b^3d+6b\,c^2}$
$+\overset{5}{\mathfrak{A}}.5b^4c$
$+\overset{6}{\mathfrak{A}}.b^6$

$+\overset{1}{\mathfrak{A}}.h$
$+\overset{2}{\mathfrak{A}}.\overline{2bg+2cf+2de}$
$+\overset{3}{\mathfrak{A}}.\overline{3b^2f+3b.2c.e+3b.d^2+3c^2d}$
$+\overset{4}{\mathfrak{A}}.\overline{4b^3e+6b^2.2cd+4b.c^3}$ $x^{7\mu}$
$+\overset{5}{\mathfrak{A}}.\overline{5b^4d+10b^3c^2}$
$+\overset{6}{\mathfrak{A}}.6b^5c$
$+\overset{7}{\mathfrak{A}}.b^7$

$+\overset{1}{\mathfrak{A}}.k$
$+\overset{2}{\mathfrak{A}}.\overline{2bh+2cg+2df+e^2}$
$+\overset{3}{\mathfrak{A}}.3b^2g+3b.2cf+3b.2de+3c^2e+3cd^2$
$+\overset{4}{\mathfrak{A}}.\overline{4b^3f+6b^2.2ce+6b^2d^2+4b.3c^2d+c^4}$ $x^{8\mu}$
$+\overset{5}{\mathfrak{A}}.\overline{5b^4e+10b^3.2cd+10.b^2c^3}$
$+\overset{6}{\mathfrak{A}}.\overline{6b^5d+15.b^4.c}$
$+\overset{7}{\mathfrak{A}}.7.b^6.c$
$+\overset{8}{\mathfrak{A}}.b^8$

$+\overset{1}{\mathfrak{A}}.l+\overset{2}{\mathfrak{A}}.\overline{2bk+2ch+2dg+2ef}+\overset{3}{\mathfrak{A}}.\overline{3b^2h+3b.2cg+3b.2df+3bc^2+3c^2f+3c.2de+d}$
$+\overset{4}{\mathfrak{A}}.\overline{4b^3g+6b.2cf+6b^2.2de+4b.3c^2e+4b.3c.d^2+4c^3.d}$
$+\overset{5}{\mathfrak{A}}.\overline{5b^4.f+10b^3.2ce+10.b^3.d^2+10.b^2.3c^2.d+5b.c^4}+\overset{6}{\mathfrak{A}}.\overline{6.b^5e+15.b^4.2cd+20b^3.c^3}$ $x^{9\mu}$ + &c.
$+\overset{7}{\mathfrak{A}}.\overline{7b^6.d+21.b^5.c^2}+\overset{8}{\mathfrak{A}}.8b^7.c+\overset{9}{\mathfrak{A}}.b^9$

$$=\overline{a+bx^{\mu}+cx^{2\mu}+dx^{3\mu}+ex^{4\mu}+fx^{5\mu}\ \&c}\,\big|^{\lambda}.$$

Si formulæ hujuſce membra ſingula cum quantitate $x^{\nu\lambda}$ multiplices, hoc eſt ſi indices poteſtatum literæ x quantitate $\nu\lambda$ augeas, efficietur formula generalis poteſtatum ſeriei $ax^{\nu}+bx^{\nu+\mu}+cx^{\nu+2\mu}+dx^{\nu+3\mu}+ex^{\nu+4\mu}$ &c. ſive formula generalis harum quantitatum $\overline{ax^{\nu}+bx^{\nu+\mu}+cx^{\nu+2\mu}+dx^{\nu+3\mu}+ex^{\nu+4\mu}\ \&c}\,\big|^{\lambda}$. Quæ denique pro ſymbolis $\overset{1}{\mathfrak{A}}$, $\overset{2}{\mathfrak{A}}$, $\overset{3}{\mathfrak{A}}$, $\overset{4}{\mathfrak{A}}$, &c. hiſce ordine reſtituti, $\lambda\alpha^{\lambda-1}$, $\lambda.\frac{\lambda-1}{2}\alpha^{\lambda-2}$, $\lambda.\frac{\lambda-1}{2}.\frac{\lambda-2}{3}\alpha^{\lambda-3}$, $\lambda.\frac{\lambda-1}{2}.\frac{\lambda-2}{3}.\frac{\lambda-3}{4}\alpha^{\lambda-4}$, in formulam generalem Moivræanam migraverit.

SAMUELIS HORSLEII

DE

GEOMETRIA FLUXIONUM

LIBER SINGULARIS.

SIVE ADDITAMENTUM TRACTATUS NEWTONIANI

DE METHODO RATIONUM PRIMARUM ET ULTIMARUM.

GEOMETRIA FLUXIONUM.

DEFINITIONES.

I.

NASCI dicuntur magnitudines motu genitæ, dum ſenſim creſcunt.

II.

Evaneſcere, dum ſenſim minuuntur.

III.

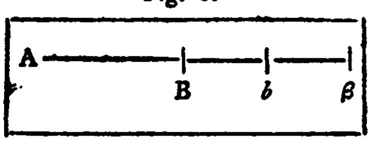
Communi verbo *fluere* dicuntur, quæ vel naſcuntur, vel evaneſcunt.

IV.

Magnitudinum fluentium velocitates creſcendi vel decreſcendi vocantur *fluxiones*.

Cæterùm hoc ipſum, velocitas creſcendi decreſcendive, quo ſenſu dicitur, exemplis declarandum eſt.

Fig. 1.



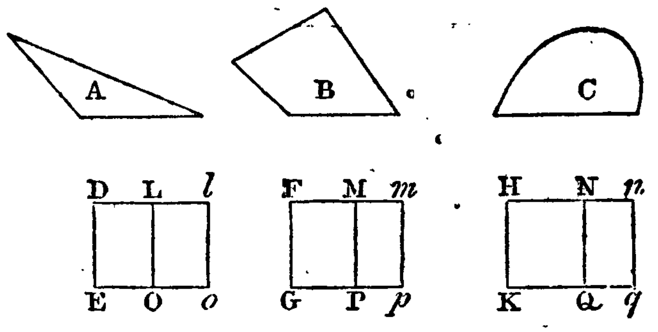
Puta rectam AB motu puncti B verſus *b*, β generari, ac perpetim augeri; velocitas creſcendi rectæ AB, cùm aſſignatam quamvis longitudinem, puta A*b*, adepta ſit, ea ipſa eſt quâ punctum mobile B è loco *b* egreditur. Et ſi regreſſu puncti B à locis β, *b* verſus A, recta AB perpetim decreſcat, velocitates receſſûs puncti B à locis β, *b* velocitates erunt decreſcendi rectæ AB, cùm è majore aliquâ longitudine in illas Aβ, A*b* decreverit. Fluxio igitur rectæ AB eſt velocitas, quâ punctum B, ſive progrediendo, ſive recedendo è loco B diſcedit. Et fluxio ejuſdem rectæ jam factæ A*b*, ea eſt, quâ punctum mobile B, progrediendo ſive recedendo, è loco *b* diſcedit. Et velocitas puncti B ex alio quovis loco β diſceſſuri, fluxio eſt rectæ AB longitudinem Aβ adeptæ.

Superficierum fluentium velocitates creſcendi vel decreſcendi ita æſtimantur, ſi ad longitudinem aliquam datam, hoc eſt, pro arbitrio ſumendam, ſed quæ ſemel ſumpta non debet mutari, ſuperficies quæ fluant ſemper applicentur. Cujuſque enim velocitas creſcendi vel decreſcendi eadem

eſt,

Definitiones.

eſt, ac lateris ex perpetuâ applicatione oriundi. Puta ſuperficies A, B, C laterum incrementis vel decrementis perpetim augeri, vel minui. Exponatur recta DE datæ alicujus magnitudinis, et rectæ DE æquales exponantur aliæ FG, HK. Sint A, B, C, ſuperficierum fluentium, A, B, C, magnitudines, quæ vel initio fluxûs fuere, vel fluendo ſimul extiterunt.



Et ad rectam DE applicetur rectangulum EL ſpatio A æquale. Ad rectam verò FG applicetur rectangulum GM ſpatio B æquale. Ad rectam denique HK, applicatum puta rectangulum KN ſpatio C æquale. Ad eaſdem rectas DE, FG, HK, applicata puta rectangula E*l*, G*m*, K*n*, aliis ſuperficierum fluentium A, B, C magnitudinibus, fluendo ſimul factis, æqualia. Atque hoc ſemper fieri intelligatur, ut ſuperficiebus A, B, C creſcentibus vel decreſcentibus, rectangula E*l*, G*m*, K*n* ſimul ac pariter cum illis fluant. Fluunt igitur latera ex applicationibus orta. Rectæ inquam DL, FM, HN fluunt. Harum verò rectarum atque ſuperficierum fluentium creſcendi decreſcendive velocitates eædem cenſendæ ſunt; nimirum ſuperficiei A, rectæque DL; ſuperficiei B, rectæque FM; ſuperficiei C, rectæque HN.

Solidorum fluentium creſcendi decreſcendive velocitates ad ſimilem modum æſtimare licet. Nimirum, ſi ad ſuperficiem aliquam datam, hoc eſt, pro arbitrio ſumendam, ſed quæ ſemel ſumpta mutari non debet, Solidorum, quotquot fluant, applicatio fiat. Cujuſque enim creſcendi decreſcendive eadem eſt velocitas, ac lateris ex perpetuâ applicatione oriundi.

Quantitatum Algebraicarum, quæ trinam dimenſionem, uti loquuntur Algebraiſtæ, excedunt, velocitates creſcendi decreſcendive æſtimantur, adhibendo interpretationem ſymbolorum Geometricam. Verbi causâ, ſi quantitatem Algebraicam $xyzus$, è fluentibus x, y, z, u, s factam, hoc modo ſcribas, $\frac{xyzus}{1}$; tum pro unitate eam rectæ alicujus datæ poteſtatem ſubſtituas, quæ quantitatem $\frac{xyzus}{1}$ ad Solidi, Superficiei, vel Rectæ ordinem deprimat; cujus velocitatem creſcendi decreſcendive eo, quo tradidimus modo, æſtimare liceat.

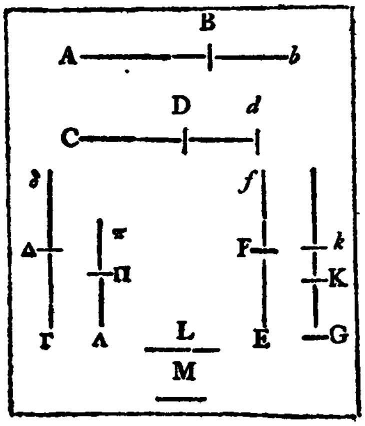
V.

Fluxionum nomine veniunt non modò velocitates illæ creſcendi decreſcendive, quarum illud maximè proprium eſt; verùm etiam, ex tranſlatione, magnitudines Geometricæ cujuſcunque generis, quæ velocitatum illarum inter ſe rationes ſervant.

Fluant

Fluant rectæ duæ AB, CD. Sint aliæ rectæ EF, GK, quarum EF ſit ad GK, ut velocitas quâ punctum B, quod motu ſuo rectam AB generat, è loco A egreditur, ad velocitatem quâ punctum D, cujus motu recta altera CD generatur, egreditur è loco C. Tum creſcere vel decreſcere intelligantur rectæ EF, GK, eâ quidem lege, ut fluentibus AB, CD, EF, GK, ſi capiantur punctorum mobilium B, F loca quælibet, *b*, *f*, quæ ſimul ea appulerint, item mobilium D, K, loca quæ ſimul ea appulerint *d*, *k*; EF ſit ad E*f*, ut velocitas motûs puncti B è loco primo A, ad velocitatem motûs ejuſdem puncti B è loco *b*; ſimiliter ſit GK ad G*k*, ut velocitas motûs puncti D è loco ſuo primo C ad velocitatem motûs ejuſdem puncti è loco *d*. Ita verò neceſſario fiet, ut E*f* ſit etiam ad G*k*, ut velocitas motûs puncti B è loco *b* ad velocitatem motûs puncti D è loco *d*. DEFINITIONES.

Fig. 3.



Rectæ E*f*, G*k*, ad hunc modum mutabiles, fluxiones dicuntur rectarum A*b*, C*d*, atque omnium adeo quantitatum, quarum velocitates creſcendi decreſcendive earum inter ſe rationes induant. Sunt enim illæ rectæ fluxionum, quarum inter ſe proportiones ſervant, ſymbola quaſi Geometrica. Ac pari jure aliæ quævis magnitudines Geometricæ, quæ ita creſcunt vel diminuuntur, ut rectarum E*f*, G*k* proportiones inter ſe ſemper ſervent, illis utique fluentium magnitudinibus inter ſe collatis, quæ ſimul fluendo exſiſtunt, rectarum A*b*, C*d*, aliarumque omnium magnitudinum, quæ pariter cum hiſce A*b*, C*d* fluunt, fluxiones ſunt dicendæ. Hujuſcemodi verò magnitudines notis $\dot{x}$, $\dot{y}$, $\dot{z}$ Newtonus deſignavit, literis x, y, z fluentes deſignantibus.

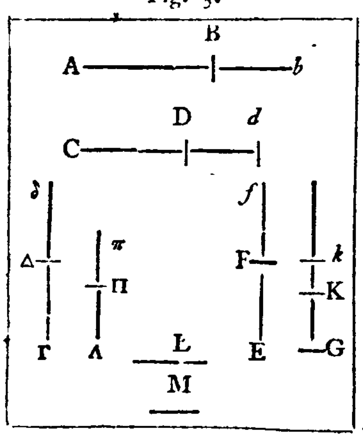
Ex his autem liquet, magnitudinum fluentium fluxiones, quæ ſenſu definitionis quintæ eo nomine vocantur, magnitudines eſſe Geometricas, quæ vel quantitate ſint fixâ et permanente, vel ipſæ fluant. Fixâ quidem quantitate manent, quarum fluentes æquabili velocitate fluunt. Sin fluenti velocitas creſcendi vel decreſcendi utcunque ſit mutabilis, in hoc caſu fluxio ipſa fluet; et creſcendo quidem, ſi velocitas fluentis creſcat, decreſcendo, ſi illa ſenſim minuatur. Sic enim ſi puncti B, rectam AB generantis, velocitates motûs è locis A, *b* æquales ſint, æquales erunt rectæ EF, E*f*, quæ velocitatum illarum inter ſe proportiones ſervant. Ac ſi velocitas motûs puncti B ex alio quovis loco rectæ AB eadem maneat, rectæ etiam EF magnitudo fixa manet. Sin velocitas motûs puncti B è loco *b* quàm è loco A major ſit, recta E*f* rectâ EF major erit. Et ſi velocitas motûs puncti B, longiùs à loco A progredientis, ſenſim augeatur, recta quoque EF ſenſim creſcet. E contrario decreſcet recta EF, ſi velocitas motûs puncti B, progrediente puncto illo, ſenſim minuatur.

Puta igitur rectas AB, CD motu inæquabili punctorum B, D hanc et illam generari. Fluent igitur rectæ EF, GK; atque harum velocitates creſcendi decreſcendive aliis rectis, L, M, exponi poterunt. Rectæ autem

DEFINITIONES.

L, M, atque aliæ omnes magnitudines Geometricæ, quæ ita creſcant vel diminuantur, ut harum rectarum inter ſe proportiones ſervent, rectarum EF, GK, ſenſu definitionis quintæ, fluxiones erunt. Hujuſmodi verò quantitates, ad fluentes primas referendo, fluxiones ſecundas Newtonus nominavit, et ſymbolis $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ deſignari juſſit. Rurſum magnitudines illæ $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ vel quantitate ſint fixâ quæque et permanente, vel ipſæ fluant; prout quantitates $\dot{x}$, $\dot{y}$, $\dot{z}$ velocitate unaquæque æquabili fluat, vel utcunque mutabili. Si fluant magnitudines $\ddot{x}$, $\ddot{z}$, $\ddot{y}$, erunt et harum fluxiones, ſenſu definitionis quintæ; magnitudines nimirum Geometricæ, quæ velocitatum, quibuſcum magnitudines $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ creſcunt vel decreſcunt, proportiones inter ſe, fluendo vel permanendo, ſervent. Tales autem magnitudines fluentium x, y, z fluxiones tertias Newtonus nominavit, atque ſymbolis $\dddot{x}$, $\dddot{y}$, $\dddot{z}$ deſignari voluit. Fluxionum quarti, quinti, cujuſvis demum ordinis, ſimilis eſt interpretatio.

Fig. 3.



Jam verò ſint ΓΔ, ΛΠ magnitudines Geometricæ, quæ eâ lege fluant, ut velocitates quibuſcum illæ creſcunt vel decreſcunt, magnitudinum fluentium AB, CD inter ſe proportiones ſervent, eis utique fluentium quantitatibus inter ſe collatis, quæ fluendo ſimul exſiſtunt. Sunt igitur magnitudines AB, CD magnitudinum ΓΔ, ΛΠ, ſenſu definitionis quintæ, fluxiones. E diverſo ſunt hæ illarum fluentes, quas ſymbolis $\grave{x}$, $\grave{y}$ Newtonus deſignavit, literis x, y, magnitudines AB, CD deſignantibus.

Rurſum magnitudines Geometricæ quarum velocitates creſcendi decreſcendive magnitudinum ΓΔ, ΛΠ proportiones inter ſe ſervant, iis utique fluentium quantitatibus, quæ fluendo ſimul exſiſtunt inter ſe collatis, rectarum ΓΔ, ΛΠ fluentes ſunt, rectarum verò AB, CD fluentes ſecundæ vocandæ, et more Newtoniano ſymbolis $\grave{\grave{x}}$, $\grave{\grave{y}}$ deſignantur.

In ſummâ, ex quâvis magnitudine fluente (puta x) tanquam radice, fluentium et fluxionum in utramque partem infinita eſt progreſſio, $\overset{\backslash n}{x}$—$x$, $\overset{\backslash\backslash\backslash\backslash}{x}$, $\overset{\backslash\backslash\backslash}{x}$, $\overset{\backslash\backslash}{x}$, $\grave{x}$, x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$—$\overset{\cdot n}{x}$, quæ tamen omnes vi definitionis quintæ ad magnitudines Geometricas revocantur.

VI.

Similiter, vel *eodem modo* fluere dicuntur quantitates, quæ ſingulæ ſimul naſcuntur, vel ſimul evaneſcunt.

VII.

Contrariè fluere dicuntur, quarum naſcente alterâ, altera evaneſcit.

THE-

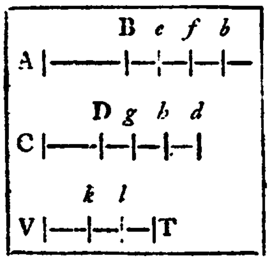
THEOREMA PRIMUM.

Theor. I.

Magnitudinum, quæ fluunt, Fluxiones inter ſe proportiones habent, quæ incrementorum naſcentium ſunt primæ, vel evaneſcentium ultimæ.

Sint lineæ primò magnitudines quæ fluunt. Fluant igitur lineæ AB, CD, et fluendo longitudines illas AB, CD ſint adeptæ. Fluendo autem perſeverantes, dato quovis tempore longitudines novas A*b*, C*d* adipiſcantur, ut ſint B*b*, D*d* incrementa linearum fluentium, fluendo ſimul facta. Dico fluxionem rectæ AB ad fluxionem rectæ CD proportionem habere, quæ prima eſt naſcentis incrementi B*b* ad incrementum naſcens D*d*; vel, ſi regreſſu punctorum B, D verſus A, C lineæ illæ decreſcant, quæ ultima eſt evaneſcentis incrementi B*b* ad D*d* ſimul evaneſcens.

Fig. 4.

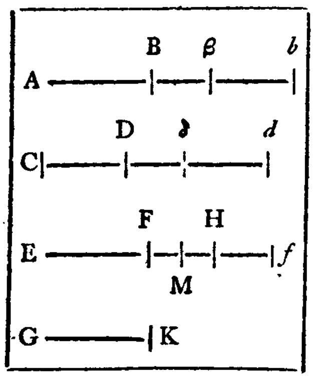


Rectæ AB, CD vel utraque fluunt velocitate æquabili, vel altera quidem æquabili, mutabili altera; vel utraque denique mutabili. Fluant primò velocitate utraque æquabili. Spatia igitur B*b*, D*d* velocitatibus æquabilibus eodem tempore confecta, erunt inter ſe ut velocitates. Et ſi dividatur B*b* in partes quot volueris æquales B*e*, *ef*, *fb*, tum ſi D*d* in partes totidem æquales dividatur D*g*, *gh*, *hd*; erunt B*e*, D*g* rectarum AB, CD incrementa, incrementis B*b*, D*d* minora, fluendo ſimul facta. Nam ſi dividatur tempus totum VT, quo ſpatia B*b*, D*d* ſunt confecta, in particulas V*k*, *kl*, *l*T inter ſe æquales, numero autem totidem quot ſunt partes in quas linearum B*b*, D*d* utraque diviſa eſt, erit B*b* ad B*e*, ut tempus VT ad tempus V*k*. Erit etiam D*d* ad D*g*, ut tempus VT ad tempus V*k*. Spatia verò, velocitatibus æquabilibus deſcripta, ſunt inter ſe ut tempora, quibus confecta ſunt. Eſt igitur B*b* ad B*e* ut ſpatium puncto mobili B, datâ velocitate lato, tempore VT confectum, ad ſpatium eodem puncto eâdem velocitate tempore V*k* confectum. Eſt autem B*b* illud ipſum ſpatium, quod punctum B tempore VT confecit. Erit igitur B*e* illud ipſum ſpatium, quod punctum B tempore V*k* confecerit. Simili ratione concludetur eſſe D*g* ſpatium, quod punctum D, datâ velocitate ſuâ latum, eodem tempore V*k* confecerit. Sunt igitur B*e*, D*g* rectarum AB, CD incrementa fluendo ſimul facta, ſive ſpatia punctis B, D ſimul confecta. Erunt igitur inter ſe ut punctorum velocitates. Neque huic rationi officiet temporis V*k* magnitudo, magna illa parvave fuerit. Numero igitur particularum V*k*, *kl*, *l*T in infinitum creſcente, magnitudinibus ſingularum infinitè decreſcentibus, ſi rectarum B*b*, D*d*, ac temporis VT diviſiones ſimiles ſemper fiant (ex quo ſcilicet numerus partium B*e*, *ef*, *fb*, necnon D*g*, *gh*, *hd* infinitè augebitur, magnitudinibus ſingularum infinitè decreſcentibus) manebit rectarum B*b*, D*d* partibus ſimul genitis, B*e*, D*g* data velocitatum ratio, quibuſcum puncta

Theor. I. ta B, D æquabiliter feruntur. Quare data illa velocitatum ratio naſcentium B*e*, D*g* prima erit, vel evaneſcentium ultima. In hoc caſu igitur velocitates motuum punctorum B, D, hoc eſt, fluxiones rectarum AB, CD inter ſe proportionem habent incrementorum naſcentium primam, vel evaneſcentium ultimam. Q. E. D.

Cas. 2. Sed fluat AB velocitate mutabili, fluente CD æquabiliter. Velocitas puncti B progredientis vel ſenſim augetur, vel ſenſim langueſcit. Primò augeatur. Exponatur recta EF datæ cujuſvis longitudinis. Sit alia E*f* ad quam EF proportionem habeat, quam velocitas motûs puncti B è loco B egredientis ad velocitatem motûs ejuſdem puncti B è loco aliquo *b* ulteriori. Sit etiam GK recta, ad quam EF proportionem habeat velocitatis motûs puncti B, è loco B, ad velocitatem æquabilem puncti D rectam CD generantis. Jam B*b* major erit, quàm ut habeat ad D*d* proportionem quam EF ad GK; minor verò, quàm ut habeat ad eandem D*d* proportionem quam E*f* ad GK. Nam ſi punctum B, id omne temporis quo rectam B*b* generavit, cum velocitate illâ, quâ è loco B primùm egreſſum eſt, æquabiliter latum eſſet, rectam utique generâſſet illâ B*b* minorem; quæ quidem ad D*d* rationem habuiſſet quam EF ad GK. Sin punctum B cum majore illâ velocitate, quâ è loco *b* egreditur, è loco B exiiſſet, tempus, quo B*b* reverâ generavit, motu æquabili perſeverans, majorem generâſſet; quæ ad D*d* rationem habuiſſet quam E*f* ad GK. Major igitur eſt B*b* quàm ut habeat ad D*d* rationem rectæ EF ad GK, ſed minor eadem, quàm quæ rationem ad eandem habeat quam E*f* ad GK. Regredientibus autem punctis B, D à locis *b*, *d* verſus B, D, ratio evaneſcentis B*b* ad evaneſcentem D*d* ſenſim minuet; et rationem rectæ EF ad GK, quam ſemper ſuperat, propiùs tamen acceſſura eſt, quàm pro datâ quâvis differentiâ. Non enim. Erit igitur ratio aliqua minoris quidem quàm rectæ E*f* ad GK, majoris verò quàm EF ad GK, quâ ratio evaneſcentis B*b* ad evaneſcentem D*d* non minor fiet. Ejuſmodi ſit ratio rectæ EH ad GK. Recta igitur EH minor eſt quàm E*f*, major autem eadem quàm EF. Jam cùm puncti B velocitas, è loco B progredientis, ſenſim creſcat; ſi detur velocitas major quidem quàm illa, quæ eſt motûs puncti B è loco B, minor verò quàm illa, quæ eſt motûs puncti B è loco *b*; erit locus aliquis locorum B, *b* intermedius, ex quo motûs puncti B velocitas datâ illâ minor erit. Detur igitur velocitas, quæ habeat ad velocitatem puncti D rectam CD generantis, proportionem quam EH ad GK. Hæc major erit velocitate quæ eſt motûs puncti B è loco B, minor verò illâ quæ eſt motûs puncti B è loco *b*. Sit β locus locorum B, *b* intermedius, ex quo velocitas motûs puncti B datâ minor ſit. Sit Dδ ſpatium puncto D confectum, dum punctum B ſpatium Bβ abſolvat. Sit EM recta, quæ habeat ad GK rationem, quam velocitas motûs puncti B è loco β ad velocitatem motûs puncti D æquabilem. Erit illa EM minor quàm ut habeat ad GK rationem quam EH ad GK. Omnino igitur minor quàm ut habeat ad GK rationem quam Bβ ad Dδ; cùm ratio illius Bβ ad Dδ ratione EH ad GK minor eſſe nequeat (id enim poſuimus).

Fig. 5.



Quare

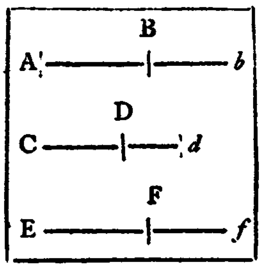
Quare Bβ major eſt, quàm ut habeat ad Dδ rationem quam EM ad GK. Theor. I.
Sunt autem Bβ, Dδ rectarum AB, CD incrementa, fluendo ſimul facta, quorum illud velocitate generatum eſt creſcente, hoc æquabili. Sunt etiam rectæ EF, EM ad rectam GK, ſicut velocitates motûs puncti B è locis B, β ad velocitatem æquabilem puncti D. Ergo recta Bβ major quidem erit, quàm ut ſit ad Dδ ut EF ad GK, ſed minor eadem quàm ut ſit ad Dδ ut EM ad GK. Sed major jam oſtenſa eſt Bβ quàm pro hâc ratione. Simul utique major et minor. Quod eſt abſurdum.

Non igitur eſt ratio ulla minoris quidem quàm E*f* ad GK, majoris verò quàm EF ad eandem GK, quâ ratio evaneſcentis B*b* ad evaneſcentem D*d* non ſit minor; quæ tamen rationem EF ad GK ſemper exſuperat. Quare ratio EF ad GK, quæ eſt fluxionis rectæ AB ad fluxionem rectæ CD, evaneſcentis B*b* ad evaneſcentem D*d* erit ultima, eademque naſcentium prima.

Quòd ſi velocitas motûs puncti B progredientis ſenſim langueſcat, ſimilis erit et in hoc caſu Theorematis demonſtratio.

Cas. 3. Sed fluant AB, CD velocitate utraque mutabili. Sic etiam dico fluxionem rectæ AB ad fluxionem rectæ CD proportionem habere, quæ prima eſt naſcentis incrementi B*b* ad naſcens D*d*, vel ultima evaneſcentis B*b* ad D*d* ſimul evaneſcens. Fluat enim tertia aliqua recta EF æquabiliter. Et ſit F*f* incrementum quod recta EF acceperit, quo tempore rectis AB, CD acceſſerint B*b*, D*d*. Tum (per Caſ. præc.) erit B*b* ad F*f* ultimò, ut fluxio rectæ AB ad fluxionem rectæ EF, et F*f* ad D*d* ultimò, ut fluxio rectæ EF ad fluxionem rectæ CD. Quare ex æquo erit B*b* ad D*d* ultimò, ut fluxio rectæ AB ad fluxionem rectæ CD. Q. E. D.

Fig. 6.



CASUS GENERALIS SECUNDUS.

Sint jam Superficies magnitudines quæ fluant. Fluant igitur ſuperficies A, B (vid. fig. Def. 4.) Exponantur rectæ æquales DE, FG; ad quas ſuperficierum fluentium fiat applicatio, ut ſint rectangula EL, GM, magnitudinibus quibuſvis fluentium, fluendo ſimul factis, æqualia, item rectangula E*l*, G*m* aliis quibuſvis earundem fluentium magnitudinibus, fluendo ſimul factis. Jam fluxiones ſuperficierum fluxionibus laterum ex applicatione oriundorum, linearum inquam DL, FM fluxionibus ſunt æquales. (Vid. Def. 4 & 5.) Rectarum verò DL, FM fluxiones inter ſe proportiones habent, quæ primæ ſunt incrementorum naſcentium L*l*, M*m* (per Caſ. Gen. 1.) Hoc eſt, quæ ſunt primæ rectangulorum naſcentium O*l*, P*m*, vel eorundem evaneſcentium ultimæ. Hiſce verò rectangulis ſuperficierum fluentium incrementa, fluendo ſimul facta, ſemper ſunt æqualia. Eædem igitur erunt naſcentium rationes primæ, vel evaneſcentium ultimæ. Fluentibus igitur ſuperficiebus A, B, fluxiones inter ſe rationes habent, quæ primæ ſunt incrementorum naſcentium, vel evaneſcentium ultimæ. Q. E. D.

 CASUS

CASUS GENERALIS TERTIUS.

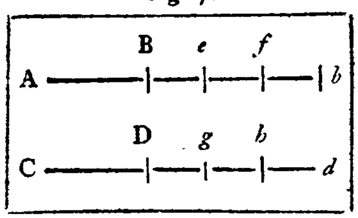
Si Solida ſint magnitudines, quæ fluunt, ſimilis erit et in hoc caſu propoſitionis probatio, Solidorum fluxionibus ad Rectarum fluxiones revocatis.

Cor. Hæc ita ſe habent, ſi magnitudines ſimiliter fluant. Sed et contrariè fluentium fluxiones inter ſe rationem habent, quæ prima eſt naſcentis incrementi magnitudinis creſcentis ad decrementum naſcens magnitudinis decreſcentis. Quod iiſdem planè argumentis effici poteſt.

THEOREMA II.

Magnitudides inter ſe æquales ſi fluant, et fluendo æqualitatem, quam initio fluxûs habuere, inter ſe ſemper ſervent, harum fluxiones ſemper erunt inter ſe æquales. Viciſſim, quarum magnitudinum fluxiones ſemper ſunt æquales, hæ, dummodo homogeneæ ſint, et ſimiliter fluant, vel ſemper inter ſe æquales erunt, vel altera alterâ major erit datâ; utique ſi conferantur fluentium quantitates, quæ fluendo ſimul exſiſtunt.

Fig. 7.

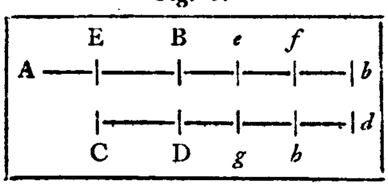


Sint magnitudines æquales AB, CD, quæ fluant, et fluendo æqualitatem inter ſe ſemper ſervent. Dico harum fluxiones ſemper eſſe inter ſe æquales. Sint B*b*, D*d* magnitudinum fluentium incrementa quæ ſimul fluendo exſiſtunt, tempore quovis finito genita. Erunt igitur A*b*, C*d* inter ſe æquales. Et propter æquales AB, CD, erunt etiam B*b*, D*d* inter ſe æquales. Nec huic rationi officient incrementorum B*b*, D*d* magnitudines, magna ea parvave fuerint, dummodo fluendo ſimul exſiſtant. Quare et naſcentium inter ipſa prima, vel evaneſcentium ultima ratio æqualitas erit. Sed incrementorum naſcentium ratio inter ipſa prima, vel evaneſcentium ultima eſt fluxionum ratio. Fluxiones igitur ſunt inter ſe æquales. Q. E. D.

Jam verò magnitudinum fluentium AB, CD fluxiones ſemper ſint æquales, et ſint fluentes magnitudines homogeneæ, et ſimiliter fluant. Dico fluentes vel ſemper inter ſe æquales eſſe, vel alterâ alteram datâ majorem, ſi quantitates, quæ fluendo ſimul exſiſtunt, conferantur. Sint AB, CD magnitudines quas initio fluxûs fluentes habuerunt, ſive priuſquam fluere inceperint. Sint B*b*, D*d* incrementa ſimul genita. Horum utrumque in partes quotvis dividas, numero æquales, temporibus iiſdem ordine genitas; nempe B*b* in partes B*e*, *ef*, *fb*; D*d* in partes totidem D*g*, *gh*, *hd*. Jam ſi partium utriuſque numerus infinitè augeatur, magnitudinibus ſingularum infinitè decreſcentibus, erunt partium ſimul genitarum rationes naſcentium primæ, ſive evaneſcentium ultimæ, eædem quæ fluxionum rationes (per Theor. I. hujus). Hoc eſt, erit B*e* ad D*g* ultimò ut fluxio magnitudinis AB ad fluxionem magnitudinis CD. Et *ef* ad *gh* ultimò ut fluxio magnitudinis A*e* ad fluxionem magnitudinis C*g*. Et *fb* ad *hd* ultimò ut fluxio magnitudinis A*f* ad fluxionem magnitudinis C*h*. Eſt autem æqualitas fluxionum fixa ratio (id enim poſuimus) Quare partium B*e*, *ef*, *fb* ad partes D*g*, *gh*, *hd* naſcentium

naſcentium ad naſcentes, vel evaneſcentium ad evaneſcentes, eſt æqualitas ratio prima, vel ultima, communis. Ergo magnitudines totæ Bb, Dd ſunt inter ſe æquales (per Cor. Lem. IV. Newton. de Rat. Prim. et Ult.) Datæ autem AB, CD (datas enim ſemper intelligo fluentium initio fluxûs magnitudines, quod ſemel monuiſſe ſatis eſto) datæ inquam AB, CD vel æquales ſunt inter ſe, vel inæquales. Si æquales ſint, erunt Ab, cd inter ſe æquales. Theor. III.

Fig. 8.

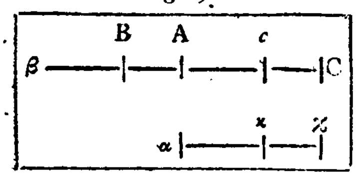


Si inæquales ſint AB, CD, altera ex illis major erit. Sit AB major, et auferatur BE minori CD æqualis. Ergo data erit AE, et Eb, cd inter ſe æquales erunt. Ergo Ab datâ major eſt quàm cd. Sunt igitur Ab, cd vel æquales inter ſe, vel altera alterâ eſt datâ major. Q. E. D.

THEOREMA III.

Si duæ magnitudines eâ lege fluant, ut ſimul ſumptæ, jungendo utique quæ ſimul fluendo exſiſtunt, datæ cuidam æquales ſint, fluxiones earum æquales quidem, ſed contrariæ erunt. Et ſi duarum fluentium fluxiones inter ſe ſemper æquales ſed contrariæ ſint, fluentium ſumma, jungendo utique quæ ſimul fluendo exſiſtunt, datæ cuidam æqualis erit.

Fig. 9.



Duæ magnitudines, quæ initio fluxûs ſint AB, AC, junctæque ſummam efficiant BC, eâ lege fluant, ut datæ BC earum ſumma ſemper ſit æqualis. Dico earum fluxiones inter ſe æquales, ſed contrarias eſſe. Et contrarias quidem eſſe fluentium AB, AC fluxiones, hoc eſt, magnitudines illas contrariè fluere, manifeſtum eſt; ſiquidem earum ſummæ ſua conſtet magnitudo. Nam ſi earum altera, puta AB, creſcat, opus eſt decreſcat altera, ne creſcente utrâque ſumma creſcat. Viciſſim, ſi altera, puta AB, decreſcat, opus erit creſcat altera, ne utrâque decreſcente ſumma decreſcat. Fluunt igitur AB, AC modis inter ſe contrariis. Dico earum fluxiones inter ſe æquales eſſe. Nam quo tempore AB fluendo facta eſt Aβ, contrariè fluendo altera AC fiat Ac. Duarum igitur Aβ, Ac ſumma βc datæ BC æqualis erit. Ablatâque communi Bc, erunt βB, cC inter ſe æquales. Hoc eſt, illud quo aucta eſt AB æquale erit ei quo imminuta eſt AC, vel viciſſim. Neque huic rationi officiet illarum Bβ, cC magnitudines, magnæ parvæve fuerint, dummodo fluendo ſimul genitæ. Quare et naſcentium Bβ, cC prima inter ipſas ratio æqualitas erit. Quæ verò naſcentibus Bβ, cC inter ipſas eſt prima, eadem magnitudinum AB, AC fluxionibus inter ipſas ratio erit. (Th. I. Cor.) Quare et fluxiones illæ inter ſe æquales erunt. Q. E. D.

Rurſum magnitudinum duarum AB, AC contrariè fluentium, ut creſcente alterâ, altera decreſcat, intelligantur fluxiones ſemper inter ſe æquales. Dico fluentium ſummam, jungendo utique quæ ſimul fluendo exſiſtunt, datæ cuidam æqualem eſſe; ei nimirum, quam magnitudines initio fluxûs

THEOR. III & IV.

fluxûs compoſitæ conficiebant. Sint enim Aβ, Ac fluentium quantitates quævis, quæ ſimul fluendo exſiſtunt. Dico harum ſummam βc magnitudini BC æqualem eſſe. Sit enim $\alpha\varkappa$ magnitudo quædam, quæ eâ lege fluat, ut fluenti Aβ addita, jungendo utique quæ ſimul fluendo exſiſtunt, ſummam reddat datæ BC æqualem. Fluentium igitur Aβ, $\alpha\varkappa$ fluxiones contrariæ quidem, ſed inter ſe æquales erunt; per ea quæ modò oſtendimus. Fluentium autem Aβ, Ac fluxiones ſunt inter ſe æquales; ſic enim poſuimus. Quare fluentium Ac, A$\varkappa$ fluxiones inter ſe æquales erunt. Et fluentes illæ, cùm utraque earum contrario modo fluat atque Aβ, ſimiliter utique fluunt. Quum verò $\alpha\varkappa$ eâ ſemper lege fluat, ut fluenti Aβ addita ſummam reddat datæ BC æqualem, ſi Aβ refluendo fiat AB, illa autem $\alpha\varkappa$ refluendo ſimul fiat $\alpha\chi$, erunt $\alpha\chi$, AC inter ſe æquales. Magnitudines igitur AC, $\alpha\chi$, quarum ſimiliter fluentium fluxiones ſemper ſunt inter ſe æquales; hæ, cùm initio fluxûs inter ſe æquales ſint, manebunt ſemper quidem inter ſe æquales, ſi conferantur utique earum quantitates quæ fluendo ſimul exſiſtunt (Th. II) At ſimul fluendo exſiſtunt Ac, $\alpha\varkappa$, nempe cùm earum utraque ſimul cum Aβ fluendo exſiſtat. Sunt igitur Ac, $\alpha\varkappa$ inter ſe æquales. Additâque communiter Aβ, duæ Aβ, Ac ſimul ſumptæ duabus Aβ, $\alpha\varkappa$ ſimul ſumptis æquales erunt. Sed duæ Aβ, Ac ſimul ſumptæ ſummam efficiunt βc. Et duæ Aβ, A$\varkappa$ ſimul ſumptæ datæ BC æquales ſunt. Id enim poſuimus. Quare βc datæ BC eſt æqualis. Q. E. D.

SCHOLIUM.

Quantitatum contrariè fluentium fluxiones algebraicè hoc modo deſignari ſolent: $\dot{x} = -\dot{y}$. Et quoties calculis artificioſè ſubductis, iſtiuſmodi tandem æquatio evaſerit, indicio eſt, quantitatem $x + y$ permanenti cuidam æqualem eſſe.

THEOREMA IV.

Magnitudines, quæ datam inter ſe rationem habent, ſi fluant, et fluendo rationem datam, quam initio fluxûs habuere, inter ſe ſemper ſervent, harum fluxiones datam fluentium ſuarum rationem inter ſe ſemper habent. Viciſſim, quarum magnitudinum fluxiones datam inter ſe rationem ſemper ſervant, hæ dummodo homogeneæ ſint, et ſimiliter fluant, vel datam fluxionum ſuarum rationem inter ſe ſemper habent, vel altera alterâ major erit datâ quàm pro fluxionum ratione; utique ſi conferantur fluentium quantitates illæ, quæ fluendo ſimul exſiſtunt.

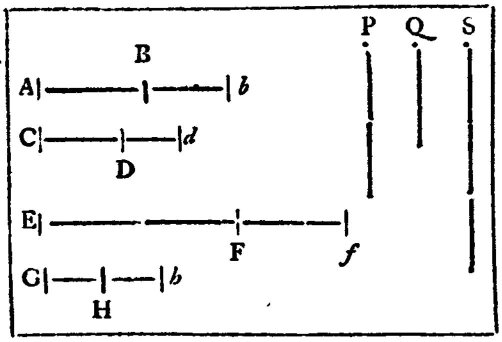
Demonſtratur hæc Propoſitio ad modum ſecundæ, imò iiſdem planè verbis, ſi in illius demonſtratione pro *ratione æqualitatis ratio data* ubique ſupponatur.

THEOREMA V.

Fluentibus magnitudinibus homogeneis, fluxionum ſumma ſummæ erit fluxio, differentia differentiæ.

Fluant magnitudines homogeneæ AB, CD. Fluat tertia EF cum prioribus homo-

Fig. 10.

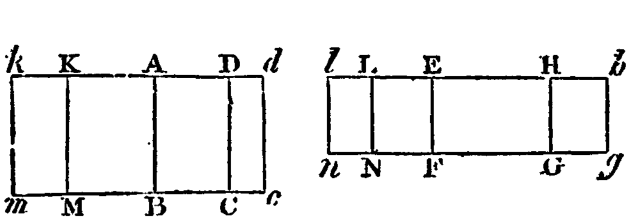


homogenea, eâ lege ut EF duarum AB, CD ſummæ ſemper ſit æqualis, nimirum ſi conferantur fluentium magnitudines fluendo ſimul factæ. Rurſum fluat alia GH cum prioribus etiam homogenea, eâ lege ut GH duarum AB, CD differentiæ ſemper ſit æqualis, ſi magnitudines fluendo ſimul factæ conferantur. Dico fluxionum AB, CD ſummam magnitudinis EF fluxionem eſſe, earundem verò fluxionum differentiam magnitudinis GH eſſe fluxionem. Sint enim P, Q magnitudinum AB, CD fluxiones, et S fluxio magnitudinis EF. Sint AB, CD, EF magnitudines fluendo ſimul factæ; et ſint B*b*, D*d*, F*f* magnitudinum fluentium AB, CD, EF incrementa fluendo ſimul facta. Habet igitur S ad P rationem naſcentis F*f* ad naſcens B*b* primam (per Prop. I. hujus). Habet quoque S ad Q rationem naſcentis F*f* ad naſcens D*d* primam. Habet igitur S ad duas P, Q ſimul ſumptas rationem eam, quæ prima eſt naſcentis F*f* ad naſcentia B*b*, D*d* ſimul ſumpta. Quoniam verò AB, CD ſimul ſumptæ magnitudini EF æquales ſunt, et A*b*, C*d* ſimul ſumptæ magnitudini E*f*, erunt incrementa B*b*, D*d* ſimul ſumpta incremento F*f* æqualia. Neque huic rationi officient incrementorum magnitudines, magna illa parvave fuerint. Quare et naſcentis F*f* ad naſcentia B*b*, D*d* ſimul ſumpta æqualitas prima ratio erit. Sed prima illa ratio magnitudinis eſt S ad duas P, Q ſimul ſumptas; ex priùs oſtenſis. Magnitudo igitur S duabus P, Q ſimul ſumptis eſt æqualis. Hoc eſt, fluxio magnitudinis EF ſummæ fluxionum AB, CD. THEOR. V & VI.

Et ſimili prorſus argumento concludetur fluxionem magnitudinis GH duarum P, Q differentiæ eſſe æqualem. Q. E. D.

THEOREMA VI.

Si rectangulis fluentibus unum tantummodò latus unicuique fluat, arearum fluxiones inter ſe rationem habent compoſitam è ratione datâ laterum permanentium, et è ratione fluxionum laterum quæ fluunt, vel datâ vel mutabili.

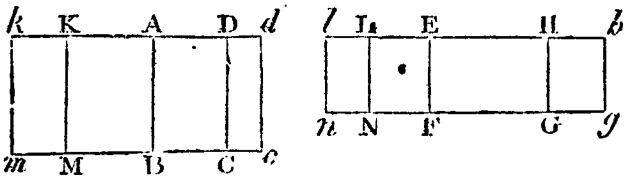


Rectangulorum ABCD, EFGH intelligantur latera AB, EF permanere, alteris AD, EH fluentibus. Dico fluxionem rectanguli AC ad fluxionem rectanguli EG rationem habere compoſitam è ratione datâ datæ AB ad datam EF, et ratione fluxionis rectæ AD ad fluxionem rectæ EH; quæ vel data erit vel mutabilis, prout rectarum AD, EH vel utriuſque vel alterius tantùm fluxus ſit æquabilis. Rectarum AD, EH longitudines, dato quovis tempore, ex illis

AD,

Theor. VI & VII.

AD, EH in alias Ad, Eh mutentur; ut ſint Dd, Hh incrementa rectarum fluentium fluendo ſimul facta. Erunt igitur rectangula Dc, Hg incrementa arearum fluentium fluendo ſimul facta. Arearum igitur fluxiones eam inter ſe rationem habebunt, quæ rectangulorum Dc, Hg vel naſcentium prima eſt, vel



evaneſcentium ultima. Hoc eſt, quæ compoſita eſt è ratione datâ rectæ AB ad rectam EF, cum illâ quæ rectæ Dd ad rectam Hh, naſcentis ad naſcentem, vel evaneſcentis ad evaneſcentem, prima ultimave erit. Sed ratio quæ rectis Dd, Hh vel naſcentibus inter ipſas prima, vel evaneſcentibus erit ultima, ea rectarum AD, EH fluxionum ratio eſt. Arearum igitur fluentium AC, EG fluxiones inter ſe rationem habent compoſitam è ratione datâ rectæ AB ad rectam EF, et ratione fluxionum laterum quæ fluunt, AD, EH, vel datâ vel mutabili.

Cor. In rectis DA, HE capiantur AK, EL, quæ ſint inter ſe ſicut velocitates, quibus puncta A, E, quorum motu rectæ AD, EH generantur, è locis ſuis primis A, E egrediuntur. Et fluentibus rectis AD, EH, rectæ AK, EL vel permaneant, vel fluant; eâ quidem lege, ut velocitatum fluendi rectarum AD, EH rationes inter ſe, vel permanendo vel fluendo, ſervent. Si punctorum A, E, K, L, capiantur loca quæ ſimul ea appulerint, d, h, k, l, erunt Ak, El fluxiones rectarum Ad, Eh, ſenſu definitionis quintæ; et ſi rectangula Am, En compleantur, erunt rectangula illa rectangulorum Ac, Eg eodem ſenſu fluxiones; ut ex hoc Theoremate ſatis patet. Jam ſi rectas datas AB, EF literæ a, b deſignent, mutabiles autem Ad, Eh literæ x, y; rectangulorum ax, by, erunt rectangula $a\dot{x}$, $b\dot{y}$ fluxiones: notis utique $\dot{x}$, $\dot{y}$ rectas Ak, El deſignantibus, vel datas quidem illas vel mutabiles, prout ratio fluendi rectarum AD, EH poſtulaverit.

THEOREMA VII.

Parallelopipedis rectangulis, ſi, fluentibus altitudinibus, baſes permaneant, Solidorum fluxiones inter ſe rationem habent è ratione datâ baſium permanentium, et ratione fluxionum altitudinum, ſive datâ, ſive mutabili, componendam. Sin altitudinibus permanentibus baſes fluant, Solidorum fluxiones rationem inter ſe habebunt è ratione altitudinum datâ, et ratione fluxionum baſium, ſive datâ, ſive mutabili, componendam.

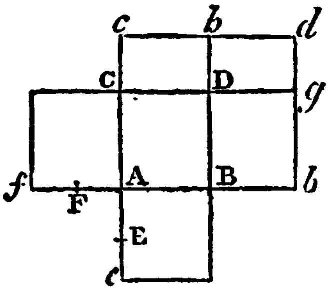
Demonſtratur caſus uterque ad exemplum præcedentis, colligendo rationes primas incrementorum ſolidorum naſcentium, ſive evaneſcentium eorundem ultimas.

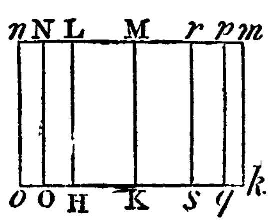
Cor. Parallelopipedorum rectangulorum abx, cdy ſunt parallelopipeda $ab\dot{x}$, $cd\dot{y}$ fluxiones, adhibitâ ſimili atque in præcedentibus notarum interpretatione.

THEOREMA

THEOREMA VIII.

Si rectanguli latus utrumque fluat, et ſumantur rectæ, quæ, ſenſu definitionis quintæ, laterum ſint fluxiones; rectangulorum ſub latere utroque et alterius lateris fluxione ſumma, ſiquidem latus utrumque eodem modo fluat, ſin latera contrariè fluant, horum rectangulorum differentia, rectanguli fluxio erit.





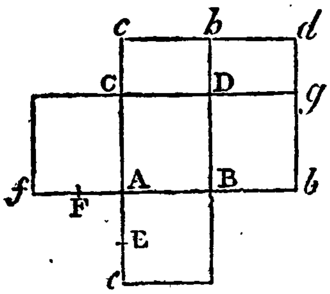
Rectanguli AD latera AB, AC fluant, ac primùm quidem eodem modo, hoc eſt, ut vel creſcat utrumque, vel utrumque decreſcat. Capiantur rectæ AF, AE, quarum AF ſit ad AE, ut velocitas, quâ punctum generans rectam AB è loco A egreditur, ad velocitatem quâ punctum generans rectam AC ex eodem loco A decedit. Progredientibus autem punctis B, C, rectæ AF, AE vel permaneant, vel eâ lege fluant, ut velocitatum fluendi rectarum AB, AC, rationes inter ſe, vel permanendo, vel fluendo, ſervent. Si capiantur punctorum B, C, F, E loca quævis B, C, *f*, *e*, quæ ſimul ea appulerint, erunt A*f*, A*e* rectarum AB, AC, ſenſu definitionis quintæ, fluxiones. Completiſque rectangulis AD, C*f*, B*e*, dico rectangula C*f*, B*e*, ſimul ſumpta, rectanguli AD eſſe fluxionem.

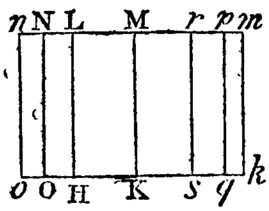
Exponatur recta LH datæ cujuſvis magnitudinis, cui recta LM ad perpendiculum inſiſtens eâ lege fluat, ut rectangula AD, LHKM ſemper ſint inter ſe æqualia, ſi ſumantur punctorum B, C, M loca illa quæ ſimul ea appulerint. In rectâ ML capiatur LN, quæ habeat ad AF, AE rationes eas quas velocitas, quâ punctum L, rectam LM generans, è loco L egreditur, ad velocitates, quibus puncta rectas AB, AC generantia è loco A diſcedunt. Et LN aut permaneat, aut eâ lege ſemper fluat, ut rectæ LN longitudines ſint inter ſe ut puncti M velocitates, ſi capiantur punctorum M, N loca, quæ ſimul ea appulerint, M, *n*. Erit igitur L*n*, ſenſu definitionis quintæ, fluxio rectæ LM. Completoque rectangulo H*n*, erit rectangulum H*n* fluxio rectanguli LK (per Th. VI. Cor.) Rectangulorum autem LK, AD fluxiones ſemper ſunt inter ſe æquales, ſi capiantur punctorum B, C, M loca quæ ſimul ea appulerint (per Th. II. hujus). Rectangulum igitur H*n* rectanguli AD fluxioni æquale erit.

Sint *b*, *c*, *m* punctorum B, C, M loca quævis, quæ ſimul ea appulerint, prioribus B, C, M ulteriora, et compleantur rectangula A*d*, L*k*. Rectanguli verò A*d* lateribus *bd*, *cd*, occurrant CD, BD productæ in *g*, *h* punctis. Rectangulum H*n* ad rectangulum C*f* rationem habet è rationibus HL ad AC et L*n* ad A*f* compoſitam. Sed cùm L*n*, A*f* fluxiones ſint rectarum LM, AB, quarum M*m*, B*b* ſunt incrementa fluendo ſimul facta; ratio L*n* ad A*f*, naſcentis M*m* ad naſcentem B*b* prima erit (per Th. I. huj.) Rectangulum igitur H*n* ad rectangulum C*f* rationem habet è ratione datæ HL ad AC,

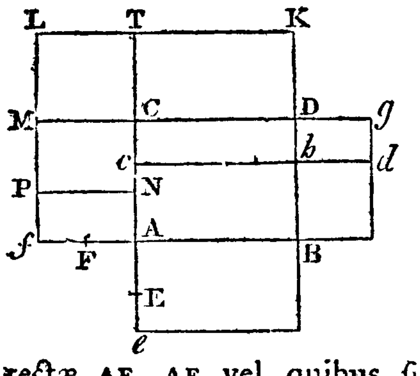
 ſive

Theor. VIII.





ſive MK ad BD, naſcentiſque M*m* ad naſcentem B*b* primâ compoſitam. Sed ratio ex hiſce compoſita prima eſt rectanguli naſcentis M*k* ad rectangulum naſcens B*g*. Rectangulum igitur H*n* ad rectangulum *cf* rationem habet, quæ prima eſt naſcentis M*k* ad naſcens B*g*. Simili argumento concludetur, rectangulum H*n* ad rectangulum B*e* rationem habere, quæ naſcentis M*k* ad naſcens D*c* prima eſt. Rectangulum igitur H*n* ad rectangula duo *cf*, B*e* ſimul ſumpta rationem habet, quæ prima eſt naſcentis M*k* ad duo naſcentia B*g*, D*c* ſimul ſumpta (Elem. V. 24.) hoc eſt, ſi ad rectam MK applicetur MK*qp* duobus B*g*, D*c* ſimul ſumptis æquale, quæ prima eſt naſcentis M*k* ad naſcens M*q*. A rectangulo M*q* auferatur MK*sr* rectangulo D*c* æquale. Jam cùm rectangula L*k*, A*d* ſint inter ſe æqualia, necnon LK, AD (id enim poſuimus) rectangulum M*k* gnomoni B*bdc*C æquale erit. Ablatiſque æqualibus M*s*, D*c*, rectangula *rk*, B*d* erunt inter ſe æqualia. Rurſum cùm rectangulum M*q* duobus B*g*, D*c* ſimul ſumptis ſit æquale, ablatis æqualibus M*s*, D*c*, erunt *rq*, B*g* inter ſe æqualia. Quare *rk* erit ad *rq* ut B*d* ad B*g*. (El. V. 7.) Atque inter omnes rectangulorum *rk*, *rq*, B*d*, B*g* magnitudines, quæ ſimul fluendo exſiſtunt, hæc analogia obtinebit. Quare et naſcentium *rk*, *rq* prima inter ipſa ratio eadem erit, quæ naſcentium B*d*, B*g* prima. Sed naſcentibus B*d*, B*g* prima inter ipſa ratio eſt æqualitas. Naſcentibus igitur *rk*, *rq* eſt ipſa æqualitas prima inter ipſa ratio. Additoque communiter M*s*, naſcentium M*k*, M*q* ratio inter ipſa prima æqualium erit. Sed naſcentis M*k* ad naſcens M*q* prima ratio, rectanguli eſt H*n* ad duo rectangula *cf*, B*e* ſimul ſumpta (ex prius oſtenſis). Rectangulum igitur H*n* æquale eſt duobus rectangulis *cf*, B*e* ſimul ſumptis. Idem verò rectangulum H*n* fluxioni rectanguli AD jam olim oſtenſum fuit æquale. Quare eidem fluxioni rectangula duo *cf*, B*e* ſimul ſumpta ſunt æqualia. Q. E. D.



Cas. 2. Si creſcente latere uno, puta AB, latus alterum AC decreſcat, ſit T punctum in rectâ AC datum, ex quo punctum C rectâ verſus A movere incipiat, quo tempore punctum A rectam AB generans è loco ſuo primo A diſcedit. Sumantur rectæ AF, AE, quarum AF ſit ad AE, ut velocitas, quâ punctum A è loco A diſcedit, ad velocitatem quâ punctum C, verſus A movendo, egreditur è loco T. Progredientibus autem punctis B, C, rectæ AF, AE vel quibus ſint longitudinibus permaneant, vel eâ quidem lege fluant, ut velocitatum fluendi rectarum AB, AC rationes, permanendo vel fluendo, inter ſe ſemper ſervent. Si capiantur punctorum B, C, F, E loca quævis B, C, *f*, *e*, quæ ſimul ea appulerint, erunt

A*f*, A*e* rectarum AB, AC fluxiones. Completiſque rectangulis AD, *cf*, B*e*, dico rectangulorum *cf*, B*e* in hoc caſu differentiam, rectanguli AD fluxionem eſſe. Compleantur enim rectangula TB, AL. Rectangulum AD duorum TB, TD differentia eſt. Fluxio igitur rectanguli AD differentia eſt fluxionum duorum TB, TD. (Th. V.) Sed cùm AT data ſit, et A*f* rectæ AB fluxio, rectangulum T*f* fluxio erit rectanguli TB (Th. VI. Cor.) Rurſum cùm rectæ TK, AB, oppoſita utique parallelogrammi latera, ſemper ſint inter ſe æquales; creſcente hâc, illa quoque paribus incrementis creſcet. Et cùm motus puncti C verſus A quicquid rectæ AC detraxerit, id omne rectæ TC apponet, decreſcente AC recta TC creſcet, atque ita quidem, ut inter ſe ſemper ſint æquales hujus et illius fluxiones. Rectis igitur AB, AC contrariè fluentibus, rectæ TK, TC eodem modo fluunt. Et fluentis TK eſt A*f* vel illi æqualis TL fluxio, fluentis autem TC fluxio A*e*. Rectangulorum igitur CL, B*e* ſumma eſt rectanguli TD fluxio; hoc eſt, ſi ad CM applicetur rectangulum CP rectangulo B*e* æquale, ut ſit TP duorum CL, B*e* ſumma, hoc ipſum TP fluxio erit rectanguli TD (per Caſ. 1. huj.) Quare duorum TP, T*f* differentia, hoc eſt AP, rectanguli AD fluxio erit. Sed cùm CP, B*e* ſint inter ſe æqualia (id enim factum) erit AP duorum *cf*, B*e* differentia. Duorum igitur *cf*, B*e* differentia fluxio eſt rectanguli AD. Q. E. D. THEOR. VIII.

COR. 1. Rectanguli xy, ſi latera, x, y, eodem modo fluant, fluxio erit $x\dot{y}+y\dot{x}$. Sin latera contrariè fluant, ut creſcente x, y decreſcat, erit $y\dot{x}-x\dot{y}$ rectanguli fluxio; negatâ nimirum illâ fluxionis parte, quæ ex fluxione lateris decreſcentis provenerit.

COR. 2. Si laterum x, y magnitudines quæ fluendo ſimul exſiſtunt, ſemper ſint æquales, rectangulum xy fit quadratum ex x, et fluxio fit $2x\dot{x}$, quadrati fluxio.

COR. 3. Si rectanguli, lateribus fluentibus, magnitudo manet, latera contrariè fluunt, et eorum fluxiones ipſorum laterum inter ſe rationem gerunt. Vel ſi rectanguli latera contrariè fluant, et tali lege ut eorum fluxiones ipſorum laterum inter ſe rationem gerant, hujus rectanguli magnitudo manebit.

Rectangulum xy, lateribus x, y fluentibus, dato ſpatio æquale maneat. Dico latera x, y, modis contrariis fluere, et $\dot{x}$ eſſe ad $\dot{y}$ ſicut x ad y. Ac primùm latera x, y modis contrariis fluere manifeſtum eſt, ne ſimiliter illis fluentibus rectangulum ipſum, quod manere poſuimus, fluat. At verò lateribus rectanguli xy contrariè fluentibus, fluxio rectanguli rectangulorum $\dot{x}y$, $\dot{y}x$ differentiæ æqualis erit (Th. VIII. Caſ. 2). Sed rectanguli, cujus magnitudo manet, fluxio nulla eſt. Nulla igitur rectangulorum $\dot{x}y$, $\dot{y}x$ differentia. Rectangula igitur illa inter ſe æqualia. Unde $\dot{x}$ erit ad $\dot{y}$ ut x ad y. (El. VI.) Q. E. D.

E contrario, ſi rectarum x, y contrariè fluentium fluxiones rationem inter ſe rectarum ipſarum ſervent, dico rectanguli xy magnitudinem manere. Cùm enim ſit $\dot{x}$ ad $\dot{y}$ ut x ad y rectangula $y\dot{x}$, $x\dot{y}$ erunt inter ſe æqualia. Quare nulla erit horum differentia. Rectanguli igitur xy fluxio nulla. Cujus autem magnitudinis fluxio nulla, ea utique manet. Q. E. D.

COROLLARIA.

COR. 4. Parallelopipedi rectanguli axy, ſi latera x, y eodem modo fluant, tertio a permanente, ſolidorum duorum $a x\dot{y}$, $a y\dot{x}$ ſumma fluxio erit. Sit enim z recta eâ lege mutabilis, ut rectangula az, yx ſemper ſint inter ſe æqualia, ſi conferantur illorum magnitudines quæ fluendo ſimul exſiſtunt. Solida igitur a^2z, ayx ſemper erunt inter ſe æqualia. Ergo et horum fluxiones ſemper inter ſe æquales (Th. II.) Solidi autem a^2z ſolidum $a^2\dot{z}$ fluxio erit (Th. VII. Cor.) Idem igitur ſolidi ayx fluxio. Eſt autem rectangulum $a\dot{z}$ fluxio rectanguli az (Th. VI. Cor.) et rectangulorum az, yx fluxiones ſunt ſemper inter ſe æquales (Th. II.) Et rectanguli yx eſt $y\dot{x}+x\dot{y}$ fluxio (Th. VIII.) Quare $a\dot{z}=y\dot{x}+x\dot{y}$. Solidum igitur $aa\dot{z}$ duobus ſolidis $ay\dot{x}$, $ax\dot{y}$ ſimul ſumptis æquale eſt. Ergo $ay\dot{x}+ax\dot{y}$ ſolidi axy fluxio erit.

Sin creſcente x, y decreſcat, erit $ay\dot{x}-ax\dot{y}$ ſolidi axy fluxio. Quod ſimili argumento ex caſu ſecundo propoſitionis noviſſimæ licet concludere. Q. E. D.

COR. 5. Parallelopipedi rectanguli xyz, cujus latera x, y, z eodem modo fluunt, ſolidorum trium $xy\dot{z}$, $xz\dot{y}$, $yz\dot{x}$ ſumma fluxio erit.

Deſignet litera a rectam quamvis datam; u aliam, eâ lege mutabilem, ut rectangula xy, au ſemper ſint inter ſe æqualia, ſi conferantur magnitudines eorum, quæ fluendo ſimul exſiſtunt. Solida igitur auz, xyz erunt ſemper inter ſe æqualia: unde horum fluxiones inter ſe æquales (Th. II.) Solidi autem auz fluxio eſt duorum $au\dot{z}$, $az\dot{u}$ ſumma (per Cor. præc.) hoc eſt, duorum $xy\dot{z}$, $az\dot{u}$ ſumma. Et rectangulum $a\dot{u}$ fluxio eſt rectanguli au (Th. VI. Cor.) fluxioni igitur rectanguli xy æquale; hoc eſt, $a\dot{u}=x\dot{y}+y\dot{x}$. Ergo $az\dot{u}=xz\dot{y}+yz\dot{x}$; additoque utrinque ſolido $xy\dot{z}$, $xy\dot{z}+az\dot{u}$, hoc eſt fluxio ſolidi auz, ſive ſolidi xyz, $=xy\dot{z}+xz\dot{y}+yz\dot{x}$. Q. E. D.

Sin latera x, y, z, modis diverſis fluant, ſolidi fluxio ex iiſdem partibus conſtabit; ſed negatis iis, quæ ex fluxionibus laterum decreſcentium factæ fuerint. Nempe, ſi creſcentibus x, y, tertium z decreſcat, erit $xz\dot{y}+yz\dot{x}-xy\dot{z}$ ſolidi xyz fluxio. Si creſcente x, duo y, z decreſcant, erit $yz\dot{x}-xy\dot{z}-xz\dot{y}$ ſolidi xyz fluxio.

COR. 6. Si laterum x, y, z magnitudines ſemper ſint inter ſe æquales parallelopipedon xyz ſit cubus ex x, et fluxio ſit $3x^2\dot{x}$, quæ proinde cubi fluxio erit.

COR. 7. Solidi $\frac{xyzu}{a}$, ſi quatuor x, y, z, u eodem modo fluant, fluxio erit $\frac{xyz\dot{u}+xyu\dot{z}+xzu\dot{y}+yzu\dot{x}}{a}$.

Sit s recta aliqua eâ lege mutabilis, ut rectæ $\frac{xy}{a}$ ſemper ſit æqualis. Tum $xy=as$, et $\frac{xyzu}{a}=szu$. Horum igitur fluxiones ſunt inter ſe æquales. Solidi verò szu fluxio eſt $sz\dot{u}+su\dot{z}+zu\dot{s}$ (per Cor. 4.) hoc eſt $\frac{xyz\dot{u}+xyu\dot{z}+azu\dot{s}}{a}$. Sed propter rectangula as, xy inter ſe æqualia erit $a\dot{s}=x\dot{y}+y\dot{x}$, & $\dot{s}=\frac{x\dot{y}+y\dot{x}}{a}$; unde $zu\dot{s}=\frac{xzu\dot{y}+yzu\dot{x}}{a}$; additiſque utrinque $\frac{xyz\dot{u}+xyu\dot{z}}{a}$, ſit $\frac{xyz\dot{u}+xyu\dot{z}}{a}+zu\dot{s}=\frac{xyz\dot{u}+xyu\dot{z}+xzu\dot{y}+yzu\dot{x}}{a}$. Sed horum prius ſolidi szu, ſive $\frac{xyzu}{a}$, fluxioni jam ſuprà oſtenſum eſt æquale. Ergo et poſterius. Q. E. D. Jam

Jam ſi horum æſtimatio Arithmetica ineunda ſit, poſito $a=1$, fluxio quantitatis $xyzu$ erit $xyz\dot{u}+xyu\dot{z}+xzu\dot{y}+yzu\dot{x}$; notis nimirum x, y, z, u, $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{u}$ non jam lineas rectas, aliaſve magnitudines Geometricas, deſignantibus, ſed Geometricarum per numeros æſtimationes ex datâ menſurâ lineari a. COROLLARIA.

Si illarum quatuor x, y, z, u magnitudines fluendo ſimul factæ ſemper ſint inter ſe æquales, quantitas $xyzu$ poteſtas ſit biquadratica à radice x, et fluxio ſit $4x^3\dot{x}$, quæ proinde poteſtatis biquadraticæ fluxio erit.

COR. 8. Simili modo concludetur eſſe $5x^4\dot{x}$ fluxionem quantitatis x^5. Et cùm ſimilium Theorematum infinita ſit progreſſio, fluxio quantitatis x^n erit $nx^{n-1}\dot{x}$.

COR. 9. Fluxio quantitatis $x^{\frac{m}{n}}$ eſt $\frac{m}{n}x^{\frac{m-n}{n}}\dot{x}$. Ponatur enim $x^{\frac{m}{n}}=s$. Harum igitur fluxiones ſunt inter ſe æquales, hoc eſt $\dot{\overline{x^{\frac{m}{n}}}}=\dot{s}$. (Th. II.) Sed quantitatum æqualium, $x^{\frac{m}{n}}$, s, poteſtates homologæ erunt inter ſe æquales. Ergo $x^m=s^n$, et $\frac{x^m}{s}$, ſive $x^{m-\frac{m}{n}}=s^{n-1}$. Duarum autem x^m, s^n, quæ ſemper inter ſe æquales ſunt, fluxiones erunt inter ſe æquales (Th. II.) hoc eſt $mx^{m-1}\dot{x}=ns^{n-1}\dot{s}$ (per Cor. præc.) vel pro s^{n-1} ſubrogato $x^{m-\frac{m}{n}}$, $mx^{m-1}\dot{x}=nx^{m-\frac{m}{n}}\dot{s}$. Æquationis autem noviſſimæ parte utrâque cum quantitate $x^{\frac{m}{n}-m}$ multiplicatâ, proveniet $mx^{\frac{m-n}{n}}\dot{x}=n\dot{s}$. Ergo $\frac{m}{n}x^{\frac{m-n}{n}}\dot{x}=\dot{s}=\dot{\overline{x^{\frac{m}{n}}}}$. Q. E. D.

COR. 10. Quantitatibus x^n, y^m eodem modo fluentibus, quantitatis x^ny^m fluxio erit $ny^mx^{n-1}\dot{x}+mx^ny^{m-1}\dot{y}$. Sin contrariè fluant quantitates x^n, y^m, ut creſcente x^n, y^m decreſcat, quantitatis x^ny^m quantitas $ny^mx^{n-1}\dot{x}-mx^ny^{m-1}\dot{y}$ fluxio erit. Primùm eodem modo fluant. Ponatur $x^n=z$, et $y^m=u$. Erit $nx^{n-1}\dot{x}=\dot{z}$, et $my^{m-1}\dot{y}=\dot{u}$ (Th. II.) Erit etiam $x^ny^m=zu$, et $\dot{\overline{x^ny^m}}=u\dot{z}+z\dot{u}=ny^mx^{n-1}\dot{x}+mx^ny^{m-1}\dot{y}$. Q. E. D. Et alterius caſus ſimilis erit demonſtratio, ex eo ſcilicet, quod $\dot{\overline{x^ny^m}}$ æquabitur $u\dot{z}-z\dot{u}$.

COR. 11. Quantitatis fractæ $\frac{x^n}{a}$, cujus, denominatore permanente, numerator fluit, quantitas fracta $\frac{nx^{n-1}\dot{x}}{a}$ fluxio erit. Quantitas enim x^n ad quantitatem $\frac{x^n}{a}$ rationem datam habet quantitatis a ad unitatem. Ergo fluxio quantitatis x^n ad fluxionem quantitatis $\frac{x^n}{a}$ eandem datam rationem

habet

COROLLARIA. habet quantitatis a ad unitatem (Th. IV.) hoc eſt $nx^{n-1}\dot{x} : \dot{\overline{\frac{x^n}{a}}} = a : 1$. Ergo $\dot{\overline{\frac{x^n}{a}}} = \frac{nx^{n-1}\dot{x}}{a}$. Q. E. D.

COR. 12. Quantitatis fractæ $\frac{a}{y^m}$, cujus, numeratore permanente, denominator fluit, quantitas fracta $\frac{-ma\dot{y}}{y^{m+1}}$ fluxio erit; ſigno — id nimirum ſignificante, contrariis ſemper modis fluere quantitatem fractam et denominatorem ejus; nempe decreſcere $\frac{a}{y^m}$ ſi y^m creſcat, ſi hæc decreſcat, illam creſcere.

Ponatur $\frac{a}{y^m} = s$. Tum $a = y^m s$. Quantitas igitur $y^m s$, ex fluentibus y^m, s, mutuâ multiplicatione facta, datæ ſemper æqualis eſt, vel quod perinde eſt, eadem ſemper manet. Hoc autem fieri nequit, niſi contrariè fluentibus illis y^m, s, quantitates $msy^{m-1}\dot{y}$, $y^m\dot{s}$ inter ſe æquales ſint. Sint enim inæquales, et inæqualium ſit z differentia. Erit igitur z fluxio quantitatis $y^m s$, è duabus y^m, s, contrariè fluentibus, mutuâ multiplicatione factæ (per Cor. 10). Fluet igitur quantitas $y^m s$, ſiquidem fluxionem habeat. Non igitur eadem manet. Quod eſt abſurdum, cùm manere oſtenſa ſit. Duæ igitur $msy^{m-1}\dot{y}$, $y^m\dot{s}$ non ſunt inæquales. Æquales igitur. Duæ igitur, $\frac{ms}{y}\dot{y}$, $\dot{s}$, inter ſe æquales. Hoc eſt, ſi pro s ſcribatur $\frac{a}{y^m}$, $\frac{ma}{y^{m+1}}\dot{y} = \dot{s}$. Q. E. D.

COR. 13. Quantitatis fractæ $\frac{x^n}{y^m}$, cui fluunt numerator et denominator, quantitas fracta $\frac{nx^{n-1}\dot{x}}{y^m} - \frac{mx^n\dot{y}}{y^{m+1}}$ fluxio erit; modò quantitas ipſa fracta $\frac{x^n}{y^m}$ eodem modo quo denominator ejus fluat.

Ponatur $\frac{x^n}{y^m} = z$. Quantitatum igitur, $\frac{x^n}{y^m}$, z, fluxiones erunt inter ſe æquales, hoc eſt $\dot{\overline{\frac{x^n}{y^m}}} = \dot{z}$.

Propter æquales autem $\frac{x^n}{y^m}$, z, erunt x^n, zy^m inter ſe æquales. Et æqualium x^n, zy^m fluxiones erunt inter ſe æquales. Ergo $nx^{n-1}\dot{x} = mzy^{m-1}\dot{y} + y^m\dot{z}$; modò y^m & z eodem modo fluant.

Ergo $nx^{n-1}\dot{x} - mzy^{m-1}\dot{y} = y^m\dot{z}$.

Ergo $\frac{nx^{n-1}\dot{x}}{y^m} - \frac{mz\dot{y}}{y} = \dot{z}$.

Hoc eſt, ſi pro z reſtituatur $\frac{x^n}{y^m}$, $\frac{nx^{n-1}\dot{x}}{y^m} - \frac{mx^n\dot{y}}{y^{m+1}} = \dot{z} = \dot{\overline{\frac{x^n}{y^m}}}$.

Sin contrariè fluant quantitas ipſa fracta $\frac{x^n}{y^m}$ et denominator ejus, erit $\frac{nx^{n-1}\dot{x}}{y^m} + \frac{mx^n\dot{y}}{y^{m+1}}$ quantitatis fractæ fluxio.

COR.

COR. 14. Si quantitas P ex quantitatibus quotvis fluentibus A, B, C, D, &c. mutuâ multiplicatione facta ſit, quantitas fracta $\frac{\dot{P}}{P}$ quantitatum fractarum $\frac{\dot{A}}{A}$, $\frac{\dot{B}}{B}$, $\frac{\dot{C}}{C}$, $\frac{\dot{D}}{D}$ ſummæ æqualis erit. COROLLARIA.

Cùm enim P quantitati $A \times B \times C \times D$ æqualis ſit, erit $\dot{P} = BCD \times \dot{A} + ACD \times \dot{B} + ABD \times \dot{C} + ABC \times \dot{D}$ (per Cor. 7.)

Sed $BCD = \frac{P}{A}$, et $ACD = \frac{P}{B}$, et $ABD = \frac{P}{C}$, et $ABC = \frac{P}{D}$.

Ergo $\dot{P} = \frac{P\dot{A}}{A} + \frac{P\dot{B}}{B} + \frac{P\dot{C}}{C} + \frac{P\dot{D}}{D}$.

Ergo $\frac{\dot{P}}{P} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}$. Q. E. D.

Cor. 15. Si P quantitati fractæ æqualis ſit, cujus numerator ex quantitatibus quotcunque fluentibus A, B, C, D mutuâ multiplicatione factus eſt, denominator ex quantitatibus quotcunque fluentibus, G, H, K, L; erit $\frac{\dot{P}}{P} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} - \frac{\dot{G}}{G} - \frac{\dot{H}}{H} - \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$.

Sit $Q = P \times G \times H \times K \times L$.

Ergo $\frac{\dot{Q}}{Q} = \frac{\dot{P}}{P} + \frac{\dot{G}}{G} + \frac{\dot{H}}{H} + \frac{\dot{K}}{K} + \frac{\dot{L}}{L}$ (per Cor. præc.)

Sed cùm $P = \frac{A \times B \times C \times D}{G \times H \times K \times L}$, erit $P \times G \times H \times K \times L$, hoc eſt $Q, = A \times B \times C \times D$.

Ergo $\frac{\dot{Q}}{Q} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}$ (per Cor. præc.)

Quare $\frac{\dot{P}}{P} + \frac{\dot{G}}{G} + \frac{\dot{H}}{H} + \frac{\dot{K}}{K} + \frac{\dot{L}}{L} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}$; et eadem utrinque auferendo, $\frac{\dot{P}}{P} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} - \frac{\dot{G}}{G} - \frac{\dot{H}}{H} - \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$. Q. E. D.

Corollariis duobus noviſſimis expoſita ſunt Maclaurini Theoremata duo, quorum uſus in computationibus fluentium Algebraicis latè patet. (*Maclaurin*, *Treatiſe of Fluxions*, *Book* II. *C.* I. *Prop.* IV & V.)

ITA demum formulas, quarum ope, quantitatum omnium quæ ſub generali *Genitarum* (*) nomine Newtono ſunt comprehenſæ, Algebraicè æſtimari fluxiones ſolent, ex paucis illis doctrinæ fluxionalis deduximus principiis, quæ octo utique Theorematis concluſa, demonſtrationibus confirmavimus Newtonianis; hoc eſt, è Newtonianâ Rationum Primarum et Ultimarum Geometriâ arceſſitis. Ex quo genere rationum concluſiones, modò peritè ſint deductæ, nec ipſis Archimedeis, meo equidem judicio, aut evidentiâ aut venuſtate cedunt. Nec dubito tamen fore multos, qui noſtram in hiſce condendis diligentiam, ut nimiam, reprehendent. Sed ii

(*) Vide Princip. Lib. II. Lemma 2.

potiſſimùm

Corollaria. potiſſimùm erunt, qui cùm ea quæ tradimus ipſi ſint conſecuti, aliorum conditionis immemores, ſupervacanea putant quibus ipſi non egent; vel quorum judicia parùm reformido, homines haud indocti quidem illi ſed incurioſi et inelegantes, qui cùm omnia feſtinanter ipſi arripuerint, utpote minùs ſcilicet ingenii perpolitionem, qui potiſſimus eſt doctrinæ fructus, quàm inanem quandam ſcientiæ famam ex ſtudiis captantes, conciſum ſcribendi genus ipſi ſummoperè affectant, alios explicatiùs diſſerentes ægrè ferunt. Qui proprias ſcriptiones non tam verborum brevitate, quàm rerum prætermiſſione et tranſcurſu contrahunt. Ac veluti ſapiente nihil magis indignum exiſtimarent, quàm clarè et apertè loqui, Theoremata Geometrica ſub involucris ſymbolorum proferunt, dicam, an occultant. Horum demonſtrationes plerumque ſtudioſè celant. Aut ſi forte benigniùs agere velint, argumentorum vice, computationes edunt; ex quibus nulla certè probatio efficitur, niſi ei qui ad calculos ratiocinia revocaverit. Quem ſane nimium laborem, niſi in rebus maximis, nemo cordatior ſubierit. Vel ſiquando, in re aliquâ difficiliore, Geometricèediſſerendi facultatem atque artem oſtentare libeat, tum verò, cùm infenſam ſibi noverint Graiorum Geometriam, quam votis nunquam placavere, ad præſtigiatricem *Indiviſibilium*, quam vocant, diſciplinam confugere coacti, ſanioris Geometriæ cultoribus miſerè ſe irridendos præbent; pro veris demonſtrationibus demonſtrationum ſimulacra quædam venditantes, qualia ferè inſaniens illa ſapientia ſuppeditare ſolet. Cujus quidem individua, quæ ſemper illis ſunt in ore, quid ſint, quæ, cùm ipſa magnitudinis ſint expertia, juncta tamen magnitudines abſolvant, nondum natus eſt Oedipus, qui mortalibus expediat.

FINIS TOMI PRIMI.

CORRIGENDA

IN TEXTU.

P. 60, lin. 2, pro *unum*, lege *unam*.
P. 65, lin. 9, pro IV, lege V.
P. 73, lin. penult. pro *x*, lege *z*.
P. 246, lin. 11 & 12, delendæ ſunt voces *datam aliquam inter ipſas proportionem ſervantibus*.
P. 249, lin. 12, pro *duplicatâ*, lege *triplicatâ*

IN NOTIS

P. 237, not. (a) lin. 2, pro *ſit*, lege *ſit*.
———— lin. 6, pro *abiit*, lege *abſit*.
P. 379, not. (oo), lin. 2, pro *Not*. a, lege *Not*. x.
P. 383, lin. 21, pro *Philoſophicæ*, lege *Philoſophica*.

To the Book-binder.

Put the two Half-ſheets of Copper-plate between Page 378 and 379.

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